List of principal symbols

\( a = (M(3000)_{1}^{-1} - 0.24) \) or 0.04, whichever is the larger

\( B = \text{slope of the variation of } s \text{ with } M(3000)_{1}^{-1} \text{ for } r = 0.85 \)

\( B_{1} = \text{slope of the variation of } s_{1} \text{ with } M(3000)_{1}^{-1.5} \text{ for } r = 1 \)

\( c = \text{intercept of linear fit to } h_{T} \text{ variation with } D \text{ for } r = 0.85 \)

\( c_{1} = \text{intercept of linear fit to } h_{T} \text{ variation with } D \text{ for } r = 1 \)

\( D = \text{ground range of a single hop} \)

\( D_{\text{max}} = \text{maximum range for a single hop} \)

\( f = \text{signal frequency} \)

\( f_{e} = \text{equivalent frequency for which } h_{T} = h_{m} \text{ at oblique incidence} \)

\( f_{0} = \text{critical frequency for layer} \)

\( f_{0E} = \text{critical frequency of E-layer} \)

\( f_{0F2} = \text{critical frequency of F2-layer} \)

\( \text{FOT} = \text{optimum transmission frequency} \) (= 0.85 MUF)

\( h' = \text{virtual-reflection height for vertical incidence} \)

\( h_{p} = \text{Shimazaki's estimate of } h_{m} \)

\( h_{m} = \text{height of layer peak} \)

\( h_{mE} = \text{height of E-layer peak} \)

\( h_{mF2} = \text{height of F2-layer peak} \)

\( h_{T} = \text{mirror-reflection height for oblique incidence} \)

\( k = \text{curved-Earth correction factor} \)

\( M(D)F2 = \text{M-factor for single hop, reflected by the F2-layer to a range } D \text{ km} \) (= MUF/f0F2)

\( M(3000)_{1} = M(3000)F2 \text{-factor scaled from synthesised ionogram} \)

\( M(3000)_{0} = M(3000)F2 \text{-factor derived from oblique propagation equations} \)

\( \text{MUF} = \text{basic maximum usable frequency} \)

\( r = \text{ratio of signal frequency to MUF} \) (= \( f/MUF \))

\( R_{E} = \text{radius of curvature of Earth's surface} \)

\( s = \text{slope of linear fit to } h_{T} \text{ variation with } D \text{ for } r = 0.85 \)

\( s_{1} = \text{slope of linear fit to } h_{T} \text{ variation with } D \text{ for } r = 1 \)

\( w = \text{ground range as a fraction of its maximum value} \) (= \( D/D_{\text{max}} \))

\( x = \text{ratio of F2- and E-layer critical frequencies} \) (= \( f_{0F2}/f_{0E} \))

\( y_{m} = \text{layer semithickness} \)

\( y_{mE} = \text{E-layer semithickness} \)

\( y_{mF2} = \text{F2-layer semithickness} \)

\( \alpha = \text{slope of variation of } c \text{ with } M(3000)_{1}^{-1} \text{ for } r = 0.85 \)

\( \alpha_{1} = \text{slope of variation of } c_{1} \text{ with } M(3000)_{1}^{-1.5} \text{ for } r = 1 \)

\( \beta_{0} = \text{elevation angle of ray hop} \)

\( \delta = \text{error in simplified estimate of } h_{T} \)

\( \delta_{S} = \text{maximum of } \delta \)

\( \Delta = \text{correction to linear variation of } h_{T} \text{ with } D \text{ for } r = 1 \)

\( \varepsilon = \text{average of } \delta \)

\( \phi = \text{function of } f \text{ and } f_{0}, \text{defined by Booker and Seaton [5]} \)

1 Introduction

The mirror-reflection height is that height at which an equivalent plane mirror would have to be placed to reflect unrefracted waves with the same elevation angles at the receiver and transmitter as for the real ionosphere. It is a convenient concept in characterising the geometry of an ionospherically supported ray hop for use in propagation predictions, in the evaluation of antenna gains, path loss and group path delay [1,2].

For propagation modes reflected from the F2-layer, the mirror-reflection height is difficult to estimate accurately; in several of the larger HF propagation prediction procedures it is calculated iteratively [1,2]; alternatively equations based on some mean reference model ionosphere can be employed [3]. The advantages of the iterative procedures are that the effects of spatial variations in the height of the reflecting layer and changes in underlying ionisation can be taken into account; also that the evaluation can be continued until the required accuracy is achieved. For many applications, however, the computation must be completed in a restricted period of time or may be based on ionospheric data of accuracy which does not justify rigorous solution. In these cases the mean-ionosphere approach is favoured.

