An application of Tutte's Theorem to 1-factorization of regular graphs of high degree

David Cariolaro
Institute of Mathematics
Academia Sinica
Nankang, Taipei
11529 Taiwan

E-mail: cariolaro@math.sinica.edu.tw

Anthony J.W. Hilton*
School of Mathematical Sciences
Queen Mary University of London
London E1 4NS
U.K.

E-mail: a.hilton@qmul.ac.uk

October 16, 2006

Abstract

A well known conjecture in graph theory states that every regular graph of even order $2n$ and degree $\lambda(2n)$, where $\lambda \geq 1/2$, is

*Also at Department of Mathematics, University of Reading, Reading RG6 6AX, U.K. E-mail address: a.j.w.hilton@reading.ac.uk

1-factorizable. Chetwynd and Hilton [*1-factorizing regular graphs of high degree - an improved bound*, Discrete Mathematics, **75** (1989), 103-112] and, independently, Niessen and Volkmann [*Class 1 conditions depending on the minimum degree and the number of vertices of maximum degree,* Journal of Graph Theory, **14** (1990), 225-246] proved the above conjecture under the assumption that $\lambda \geq \frac{\sqrt{7}-1}{2} \approx 5/6$. Since these results were published no significant improvement has been done in terms of lowering the bound on λ. We shall here obtain a substantial but partial improvement on λ. Specifically, using the original Chetwynd-Hilton approach and Tutte's 1-Factor Theorem, we show that the above bound can be improved to $\lambda > \frac{\sqrt{57}-3}{6} \approx 3/4$, apart (possibly) from two exceptional cases. We then show that, under the stronger assumption that $\lambda \geq \lambda^*$, where $\lambda^* \approx 0.785$, one of the two exceptional cases cannot occur.

Keywords:1-factorization, 1-factorization conjecture, Tutte's 1-factor theorem

MSC 2000: 05C15, 05C70

1 Introduction

All graphs that we shall consider are finite, simple and undirected. Let G be a graph. The vertex set, edge set, maximum degree and minimum degree of G will be denoted by $V(G), E(G), \Delta(G), \delta(G)$, respectively. The *order* of G is the number of vertices in G. G is *regular* if $\Delta(G) = \delta(G)$, in which case the common degree of the vertices of G is called the *degree* of G and denoted by d. If $V_1 \subset V(G)$, by $G - V_1$ we shall denote the graph obtained from G by deleting all the vertices in V_1 (together with their incident edges). Similarly, if $E_1 \subset E(G)$, by $G - E_1$ we shall denote the graph obtained from G by deleting all the edges in E_1. The notation $G - X - Y$, where X and Y are sets of vertices or edges, will be used to denote the graph $(G - X) - Y$. The symbol n will be always used in this paper to denote a positive integer.

A *matching* M of G is a set of mutually nonadjacent edges of G. If M is a matching, we denote by $V(M)$ the set of vertices of G incident with edges of M. Two matchings are *(edge)-disjoint* if they have no common edge. A matching M is a *1-factor* of G if $|V(M)| = |V(G)|$, and a *near 1-factor* of G if $|V(M)| = |V(G)| - 1$. If M is a near 1-factor of G, and $v \in V(G)$ is the (only) vertex of G such that $v \notin V(M)$, we say that M *misses* vertex v, or that v is *missed* by M. A *1-factorization* of G is a set $\mathcal{F}$ of pairwise edge-disjoint 1-factors of G whose union is $E(G)$. For an introduction to 1-factorization, and undefined graph theoretic terminology, the reader is referred to Wallis [16]. Clearly, in order to have a 1-factorization, G must be regular and have even order. However this condition is certainly not sufficient, and the problem of deciding whether a given graph is 1-factorizable is a difficult problem in general, which is known to be NP-complete [7]. In apparent contrast with this result, the following well known conjecture (which first appeared in print in a 1985 paper by Chetwynd and Hilton [3], but which certainly circulated informally much earlier) claims that, for a vast class of regular graphs, the above decision problem is trivial.

Conjecture 1 ***(1-Factorization Conjecture)*** *Every regular graph of order $2n$ and degree $d \geq n$ is 1-factorizable.*

This conjecture is considered very hard and, if it could be proven, would have important consequences in graph theory as well as other branches of mathematics. We briefly summarize the history of this conjecture. It has long been known (and it may be regarded as part of "mathematical folklore") that the conjecture holds for complete graphs, i.e. when $d = 2n - 1$. Since the choice of the first 1-factor in a 1-factorization of K_{2n} is arbitrary, the conjecture also holds for $d = 2n - 2$. Chetwynd and Hilton [3] proved the conjecture for $d = 2n - 3$. Rosa and Wallis [12] settled the case $d = 2n - 4$, under the assumption that $\overline{G}$ (the complement of G) is 1-factorizable. The case $d = 2n - 4$ and $d = 2n - 5$ were settled in full generality by Chetwynd and Hilton in [3] (see also [4]). The case $d = 2n - 6$ was settled by Niessen in [9], under the assumption that $2n \geq 18$, and in full by Song [13] and Song and Yap [14], as a corollary of their determination of the chromatic index of graphs with exactly five vertices of maximum degree. To the best of our knowledge, the case $d = 2n - 7$ of the conjecture is currently open in general (it certainly holds for $n \geq 20$ by Theorem 1 below). In 1985 there was a breakthrough by Chetwynd and Hilton [3], when they proved that all regular graphs of order $2n$ and degree $d = \lambda(2n)$, where $\lambda \geq 6/7$, are 1-factorizable. This result set a completely new and more interesting challenge, namely to lower as much as possible the bound on λ, with the aim of (hopefully) reaching the target $\lambda \geq 1/2$, which would settle Conjecture 1 in its full generality. Substantial progress in this direction was obtained a few years later by Chetwynd and Hilton [5] and, independently, Niessen and

Volkmann [10], by means of the following result.