Reflection by a spherically curved parabolic model layer was initially considered by Appleton and Beynon [4]. From their equations Booker and Seaton [5] derived the
relationship for the virtual-reflection height, of a signal frequency \( f \) at vertical incidence:

\[
h' = \text{hm} + \text{ym}f(f/\text{fo})
\]

(1)

where \( \text{hm}, \text{ym} \) and \( \text{fo} \) are the layer peak height, semithickness and critical frequency, respectively. The function \( \phi \) is zero when \( f \) equals \( 0.834/\text{fo} \); hence, \( \text{hm} \) equals \( h' \) at this frequency. From Martyn’s equivalent path theorem, generalised to allow for the Earth’s curvature (radius = \( R_E \)), \( \text{hm} \) equals the mirror-reflection height, \( h_T \), for an equivalent, oblique-incidence frequency [6]:

\[
f_0 = \frac{0.834/\text{fo}}{k} \sin \left( \beta_0 + \frac{D}{2R_E} \right)
\]

(2)

where \( \beta_0 \) is the hop elevation angle and the curved-Earth correction factor, \( k \), is 1.0 for short hop length, \( D \), and about 1.2 for large \( D \). Shimazaki [3] derived a simple empirical expression for an approximation to \( \text{hm}, h_p \), using data from a global network of ionosondes:

\[
\text{hm} \approx h_p = \frac{1490}{\text{M}(3000)\text{F}2} - 176 \text{ km}
\]

(3)

the \( \text{M}(3000)\text{F}2 \)-factor being scaled routinely from ionograms. Subsequently, this equation has often been used to give estimates for \( h_p \) with the inherent assumption that the signal frequency is approximately equal to \( f_0(D) \); an application which Shimazaki had not intended. Amended coefficients were suggested by Wright and McDuffie [7]; however, eqn. 3 remains in use today as an approximate method of determining mirror-reflection heights [8].

The Shimazaki formula (eqn. 3) is based on a single parabolic reflecting layer. Hence, for F2-layer propagation, the effects of underlying ionisation beneath the F2-layer are not accounted for. A more complete model ionospheric profile, with variable amounts of underlying ionisation, has been proposed by Bradley and Dudeney [9], consisting of a combination of linear and parabolic segments to represent the E-, F1- and F2-regions. To allow exact analytic solution of ray-path parameters, Milsom [10] has fitted quasilinear and quasiparabolic segments to give a close approximation to the Bradley–Dudeney profile. In a previous paper, the present author listed Milsom’s equations for the ground range and group path of a hop reflected by such a model ionosphere, and used them to develop an algorithm for simplified, noniterative estimation of basic maximum usable frequency [11]. Allowance was made for variations in both underlying ionisation and F2 peak height. In this paper complementary algorithms for estimation of mirror-reflection height are derived and assessed.

2 Model parameters used in the study

The general form of the Milsom approximation to the Bradley–Dudeney model ionospheric profile is demonstrated by Fig. 1. The E-layer peak height, \( \text{hm}E \), and semithickness, \( \text{ym}E \), are fixed at 110 km and 20 km, respectively; hence, the entire profile is characterised by the remaining four independent variables, namely the E- and F2-layer critical frequencies, \( \text{fo}E \) and \( \text{fo}F2 \), and the F2-layer peak height and semithickness, \( \text{hm}F2 \) and \( \text{ym}F2 \). Bradley and Dudeney [8] concluded that, in practice, the ratio \( \text{hm}F2/\text{ym}F2 \) generally lies between 2.0 and 5.0. Here, as in the previous study of basic maximum usable frequency (MUF), a fixed ratio of 3.5 is adopted, thereby reducing the number of independent ionospheric variables to three.

For a given such ionospheric profile, \( h_T \) depends on these three ionospheric parameters and on the signal frequency \( f \) and the ground range, \( D \). The Milsom equations show that \( h_T \) has the functional form:

\[
h_T = h_T(\text{hm}F2, x, \text{fo}E/\text{fo}F2, f, D)
\]

(4)

where \( x \) is the ratio (\( \text{fo}F2/\text{fo}E \)). In this paper use is made of a parameter \( r \), the ratio (\( f/\text{MUF} \)). In Reference 11 it is shown that:

\[
f = r\text{fo}F2M(x, \text{hm}F2, D)
\]

(5)

Hence from eqns. 4 and 5:

\[
h_T = h_T(\text{hm}F2, x, r, D)
\]

(6)

A range of \( \text{hm}F2 \) between 250 km and 500 km is considered here, and six values, 50 km apart, are found to be adequate to characterise the behaviour of \( h_T \). The value of \( h_T \) is very dependent on \( x \) near 1.7, which is the limit of applicability of the model, but is more slowly varying at large \( x \) : of 10, 5, 3.33, 2.5, 2.22, 2.08 and 2.0 are examined. The algorithms devised here are not, therefore, designed for use with \( x \) less than 1.95. The greatly increased complexity required to evaluate \( h_T \) for any lower \( x \) is not considered justified by the model approximations used. Cases of low \( x \) are rare and are predominantly found for the daytime high-latitude ionosphere in winter at sunspot minimum.

The variation of \( h_T \) with range, \( D \), was then studied for each model ionosphere profile with signal frequency equal to various fractions, \( r \), of the basic MUF.

3 Calculation of mirror-reflection height for model ionospheric profiles

For each of the 42 model ionospheric profiles, an ordinary-wave ionogram trace was synthesised using the group-path equations [11] and an \( \text{M}(3000)\text{F}2 \)-value was scaled using the standard URSI slider. The value of MUF for a fixed \( D \) was calculated iteratively by the procedure given by Lockwood [11] and a value of \( h_T \) then calculated for that \( D \) and a fixed \( r (f/\text{MUF}) \) by iterating the elevation angle, \( \beta_0 \). In this way \( h_T(D) \) curves were calculated for various \( r \) and for each of the model profiles.
Fig. 2 shows a set of $h_T(D)$ curves for various values of $r$ with $x$ of 3.33 and $hmF2$ equal to 250 km and 500 km. The important range of $r$ between 0.6 and 1.0 is considered for both these two model profiles. It can be seen that, over a large part of this range of $r$, $h_T$ at a fixed $D$ is only weakly dependent on $r$ (for $r = 0.8 - 0.95$ when $hmF2 = 500$ km and for $r = 0.7 - 0.95$ when $hmF2 = 250$ km, except at the shortest ranges). In every case where $r \geq 0.7$, $h_T$ is greatest at all $D$ when $r$ is unity; it is much greater than for the lower $r$ at the smallest and largest distances.

The variations of $h_T$ with $x$ for $hmF2$ of 250 km and 500 km are illustrated by Fig. 3 for $r = 1.0$ and by Fig. 4 for $r = 0.85$. The $h_T$ is most dependent on $x$ near the lower limit of 2.0, and increasing the underlying ionisation (decreased $x$) results in an $h_T$ rise which is greater at large $D$ and large $hmF2$.

The variation of $h_T$ with $hmF2$ is considerably more regular than that with $r$ or $x$, as is demonstrated by Fig. 5 for $r = 0.85$ and $x = 3.33$. At a fixed $D$, $h_T$ increases approximately linearly with $hmF2$. Note that at low $hmF2$ ($\leq 350$ km), high $x$ ($\geq 3.3$) and $f = FOT$, $h_T$ equals $hmF2$ to within 10%.

4 Algorithms for rapid evaluation of mirror-reflection height

Inspection of sets of $h_T(D)$ curves like that shown in Fig. 2 reveals that $h_T$ is a complicated function of $r$. However, for the range of $r$ between about 0.75 and 0.95, $h_T$ is approximately a linear function of $D$; hence, this range can be characterised with relative simplicity. As the majority of HF communication circuits employ this range of $r$ (by operating as close as interference permits to the optimum transmission frequency, FOT, defined by $f = FOT$ when $r = 0.85$), such a simplification should have wide applications.
A second requirement for prediction of mirror-reflection height exists at the basic MUF \((r = 1.0)\). Many prediction procedures require this information in order that propagation losses at frequencies above the monthly median basic MUF can be assessed.

In the following Sections algorithms are developed for these two ranges of \(r\).