Theorem 1 *All regular graphs G of order $2n$ and degree $\lambda(2n)$, where $\lambda \geq \frac{\sqrt{7}-1}{2}$, are 1-factorizable.*

Notice that $\frac{\sqrt{7}-1}{2} \approx 0.823 \approx 5/6$.

In 1997 there was another breakthrough in 1-factorization by Perkovic and Reed [11], who proved (by probabilistic methods) that the 1-Factorization Conjecture is "asymptotically true", i.e. it is true (for any given $\epsilon > 0$) for all regular graphs of order $2n$ and degree $d \geq (1/2+\epsilon)(2n)$, provided n is sufficiently large (depending on ϵ). This result, which (although never published) had been also announced many years earlier by Häggkvist, clearly provides a strong evidence in favour of Conjecture 1.

Unfortunately, in the large time span elapsed from the publication of Theorem 1, no improvement has been made on the 1-factorization of regular graphs of order $2n$ and degree $\lambda(2n)$, in terms of lowering the bound on λ. A related result was established by the present authors in [2], where it was proven that, if $\lambda \geq \frac{3-\sqrt{3}}{2} \approx 0.64$, then G contains two (distinct) vertices x and y such that $G-x-y$ is Class 1, and, if $\lambda \geq 3/4$, then, for any pair of (distinct) vertices x and y, $G-x-y$ is Class 1. Unfortunately we were unable to deduce from this that, under the same conditions, also $G-x$ (and hence G) are Class 1. In his Ph.D. thesis [1], the first author considered the problem of improving the Chetwynd-Hilton bound on λ and obtained some partial results. The absence from the literature of any definite or even partial improvement on the Chetwynd-Hilton bound for nearly twenty years, prompts us to publish some further material from [1], which may hopefully provide some ground to other researchers for further investigations on this topic. The results which we will present have been further improved by us since the first author completed his Ph.D. thesis.

The first result that we shall present will be stated precisely in section 3 in the form of Theorem 5, but may be informally described as follows. Let G be a regular graph of order $2n$ and degree $\lambda(2n)$, where $\lambda > \frac{\sqrt{57}-3}{6} \approx 3/4$. Then we prove that G is 1-factorizable, apart from two possible well described exceptional cases, which we call *Case One* and *Case Two*. These cases are, in some sense, extremal, because a certain parameter, which will be defined later, takes its extremal values on these two cases. In all the remaining cases (i.e. for the vast majority of graphs G), our result establishes directly the existence of a 1-factorization in G (it does not provides an algorithm, though, since our proof is existential). Clearly, in order to claim the 1-factorizability of all graphs G with $\lambda > \frac{\sqrt{57}-3}{6}$, one must rule out these two cases and prove their impossibility. Unfortunately we could not solve Case Two in general, even under some stronger assumptions on λ. However, in section 4, we shall present our second result, namely a proof of the fact that, if $\lambda \geq \lambda^*$, where $\lambda^* \approx 0.785$ is defined as a root of a certain quartic polynomial, then Case One is impossible[1].

We believe that a technique similar to the one used here could lead to a proof of the impossibility of Case Two for, say, $\lambda \geq \lambda_0$, where λ_0 is well below the Chetwynd-Hilton bound of $\frac{\sqrt{7}-1}{2} \approx 0.823$. Obviously, by the results proved in this paper, such a proof would imply the truth of Conjecture 1 for all graphs G with $\lambda \geq \max\{\lambda^*, \lambda_0\}$.

2 Preliminary lemmas and results

If G is a graph and $S \subset V(G)$, we denote by $odd(S)$ the number of connected components of odd order in the graph $G - S$ (we call these *odd components*). We shall use the following well known theorem of Tutte [15].

Theorem 2 ***(Tutte's Theorem)*** *Let G be a graph. Then G has no 1-factor if and only if there exists a set $S \subset V(G)$ such that $odd(S) > |S|$.*

A set S as in the statement of Tutte's Theorem will be called a *Tutte Set*. We shall also need the following well known sufficient condition for the existence of a Hamilton cycle in a graph, due to Dirac [6].

Theorem 3 ***(Dirac's Theorem)*** *Let G be a graph of order at least three. Suppose $\delta(G) \geq |V(G)|/2$. Then G is Hamiltonian.*

The following theorem, due to Chetwynd and Hilton, was proven as the main result in [5], from which Theorem 1 follows immediately as a corollary.

Theorem 4 *Let G be a regular graph of order $2n$ and degree*

$$d \geq \frac{5}{6}(2n) - \frac{\overline{p}}{3} + \frac{1}{2}.$$

Then G is 1-factorizable.

[1] This result improves the corresponding bound given in the first author's Ph.D. thesis [1], which was $\lambda \geq 0.794$.

Here $\overline{p} = \overline{p}(G)$ is defined as $\max_{x \neq y \in V(G)} \overline{p}(x, y)$, where

$$\overline{p}(x, y) = |\{z \in V(G) : z \neq x, z \neq y, zx \notin E(G), zy \notin E(G)\}|.$$

We shall use the same approach and part of the original proof of Theorem 4. Details of this proof may be found either in Chetwynd and Hilton [5], or Wallis [16]. For some remarks, clarifications and slight improvements the interested reader is referred to [1] or [2]. However here we shall only use the following fact from the proof of Theorem 4, which has been derived from the original paper by Chetwynd and Hilton [5] (apart from a slight improvement given in [1] and [2]).

Lemma 1 *Let G be a regular graph of order $2n$ and degree $d = \lambda(2n)$, where $\lambda \geq \frac{3-\sqrt{3}}{2} \approx 0.64$. Let w, v^* be two distinct vertices such that $\overline{p}(w, v^*) = \overline{p}(G)$. There exists a subgraph H^* of $G - w$ of order $q + 1 \leq 4n - 2d - \overline{p} - 2$, a matching M_0 of H^* and $q - 2$ edge-disjoint matchings $M_1, M_2, \ldots M_{q-2}$ (each of which edge-disjoint from M_0), where each matching M_i satisfies $|M_i| \leq \frac{1}{2}(q + 1 - i)$, and a set of (not necessarily distinct) vertices ξ_i, where $\xi_i \notin V(M_i)$, such that G is 1-factorizable if the following conditions are satisfied:*

(a) there exists a set of $q-2$ pairwise edge-disjoint 1-factors $F_i, i = 1, 2, \ldots q-2$, of $G - w$ which are edge-disjoint from M_0, such that each near 1-factor F_i misses vertex ξ_i;

(b) $\bigcup_{i=1}^{q-2} F_i \supset \bigcup_{i=1}^{q-2} M_i$.

It will be useful to have at our disposal the following upper bound on q.

Lemma 2 *Using the notations and hypotheses of Lemma 1, we have $q+2 \leq (1 - \lambda^2)(2n)$.*

Proof. From Lemma 1, we have

$$q + 2 \leq 4n - 2d - \overline{p} - 1 \ . \tag{1}$$

By an easy combinatorial argument (see [5, Lemma 3]), it can be proven that $\overline{p}$ satisfies the inequality

$$\overline{p} \geq \frac{(2n - d - 1)(2n - d - 2)}{2n - 1} \ . \tag{2}$$

By (1) and (2) we have

$$q + 2 \leq 4n - 2d - 1 - \tfrac{1}{2n-1} \cdot (2n - d - 1)(2n - d - 2)$$

$$= \tfrac{1}{2n-1} \cdot [(2n-1)(4n - 2d - 1) - (4n^2 + d^2 + 2 - 4nd - 6n + 3d)]$$

which simplifies to

$$q + 2 \leq \frac{4n^2 - d - 1 - d^2}{2n - 1}.$$

Thus, in order to prove the lemma, it will suffice to verify that

$$\frac{4n^2 - d - 1 - d^2}{2n - 1} \leq (1 - \lambda^2)(2n).$$

Using $d = \lambda(2n)$, we can rewrite this (after some simplifications) as

$$2n(\lambda^2 + \lambda - 1) + 1 \geq 0.$$

This certainly holds if λ satisfies the inequality

$$\lambda^2 + \lambda - 1 \geq 0,$$

which holds if $\lambda \geq \frac{\sqrt{5}-1}{2} \approx 0.61$ (and hence holds under the stronger hypotheses of Lemma 1). $\square$

In [5] the existence of a set of near 1-factors F_i of $G - w$ satisfying the conditions (a),(b), stated in Lemma 1 was established by considering the graph

$$G_i = (G - w) - (F_1 \cup F_2 \cup \ldots \cup F_{i-1} \cup F_i \cup M_0) \tag{3}$$

and by proving the existence of a near 1-factor F_{i+1} of G_i containing the matching M_{i+1} and missing the vertex ξ_{i+1}. This is clearly equivalent to showing the existence of a near 1-factor missing vertex ξ_{i+1} in the graph

$$G_i^* = (G - w) - (F_1 \cup \ldots \cup F_i \cup M_0)) - V(M_{i+1}),$$

where $V(M_{i+1})$ denotes the vertex set of the matching M_{i+1}. Notice that, by Lemma 1, the matchings M_i satisfy the inequality

$$|V(M_i)| \leq q - i + 1 \text{ for } i = 1, \ldots, q - 2. \tag{4}$$

Chetwynd and Hilton showed that, under the assumption $\lambda \geq \frac{\sqrt{7}-1}{2}$, all graphs G_i^* above are Hamiltonian because they satisfy the conditions of Dirac's Theorem (Theorem 3), and hence have the required near 1-factor. If $\lambda < \frac{\sqrt{7}-1}{2}$, we cannot in general claim that G_i^* satisfies Dirac's condition for Hamiltonicity, and hence we have to look for other ways to prove that the conditions of Lemma 1 are satisfied. We shall prove that (under certain conditions to be specified later) the conditions of Lemma 1 are satisfied by considering the graph

$$G_i^{**} = G_i^* - \xi_{i+1}$$

and by proving (using Tutte's Theorem) that either G_i^{**} has a 1-factor or that we can redefine the near 1-factors F_i in such a way that the (new) graph G_i^{**} has a 1-factor.