### 4.1 Algorithms applicable to a range of signal frequencies about the FOT

Fig. 2 demonstrates that, at all but the shortest ranges, \(h_T\) can be described by the linear form when \(0.75 < r < 0.95\):

\[
h_T = sD + c
\]

Linear regression lines were fitted to all \(h_T(D)\) curves for \(r = 0.85\) (\(\ell = \text{FOT}\)). The slopes, \(s\), of these regression lines are plotted in Fig. 6 as a function of the inverse of the \(M(3000)\). M(3000)F2-factor scaled from the synthesised ionogram; this \(M\)-factor is referred to here as \(M(3000)\), to be consistent with Reference 11. For a fixed value of \(x\), \(s\) is roughly inversely proportional to \(M(3000)\), and is approximately given by:

\[
s = \left(\frac{1}{M(3000)_i} - 0.24\right)B \quad \text{for} \quad M(3000)_i \leq 3.57
\]

It was found that use of eqn. 8 for \(M(3000)_i\), greater than 3.57 gave large errors in \(h_T\) at large \(x\) and \(D\). This problem was overcome by the introduction of the condition:

\[
s = 0.04B \quad M(3000)_i > 3.57
\]

The slopes of the fitted lines shown in Fig. 6, \(B\), vary linearly with the inverse of the fourth power of \(x\):

\[
B = 0.03 + 14.0/x^4
\]

The intercept, \(c\), of the regression lines is also approximately inversely proportional to \(M(3000)\), at fixed \(x\) (see Fig. 7), and straight lines which pass through the point \((c = 358, 1/M(3000)_i = 0.35)\) can be adopted. Least-squares fits to these lines give the values of their slope, \(\alpha\), shown in Fig. 8 as a function of \(x\). The solid curve satisfies

\[
\alpha = 1880 - 32000/x^5
\]

and \(c\) is given by:

\[
c = 358 + \alpha\left(\frac{1}{M(3000)_i} - 0.35\right)
\]

Eqns. 7–12 can be combined into the equation:

\[
h_T = 358 - (11 - 100\alpha)\left(18.8 - \frac{320}{x^5}\right) + aD\left(0.03 + \frac{14}{x^2}\right) \text{km}
\]

where \(a = (1/M(3000)_i) - 0.24\), or 0.04 whichever is the
larger. The full eqn. 13 allows simple and rapid evaluation of \( h_r \), the accuracy of which is assessed in Section 5.

Comparison with eqn. 2 shows that the above full equation is considerably more complex than the Shimazaki equation which is widely used to predict \( h_T \). However, the full equation can be simplified, with some loss of accuracy, to separate equations for night and day by adopting representative mean values of \( x \):

\[
\begin{align*}
\text{night (} x \sim 10 \text{)} & \quad h_T = \frac{1880}{M(3000)} - 300 \text{ km} \\
\text{day (} x \sim 3 \text{)} & \quad h_T = 160 + (0.143D + 1800) \\
& \quad \times \left\{ \frac{1}{M(3000)} - 0.24 \right\} \text{ km}
\end{align*}
\]

The night equation has the same form as that of Shimazaki, with new values for the constants: the increase in \( h_T \) with \( D \) is neglected in both expressions. On the other hand, Fig. 4 demonstrates that this increase needs to be included for the \( x \) values common in the dayside ionosphere \((x < 3)\), and accordingly the day equation contains an additional distance term. The accuracy of the simplified eqns. 14 and 15 is also assessed in Section 5.

### 4.2 Algorithm applicable to a signal frequency equal to the basic MUF

Fig. 3 shows that the general form of the \( h_T(D) \) curves for \( r = 1.0 \) cannot be approximated by a single linear relationship as was possible for the \( r = 0.85 \) curves. Figs. 9 and 10 demonstrate that the increase of \( h_T \) with decreasing \( D \) at the lowest distances can be accounted for by the use of a single correction term \( \Delta \), chosen such that \((h_T - \Delta)\) has a near-linear variation with \((D/D_{\text{max}})\), where \( D_{\text{max}} \) is the maximum range (corresponding to zero elevation of the ray path, \( h_0 \)) and is given by eqns. 18 and 19 of Reference 11. Curves for \( h_T \) (solid line) and \((h_T - \Delta)\) (broken line) are given for \( \text{hmF}2 \) of 250, 350, 400 and 500 km with \( x \) of 10.0 in Fig. 9 and with \( x \) of 2.2 in Fig. 10.