The starting point of our proof is the following. Assume that $t \leq q-2$ is the largest positive integer such that there exists a set of exactly t near 1-factors $F_1, F_2, \ldots, F_t$ of $G-w$ which are mutually edge-disjoint, edge-disjoint from M_0, and such that F_i misses vertex ξ_i and $\bigcup_{i=1}^t F_i \supset \bigcup_{i=1}^t M_i$. Clearly, if $t = q-2$, there is nothing to prove, since all conditions of Lemma 1 are then satisfied. Hence we can assume, without loss of generality, that

$$t \leq q-3. \tag{5}$$

Consider the graph

$$G_t^{**} = (G-w) - (F_1 \cup F_2 \cup \ldots \cup F_t \cup M_0)) - (V(M_{t+1}) \cup \{\xi_{t+1}\}). \tag{6}$$

By assumption, G_t^{**} does not have a 1-factor, for, if it had one, then this would contradict the maximality of t.

By (4), we have

$$|V(M_{t+1})| \leq q-t. \tag{7}$$

Let

$$2n^* = |V(G_t^{**})|. \tag{8}$$

Notice that

$$2n^* = 2n - 2 - |V(M_{t+1})|\,. \tag{9}$$

By (3) and (6), we have

$$\delta(G_t^{**}) \geq \delta(G_t) - 1 - |V(M_{t+1})|, \tag{10}$$

and since, by (3), we have

$$\delta(G_t) \geq d - 1 - (t+1) = d - t - 2,$$

and taking (5), (7) and (10) into account, it follows that

$$\delta(G_t^{**}) \geq d - q - 3.$$

Using the bound given by Lemma 2 and the identity $d = \lambda(2n)$, we obtain

$$\delta(G_t^{**}) \geq (\lambda^2 + \lambda - 1)(2n) - 1. \tag{11}$$

Let τ be defined by the position

$$t = \tau(2n).$$

By Tutte's Theorem, G_t^{**} has a Tutte Set, i.e. a set $S \subset V(G_t^{**})$ such that $odd(S) > |S|$, where $odd(S)$ is the number of odd components in the graph $G_t^{**} - S$. Let $s = |S|$, and let $z = odd(S)$. We have, by what we just said,

$$z > s. \tag{12}$$

Since G_t^{**} has even order and z has clearly the same parity of $|V(G_t^{**} - S)|$, we have

$$z \equiv s \pmod{2}. \tag{13}$$

(12) and (13) imply that

$$z \geq s + 2. \tag{14}$$

Let $\{Q_1, Q_2, \ldots, Q_z\}$ be the odd components of $G_t^{**} - S$, with $|Q_1| \leq |Q_2| \leq \ldots \leq |Q_z|$. Let $q_i = |Q_i|$, for $i = 1, 2, \ldots, z$.

Since Q_1 is the smallest odd component of $G_t^{**} - S$ and using (14), we have

$$q_1 \leq \frac{2n^*}{z} \leq \frac{2n^*}{s+2}. \tag{15}$$

If $v \in V(Q_i)$, then any edge of G_t^{**} incident with v joins v to either a vertex in Q_i or a vertex in S. Therefore

$$\delta(G_t^{**}) \leq (q_i - 1) + s. \tag{16}$$

Thus

$$q_i \geq \delta(G_t^{**}) + 1 - s. \tag{17}$$

By this, (14) and the fact that $V(G_t^{**}) \supset S \cup \bigcup_{i=1}^{z} Q_i$, we have the following:

$$2n^* \geq s + \sum_{i=1}^{z} q_i \geq s + (s+2)(\delta(G_t^{**}) + 1 - s). \tag{18}$$

Now, the quadratic

$$x + (x+2)(K+1-x) = -x^2 + Kx + 2K + 2, \tag{19}$$

where $K = \delta(G_t^{**})$, is (as a function of x) symmetric with respect to the axis $x = K/2$ and increasing for $x \leq K/2$ (and hence decreasing for $x \geq K/2$.) For $x = 1$ and $x = K-1$, the quadratic takes the value $1+3K = 1+3\delta(G_t^{**})$. By (11), we have

$$1 + 3\delta(G_t^{**}) \geq 3(\lambda^2 + \lambda - 1)(2n) - 2. \tag{20}$$

The right-hand side of (20) is larger than $2n - 2$ (and hence, by (9), larger than $2n^*$) if

$$3(\lambda^2 + \lambda - 1) > 1, \tag{21}$$

i.e. if

$$3\lambda^2 + 3\lambda - 4 > 0,$$

which holds if

$$\lambda > \frac{1}{6}(\sqrt{57} - 3) \approx 0.758. \tag{22}$$

From now on we shall assume that (22) is satisfied. Under this assumption, (18) can be satisfied only (possibly) if $s = 0$ or $s \geq \delta(G_t^{**})$. This proves the following.

Theorem 5 *Let G be a regular graph of even order $2n$ and degree $\lambda(2n)$. Let $\lambda > \frac{1}{6}(\sqrt{57} - 3)$. Then either G is 1-factorizable, or G_t^{**} has a Tutte Set S such that either of the following cases occurs:*

- ***Case One*** $S = \emptyset$.
- ***Case Two*** $|S| \geq \delta(G_t^{**})$.

In the next section we shall consider Case One in detail and give a full solution of Case One under some condition on λ stronger than that of Theorem 5, but still much weaker than the Chetwynd-Hilton condition $\lambda \geq \frac{\sqrt{7}-1}{2}$. Case Two will be not dealt with in this paper, but some remarks on Case Two will be made at the end.

3 Case One

The main result of this section is the following theorem.

Theorem 6 *Let λ^* be defined as the second largest root of the polynomial $x^4 - x^3 - 4x^2 + 2x + 1$. Let G be a regular graph of order $2n$ and degree $\lambda(2n)$, where $\lambda \geq \lambda^*$. Then either G is 1-factorizable, or G_t^{**} has a Tutte Set S such that $|S| \geq \delta(G_t^{**})$.*

Notice that $\lambda^* \approx 0.785$.

To prove the theorem, we first make some preliminary observations and prove an auxiliary lemma. Our argument goes as follows. By Theorem 5, in order to prove Theorem 6, it clearly suffices to prove that G_t^{**} does not admit the empty set as a Tutte Set. Arguing by contradiction, we assume that $S = \emptyset$ is a Tutte set for G_t^{**}. By (14), G_t^{**} has $z \geq 2$ odd components $Q_1, Q_2, \ldots, Q_z$. By (17), we have

$$|Q_i| \geq \delta(G_t^{**}) + 1 \text{ for each } i. \tag{23}$$

By (11) and (23), we have

$$|Q_i| \geq (\lambda^2 + \lambda - 1)(2n), \tag{24}$$

so that, by (14) and (21), we must have $z = 2$ and there cannot be connected components of even order in G_t^{**}. Thus G_t^{**} consists of exactly two components Q_1 and Q_2, both of odd order. We rename these components A and B, respectively. There are obviously no edges in G_t^{**} joining A and B. Let $E_G(A, B)$ denote the set of edges in $G - w - M_0$ joining A and B. Let α, β be defined by the relations

$$|A| = \alpha(2n), \tag{25}$$

$$|B| = \beta(2n). \tag{26}$$

By (24), we have

$$\alpha \geq \lambda^2 + \lambda - 1, \tag{27}$$

and

$$\beta \geq \lambda^2 + \lambda - 1. \tag{28}$$

Notice that each vertex in G is non-adjacent to exactly $2n - 1 - \lambda(2n)$ other vertices of G. Hence every vertex in B must be adjacent in G to at least $|A| - (2n - 1 - \lambda(2n))$ vertices of A. It follows that

$$|E_G(A, B)| \geq |B|(|A| - (2n - 1 - \lambda(2n)) > |B|(|A| - 2n + \lambda(2n)). \tag{29}$$

Using the notations introduced by (25) and (26), we can rewrite (29) as

$$|E_G(A, B)| > \beta(\alpha - 1 + \lambda)(2n)^2. \tag{30}$$

As there are no edges joining A and B in G_t^{**}, by (6) and the definition of $E_G(A, B)$, we have

$$E_G(A, B) \subset F_1 \cup F_2 \cup \ldots, F_t. \tag{31}$$

Let, for each $i = 1, 2, \ldots, t$, the set E_i be defined as

$$E_i = E_G(A, B) \cap F_i. \tag{32}$$

Then, obviously,

$$E_G(A, B) = \bigcup_{i=1}^{t} E_i, \tag{33}$$

and, since the E_i's are disjoint, this implies that

$$|E_G(A, B)| = \sum_{i=1}^{t} |E_i|. \tag{34}$$

Call an edge $e \in E_G(A, B)$ *marginal* if $e \in \bigcup_{i=1}^{t} M_i$, and *non-marginal* otherwise. Denote the set of non-marginal edges by N, i.e. let

$$N = E_G(A, B) \setminus (\bigcup_{i=1}^{t} M_i). \tag{35}$$

We are ready to prove the following lemma.

Lemma 3 *Suppose that, using the notations introduced above, there exists an index k, $1 \leq k \leq t$, such that*

(i) $|E_k| \geq (3 - 2\lambda - 2\lambda^2)(2n)$;

(ii) $E_k \cap N \neq \emptyset$;

(iii) $|E_k| \geq 2$.

Then we have a contradiction.