The form for \( \Delta \) used in Figs. 9 and 10 is:

\[
\Delta = 23 \left( \frac{1}{w} - 1 \right)
\]

The second expression allows for the flattening of the \( h_T(D) \) curves as \( D \) approaches \( D_{\text{max}} \).

The intercept \( c_1 \) can be fitted in the same way as \( c \) in the preceding Section, giving expressions of the same form as eqns. 11 and 12 but with different constants:

\[
c_1 = 35 + a_1 \left\{ \frac{1}{M(3000)} - 0.225 \right\}
\]

where

\[
a_1 = 1785 - 4000/x^3
\]

and \( M(3000) \) is the ordinary-wave M-factor from the iteration of ray-path solutions for a range of 300 km. In general, this differs from \( M(3000) \), and is derived in the MUF algorithm procedure given in Reference 10. The slope \( s_1 \) varies linearly with \( M(3000) \) to the power \(-1.5\):

\[
s_1 = 230 + B_1(M(3000))^{-1.5} - 0.14
\]

and

\[
B_1 = 325 + 6.4 \times 10^4/x^{3.8}
\]

Eqns. 16-21 can be used to evaluate \( h_T \) for \( f = \text{MUF} \) at a given \( D \), in conjunction with the values for \( M(3000) \) and \( D_{\text{max}} \) obtained in the evaluation of MUF by the simplified algorithm by Lockwood [11]. The accuracy of the \( h_T \) values is assessed in Section 5.2.

### 5 Accuracy of mirror-reflection height algorithms

The accuracy of the algorithms presented in Sections 4.1 and 4.2 are assessed by comparing the predicted mirror-reflection height, for a given range and ionospheric profile, with results of the full calculation using the Milsom equations. The error as a percentage of the correct value, \( \delta \), is evaluated; positive values of \( \delta \) representing overestimates of mirror-reflection height.
5.1  Accuracy of predictions for the range of \( r \) between 0.75 and 0.95

Fig. 11 plots simplified predictions using eqn. 13 (broken lines) and fully calculated values (solid lines) for the mirror-reflection height, \( h_T \), and elevation angle, \( \beta_T \), as a function of distance, \( D \).

Fig. 11  Fully calculated values and approximate estimates given by eqn. 13 (\( r = 0.85 \)) for the mirror-reflection height, \( h_T \), and elevation angle, \( \beta_T \), as a function of distance, \( D \).

![Image](image-url)

Table 1: Largest percentage errors (positive values are overestimates) in using eqn. 13, \( \Delta m \), for \( r \) of 0.75, 0.80, 0.85, 0.90 and 0.95

<table>
<thead>
<tr>
<th>( D = 1000 \text{ km} )</th>
<th>( x )</th>
<th>10.0</th>
<th>5.0</th>
<th>3.33</th>
<th>2.5</th>
<th>2.22</th>
<th>2.08</th>
<th>2.0</th>
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<tbody>
<tr>
<td>250 4.9 1.8</td>
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<td>-2.6</td>
<td>-2.7</td>
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<tr>
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<tr>
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<tr>
<td>450 3.1 0.6</td>
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<td>-7.9</td>
<td>-9.1</td>
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<td></td>
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<tr>
<td>500 2.1 0.3</td>
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<td>-8.5</td>
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<th>2.08</th>
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</tbody>
</table>

largest error, \( \Delta m \), with \( r \) of 0.75–0.95 for each of the 42 model ionospheric profiles for which ray-tracing solutions are possible. The errors for \( D \) of 3000 km were largely independent of \( r \), whereas those for 1000 km are largest for the \( r = 0.95 \) case and generally decrease with decreasing \( r \). Errors are less than 5% in all but one case: when \( x \geq 2.5 \). For the full range of \( x \) considered here (2.0–10.0), values are accurate to within about 10%; worst errors arising from high \( D \), low \( x \) and high \( hmF_2 \) cases. For distances near 1000 km, eqn. 13 tends to overestimate \( h_T \) at high \( x \) and underestimate at low \( x \); for the greater ranges it tends to overestimate at low and high \( x \) and underestimates for intermediate \( x \). At the shortest ranges eqn. 13 gives underestimates, particularly for the larger \( r \).

Table 2 gives the equivalent errors to those in Table 1 for mirror-height evaluation using the Shimazaki eqn. 2.