Proof. Let $A_0 = A \cap V(E_k)$ and let $B_0 = B \cap V(E_k)$. Clearly

$$|A_0| = |B_0|. \tag{36}$$

Let α_0 be defined by the relation

$$|A_0| = \alpha_0(2n). \tag{37}$$

With a slight abuse of notation, let A_0, B_0, A, B, denote the *subgraphs* of G_t^{**} induced by A_0, B_0, A, B, respectively. As two vertices in A_0 are non-adjacent if and only if they are non-adjacent in G_t^{**}, and there are no more than $|A| - \delta(A)$ vertices in A which are non-adjacent to a given vertex of A, we have

$$\delta(A_0) \geq |A_0| - (|A| - \delta(A)). \tag{38}$$

Similarly,

$$\delta(B_0) \geq |B_0| - (|B| - \delta(B)). \tag{39}$$

Adding (38) and (39), noticing that $|V(G_t^{**})| = 2n^* = |A| + |B|$, and taking (36) into account, we have

$$\delta(A_0) + \delta(B_0) \geq 2|A_0| + 2\delta(G_t^{**}) - 2n^*. \tag{40}$$

We claim that

$$\delta(A_0) + \delta(B_0) \geq |A_0|. \tag{41}$$

By (40), this holds if

$$|A_0| + 2\delta(G_t^{**}) - 2n^* \geq 0.$$

Using the fact that $2n^* \leq 2n-2$, and using (11) and (37), it suffices to verify that

$$\alpha_0 + 2(\lambda^2 + \lambda - 1) - 1 \geq 0$$

i.e. that

$$\alpha_0 \geq 3 - 2\lambda - 2\lambda^2,$$

which is assumption (i) of the present lemma. Hence (41) holds.

Let now e_1 be a non-marginal edge of E_k (which exists by assumption (ii) of the present lemma). Let $e_1 = u_1v_1$, with $u_1 \in A$ and $v_1 \in B$. Let $e_2 = u_2v_2, e_3 = u_3v_3, \ldots, e_{|A_0|} = u_{|A_0|}v_{|A_0|}$ be the remaining edges in E_k, where $u_i \in A$ and $v_i \in B$ for each $i = 2, 3, \ldots, |A_0|$. Notice that this set of edges is non-empty by assumption (iii) of the present lemma. By (41), and the pigeon-hole principle, there exists an edge $e_j \in E_k$, where $2 \leq j \leq |A_0|$, such that $u_ju_1 \in E(A_0)$ and $v_jv_1 \in E(B_0)$ (see Fig. 1).

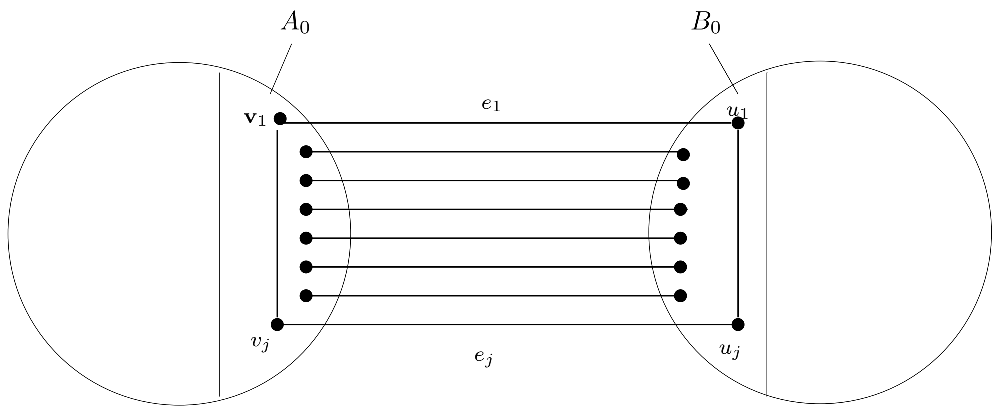

Figure 1: The key idea in the proof of Lemma 3.

The existence of the quadrilateral $v_1u_1u_jv_j$ is the critical part of this proof. Indeed we can now alter the structure of the 1-factors $F_1, F_2, \ldots, F_t$ in such a way that it will be possible to obtain a contradiction.

The alteration is done simply by "swapping" the edges of the quadrangle $v_1u_1u_jv_j$. More precisely, let $\tilde{F_k}$ be defined as follows:

$$\tilde{F_k} = (F_k \setminus \{u_1v_1, u_jv_j\}) \cup \{u_1u_j, v_1v_j\}).$$

Clearly $\tilde{F_k}$ is still a near 1-factor of G, edge-disjoint from all the F_i's (for $i \neq k$), and edge-disjoint from M_0. Consider the graph

$$\tilde{G_t^{**}} = (G-w)-(F_1 \cup F_2 \cup \ldots \cup F_{k-1} \cup \tilde{F_k} \cup F_{k+1} \cup \ldots \cup F_t \cup M_0)-(\{\xi_{t+1}\} \cup V(M_{t+1})).$$