Table 2: Largest percentage errors (positive values are overestimates) in using the Shimazaki eqn. 3, \( \Delta m \) for \( r \) of 0.75, 0.80, 0.85, 0.90 and 0.95

<table>
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<tr>
<th>( hmF_2, \text{ km} )</th>
<th>( x )</th>
<th>10.0</th>
<th>5.0</th>
<th>3.33</th>
<th>2.5</th>
<th>2.22</th>
<th>2.08</th>
<th>2.0</th>
</tr>
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<td>( D = 1000 \text{ km} )</td>
<td>250 4.0 2.5</td>
<td>-1.6</td>
<td>-9.5</td>
<td>-17.0</td>
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<td>-11.3</td>
<td>-21.0</td>
<td>-26.6</td>
<td>-35.1</td>
<td>—</td>
<td>—</td>
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</tr>
<tr>
<td>500 -6.8 -9.0</td>
<td>-14.1</td>
<td>-22.3</td>
<td>-28.7</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( D = 3000 \text{ km} )</td>
<td>250 -2.0 -3.7</td>
<td>-9.1</td>
<td>-21.4</td>
<td>-30.7</td>
<td>-40.6</td>
<td>-42.7</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>300 -2.4 -4.1</td>
<td>-12.5</td>
<td>-26.4</td>
<td>-32.1</td>
<td>-41.3</td>
<td>-44.1</td>
<td>—</td>
<td></td>
<td></td>
</tr>
<tr>
<td>350 -3.0 -5.4</td>
<td>-14.7</td>
<td>-28.5</td>
<td>-35.5</td>
<td>-41.8</td>
<td>-45.2</td>
<td>—</td>
<td></td>
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</tr>
<tr>
<td>400 -4.6 -7.9</td>
<td>-17.5</td>
<td>-31.2</td>
<td>-36.2</td>
<td>-43.3</td>
<td>—</td>
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<td></td>
</tr>
<tr>
<td>450 -5.1 -11.2</td>
<td>-19.6</td>
<td>-32.7</td>
<td>-36.6</td>
<td>-44.6</td>
<td>—</td>
<td>—</td>
<td></td>
<td></td>
</tr>
<tr>
<td>500 -6.4 -12.6</td>
<td>-21.6</td>
<td>-33.8</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td></td>
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</tr>
</tbody>
</table>

Because eqn. 2 contains no allowance for the increase of \( hmF_2 \) with \( D \), it tends to seriously underestimate, particularly at large \( D \). Errors are similar to those given in Table 1 at \( x = 10 \) but rise to over 30% for \( x \geq 2.5 \) and 45% for \( x \geq 2.0 \).

The overall behaviour of the errors for the simplified equations is compared in Fig. 12, which shows the means of the moduli of the error deviations, \( \Delta \), averaged over all six \( hmF_2 \) and as a function of \( x(e = |\langle \delta \rangle|) \). The dotted curve gives the error for the full eqn. 13 and the dashed curve that for the Shimazaki equation. The former gives mean errors which are less than 5% at all but the lowest \( x \), the latter is seriously in error (underestimates) at low \( x \), particularly for the greater range. The solid curve shows the mean errors for the night eqn. 14 and the broken curve those for the day eqn. 15; both of which tend to underestimate \( h_T \) at low \( x \) in the same way as for the Shimazaki equation. The day equation gives smaller errors than the Shimazaki, at all \( x \) below 5, and smaller errors than the night equation for \( x \) below about 4.5. Hence, if in the interests of method simplicity eqns. 14 and 15 are used instead...
of eqn. 13, the day equation should be applied when \( x \leq 4.5 \) and the night equation at all greater \( x \). Note that the errors for the night equation are slightly smaller than those for Shimazaki at all \( x \); hence, the coefficients given in eqn. 14 are preferable if the simplest form of a single equation for \( h_T \) is required. Fig. 13 demonstrates that the day equation also gives a more desirable variation in errors with \( hmF_2 \). The values of \( \delta \) are shown for \( x \) of 10 and as a function of \( hmF_2 \); the dashed and solid curves are for the Shimazaki and high equations, respectively. The high-\( x \) case is most applicable under night conditions when \( hmF_2 \) is generally high. Fig. 13 shows that the night equation is more accurate for high \( hmF_2 \). The Shimazaki equation is only more accurate for high \( x \) and high \( hmF_2 \), which is a relatively rare combination.