This graph, as we shall see, has a 1-factor F_{t+1} containing the edge e_j, and this proves that it is possible to remove $t+1$ edge-disjoint near 1-factors $F'_1, F'_2, \ldots, F'_{t+1}$ from $G - w$ (where F'_i is defined as F_i for $i \neq k$ and $\tilde{F_k}$ for $i = k$), which are mutually edge-disjoint, edge-disjoint from M_0, and such that F'_i misses vertex ξ_i and $\bigcup_{i=1}^{t+1} F'_i \supset \bigcup_{i=1}^{t+1} M_i$, thus contradicting the maximality of t (remember that the edge e_1, which is left out from $\bigcup_{i=1}^{t+1} F'_i$, is non-marginal by assumption). Thus, to complete the proof, we only need to verify that $\tilde{G_t^{**}}$ has a 1-factor containing the edge e_j. We can easily construct one such 1-factor by taking a near 1-factor of A missing vertex v_j, a near 1-factor of B missing vertex u_j, and adding the edge $u_j v_j$. To check that A and B have the desired near 1-factors, we just need to observe that they are both Hamiltonian. We give the explicit proof for A. By Dirac's Theorem, it suffices to verify that

$$\delta(A) \geq \tfrac{1}{2}|A|.$$

As $\delta(G_t^{**}) \leq \delta(A)$, and by (11), it suffices that

$$|A| \leq 2(\lambda^2 + \lambda - 1)(2n) - 2.$$

Adding $|B|$ to both sides of the previous inequality and using the fact that $|A| + |B| = 2n^*$, we can rewrite the previous inequality as

$$2n^* \leq 2(\lambda^2 + \lambda - 1)(2n) + |B| - 2.$$

Since $|B| \geq \delta(G_t^{**}) + 1 \geq (\lambda^2 + \lambda - 1)(2n)$ and by (9), it will suffice to show that

$$2n \leq 3(\lambda^2 + \lambda - 1)(2n).$$

But this inequality is guaranteed by (21), and hence the proof is completed. □

Lemma 3 enables us to conclude the proof of Theorem 6.

Proof of Theorem 6: We prove that the conditions of Lemma3 are satisfied by G, which implies a contradiction. Let N be the set of non-marginal edges, defined by (35). Let E_i be defined as in (32). Notice that E_i is an independent set of edges (i.e. a matching), for every $i = 1, 2, \ldots t$. Since, by Lemma 1, all the marginal edges are taken from a subgraph H^* of order $q+1$, it follows that the condition $|E_k| > (q+1)/2$ (together with the fact that E_k is a matching) guarantees the existence of a non-marginal edge in E_k, which is condition (ii) of Lemma 3. Since, by Lemma 2, we have

$$\frac{q+1}{2} < \frac{q+2}{2} \leq \frac{(1-\lambda^2)(2n)}{2},$$

we can claim that condition (ii) of Lemma 3 is satisfied if

$$|E_k| \geq \frac{(1-\lambda^2)(2n)}{2}.$$

Therefore, the conditions of Lemma 3 are satisfied if

$$|E_k| \geq \max\{(3-2\lambda-2\lambda^2)(2n), \tfrac{1}{2}(1-\lambda^2)(2n), 2\}. \tag{42}$$

Using (34), in order to verify that (42) is satisfied, it will suffice to prove that

$$\lceil |E_G(A,B)|/t \rceil \geq \max\{(3-2\lambda-2\lambda^2)(2n), \tfrac{1}{2}(1-\lambda^2)(2n), 2\},$$

and, using $t < q-2 \leq (1-\lambda^2)(2n)$, it will suffice to verify that

$$\frac{|E_G(A,B)|}{(1-\lambda^2)(2n)} \geq \max\{(3-2\lambda-2\lambda^2)(2n), \tfrac{1}{2}(1-\lambda^2)(2n), 1\},$$

where (in the set on the right-hand side of the above inequality) we have replaced the number 2 by the number 1 using the trivial fact that, if $x > y$ are real numbers, then the condition $y \geq 1$ guarantees the condition $\lceil x \rceil \geq 2$.

For convenience, let $\Theta = \Theta(\lambda, n)$ be defined as

$$\Theta = \max\{(3-2\lambda-2\lambda^2)(2n), \tfrac{1}{2}(1-\lambda^2)(2n), 1\}.$$

It is easy to see, by solving some simple inequalities, that

$$\Theta = \begin{cases} \frac{1-\lambda^2}{2}(2n) & \text{if } \lambda \geq \frac{\sqrt{19}-2}{3} \\ \\ (3-2\lambda-2\lambda^2)(2n) & \text{if } \lambda \leq \frac{\sqrt{19}-2}{3} \end{cases} \tag{43}$$

unless n is very small ($n \leq 3$), which is certainly a case we do not need to consider here, since we know (by the results mentioned in the Introduction) that Conjecture 1 (and consequently Theorem 6) holds for very small values of n. Notice that $\frac{\sqrt{19}-2}{3} \approx 0.786$.

Using (30), it will suffice to verify the inequality

$$\frac{\beta(\alpha - 1 + \lambda)(2n)^2}{(1-\lambda^2)(2n)} \geq \Theta,$$

which, after taking (27) and (28) into account, is satisfied if

$$(\lambda^2 + \lambda - 1)(\lambda^2 + 2\lambda - 2)(2n) \geq (1-\lambda^2)\Theta. \tag{44}$$

Now, by (43), we can claim that (44) is satisfied for $\lambda \geq \frac{\sqrt{19}-2}{3}$ if we can prove that

$$(\lambda^2 + \lambda - 1)(\lambda^2 + 2\lambda - 2)(2n) \geq \tfrac{1}{2}(1-\lambda^2)^2(2n). \tag{45}$$

After expanding, rearranging and simplifying, (45) becomes

$$\frac{1}{2}\lambda^4 + 3\lambda^3 - 4\lambda + \frac{3}{2} \geq 0.$$

Using elementary calculus (or a calculator), one can verify that the above inequality, and hence (44), is satisfied under the current assumption $\lambda \geq \frac{\sqrt{19}-2}{3}$. Therefore we are left only with the verification of the case $\lambda \leq \frac{\sqrt{19}-2}{3}$. In this case, by (43), inequality (44) is equivalent to

$$(\lambda^2 + \lambda - 1)(\lambda^2 + 2\lambda - 2)(2n) \geq (1-\lambda^2)(3 - 2\lambda - 2\lambda^2)(2n).$$

After expanding, simplifying, and changing sign, this becomes

$$\lambda^4 - \lambda^3 - 4\lambda^2 + 2\lambda + 1 \leq 0. \tag{46}$$

Let λ^+, λ^* be, respectively, the largest and second largest root of the above polynomial. It can be checked (e.g. using a calculator), that $\lambda^+ > \frac{\sqrt{19}-2}{3} > \lambda^*$, and therefore (46) is satisfied for $\lambda^* \leq \lambda \leq \frac{\sqrt{19}-2}{3}$. Combining this with what was proven above, we can claim that (44) is satisfied for all $\lambda \geq \lambda^*$. Therefore, under the same condition, the hypotheses of Lemma 3 are satisfied, which yields a contradiction by Lemma 3. This contradiction proves that $\emptyset$ cannot be a Tutte Set for G_t^{**}, and hence concludes the proof of the theorem. □

4 Conclusion

Theorem 6 leaves Case Two as the only open case for establishing the 1-factorizability of the regular graphs of order $2n$ and degree $\lambda(2n)$, with $\lambda \geq \lambda^*$. We will not solve this problem in the present paper, as we have not yet devised a general argument. It is clear from Theorem 5, however, that, if Case Two occurs, then the Tutte set S ought to be "very large", and, correspondingly, there ought to be a large number of very small odd components in $G_t^{**} - S$ (most of which singleton), and a large number of edges joining S to $G_t^{**} - S$. One possible approach could be to prove that, through a possible re-selection of the near 1-factors $F_1, F_2, \ldots F_t$, one can decrease the number of odd components of $G_t^{**} - S$, or decrease the difference $odd(S) - |S|$, and so forth. It may be useful to single out a special Tutte Set for G_t^{**} (on which further assumptions can be made), for example by considering a Gallai-Edmonds decomposition for G_t^{**} (see [8]).

Acknowledgements The authors wish to thank Prof. Hung-Lin Fu, who provided them with a stimulus for this paper to be written.

References

[1] D. Cariolaro, *The 1-factorization problem and some related conjectures,* Ph.D. thesis, University of Reading, U.K., 2004, pp.xi+141.

[2] D. Cariolaro and A.J.W. Hilton, *Class 1 graphs associated with regular graphs of high degree,* Congressus Numerantium, **159** (2002), 133-142.

[3] A.G. Chetwynd and A.J.W. Hilton, *Regular graphs of high degree are 1-factorizable,* Proc. Lond. Math. Soc., (3) **50** (1985), 193-206.

[4] A.G. Chetwynd and A.J.W. Hilton, *The edge-chromatic class of regular graphs of degree* 4 *and their complements,* Discr. Appl. Math., **16** (1987), 125-134.

[5] A.G. Chetwynd and A.J.W. Hilton, *1-factorizing regular graphs of high degree - an improved bound*, Discrete Math., **75** (1989), 103-112.

[6] G.A. Dirac, *Some theorems on abstract graphs,* Proc. London Math. Soc., **2** (1952), 69-81.

[7] I. Holyer, *The NP-completeness of edge-colouring,* SIAM J. Computing, **10** (1981), 718-720.

[8] L. Lovasz and M.D. Plummer, *Matching theory*, North-Holland Mathematics Studies 121 (1986).

[9] T. Niessen, *1-Faktorisierbarkeit regulärer Graphen hohen Grades,* Diplomarbeit, Aachen (1988).

[10] T. Niessen and L. Volkmann, *Class 1 conditions depending on the minimum degree and the number of vertices of maximum degree,* Journal of Graph Theory, (2) **14** (1990), 225-246.

[11] L. Perkovic and B. Reed, *Edge coloring regular graphs of high degree*, Discrete Math., **165/166** (1997), 567-578.

[12] A. Rosa and W.D. Wallis, *Premature sets of 1-factors or how not to schedule round robin tournaments,* Discr. Appl. Math., **4** (1982), 291-297.

[13] Z. Song, *Chromatic index critical graphs of odd order with five major vertices,* JCMCC, **41** (2002), 161-186.

[14] Z. Song and H.P. Yap, *The chromatic index critical graphs of even order with five major vertices,* manuscript.

[15] W.T. Tutte, *The factorization of linear graphs,* J. Lond. Math. Soc., **22** (1947), 459-474.

[16] W.D. Wallis, *One-Factorizations,* Kluwer, Dordrecht, 1997.