### 5.2 Accuracy of values for \( r \) of 1.0

Fig. 14 plots fully calculated \( h_T \) and \( \beta_0 \) (solid curves) and values from the simplified algorithm (eqns. 13–18). Because of the large gradient of the curves at short distance and the simplicity of the correction term \( \Delta \) (eqn. 13), large errors in \( h_T \) can arise for distances below 1000 km; Fig. 14 demonstrates that errors in \( \beta_0 \) are still small. Table 3 gives the

| Table 3: Percentage error (positive values are overestimates) in using eqns. 16–21, \( \delta \), for unity \( r \) |
|---|---|---|---|---|---|---|
| \( x \) | 1.00 | 5.00 | 3.25 | 2.22 | 2.08 | 2.00 |
| \( D = 1000 \) km | 250 | 1.9 | 1.1 | -1.0 | -0.1 | 0.7 | -4.8 | -3.1 |
| | 300 | 2.7 | 1.5 | 0.6 | 0.7 | 1.5 | 3.4 | 1.9 |
| | 350 | 4.4 | 3.1 | 1.7 | 1.9 | 4.5 | 8.2 | 7.6 |
| | 400 | 2.7 | 2.3 | 1.8 | 5.0 | 11.1 | 2.0 | |
| | 450 | 1.2 | 0.9 | -3.2 | 4.5 | 8.3 | 2.0 | |
| | 500 | -1.5 | -3.1 | -5.4 | 4.3 | 7.0 | | |
| \( D = 3000 \) km | 250 | 1.9 | 0.5 | -0.8 | -3.4 | 14.3 | 14.1 | 10.1 |
| | 350 | 4.5 | 4.0 | 3.8 | 3.9 | 15.8 | 21.0 | 31.9 |
| | 400 | 4.1 | 4.3 | 4.2 | 5.7 | 17.8 | 17.6 | |
| | 450 | 3.3 | 3.7 | 4.6 | 5.5 | 21.4 | 19.0 | |
| | 500 | 2.9 | 1.3 | -0.2 | 4.3 | | | |

percentage errors \( \varepsilon \), for \( D \) of 1000 and 3000 km. Errors are always less than 6% for \( x \geq 2.5 \); at lower \( x \) errors are larger, rising to over 30% for \( x = 2.0 \) at the larger distance because of slight deviations of \( (h_T - \Delta) \) from a linear dependence on \( D/D_{\text{max}} \), particularly for high \( hmF_2 \) and low \( x \). The algorithm would be significantly more complex if allowance were to be made for this effect.

### 6 Conclusions

From the results of ordinary-wave ray tracing through the Bradley–Dudeney model ionospheric profile using the Milsom equations, simple noniterative algorithms for rapid evaluation of mirror-reflection height have been developed. Algorithms of various complexity and accuracy are presented for the range of signal frequencies between 0.75 and 0.95 of the basic MUF. In addition, a single algorithm is given for a frequency equal to the basic MUF.

The most rigorous equations derived allow for the variation of mirror-reflection height with distance, \( M(3000)F_2 \) and \( x \), the ratio of the F- and E-layer critical frequencies; hence, all the required ionospheric inputs to the algorithm are scaled routinely from ionograms. No explicit dependence on signal frequency is included, but the equation for 0.75 to 0.95 times the basic MUF is accurate to within 6% for \( x \geq 2.5 \) and 11% for \( x \geq 2.0 \). This accuracy involves considerable complexity, as compared with the widely used equation of Shimazaki which only allows for the variation with \( M(3000)F_2 \) and has no dependence on distance or underlying ionisation; consequently, the Shimazaki equation is only accurate to within 34% for \( x \geq 2.5 \) and 45% for \( x \geq 2.0 \). For large \( x \) the full equation reduces to a comparable form to that of Shimazaki and is about 2% more accurate at all \( x \). Such an equation may be sufficiently accurate at night, when \( x \) is large, but gives large errors by day when \( x \) is smaller. A second special case of the full equation (\( x = 3 \)) leads to an expression with a non-negligible distance term which can be applied to day conditions with errors about 10% lower than those using the Shimazaki equation.

The algorithm presented for evaluation of mirror-reflection height for a signal frequency equal to the basic MUF yields values accurate to within 6% for \( x \geq 2.5 \). This
is of comparable accuracy to the lower frequency algorithm, but the largest errors rise to 32% for cases where the elevation angle falls below a few degrees and $x$ is between 2.0 and 2.5.

7 Acknowledgments

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8 References

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