Cycle decompositions of the complete graph

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Abstract

For a positive integer n, let G be K_n if n is odd and K_n less a one-factor if n is even. In this paper it is shown that, for non-negative integers p, q and r, there is a decomposition of G into p 4-cycles, q 6-cycles and r 8-cycles if $4p + 6q + 8r = |E(G)|$, $q = 0$ if $n < 6$, and $r = 0$ if $n < 8$.

1 Introduction

Is it possible to decompose K_n (n odd) or $K_n - I_n$ (n even, I_n is a one-factor of K_n) into t cycles of lengths $m_1, \ldots, m_t$? Obvious necessary conditions for finding these *cycle decompositions* are that each cycle length must be between 3 and n and the sum of the cycle lengths must equal the number of edges in the graph being decomposed. That these simple conditions are sufficient was conjectured by Alspach [3] in 1981. To date, only a few special cases have been solved, mostly where each m_i must take one of a restricted

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number of values [1, 2, 6, 8]. In particular, we note that the case where all the cycles have the same length has recently been completely solved by Alspach and Gavlas [4] and Šajna [11]. We also note that Rosa [10] has proved that the conjecture is true for $n \leq 10$, and Balister [5] has shown that the conjecture is true if the cycle lengths are bounded by some linear function of n and n is sufficiently large.

In this paper, we solve the case where each cycle has length 4, 6 or 8; for the proof we introduce an innovative extension technique for finding cycle decompositions of $K_n(-I_n)$ from decompositions of K_m, $m < n$.

Theorem 1 *Let n be a positive integer. Let p, q and r be non-negative integers. Then K_n (n odd) or $K_n - I_n$ (n even) can be decomposed into p 4-cycles, q 6-cycles and r 8-cycles if and only if*

1. $4p + 6q + 8r = \begin{cases} |E(K_n)| & \text{if } n \text{ is odd,} \\ |E(K_n - I_n)| & \text{if } n \text{ is even, and} \end{cases}$

2. *the cycles all have length at most n.*

Our novel extension technique is described in the next section. The proof of Theorem 1 is in the final section.

Definitions and notation. An edge joining u and v is denoted (u, v). A path of length $k-1$ is denoted $(v_1, \ldots, v_k)$ where v_i is adjacent to v_{i+1}, $1 \leq i \leq k-1$, but a path of length zero—that is, a single vertex—will be denoted simply v_1 rather than (v_1). A k-cycle is denoted $[v_1, \ldots, v_k]$, where v_i is adjacent to v_{i+1}, $1 \leq i \leq k-1$, and v_1 is adjacent to v_k. A *path-graph* is a collection of vertex-disjoint paths and is described by listing the paths. A path-graph containing only paths of lengths zero or one is a *matching*.

2 An extension technique

In this section we introduce a technique that we can use to obtain cycle decompositions of $K_n(-I_n)$ from cycle decompositions of $K_m(-I_m)$ when $m < n$.

First we define a different type of decomposition. Let n, s and t be non-negative integers. An (s, t)-*decomposition* of K_n may be either even or odd. An odd (s, t)-decomposition contains the following collection of subgraphs:

- path-graphs $P_1, \ldots, P_s$, and
- cycles $C_{s+1}, \ldots, C_{s+t}$;

with the following properties:

- their edge-sets partition the edge-set of K_n, and
- each vertex is in precisely s of the subgraphs $P_1, \ldots P_s, C_{s+1}, \ldots, C_{s+t}$.

An *even* (s,t)*-decomposition* of K_n is the same as an odd (s,t)-decomposition except that it also contains a matching P_0 which contains every vertex.

Example 1. We display an odd $(4,2)$-decomposition of K_7:

$$\begin{aligned} P_1 &= (1,5,2,4),(3,7) \\ P_2 &= (1,6,2,7) \\ P_3 &= (3,6,5),2 \\ P_4 &= (4,7,5),6 \\ C_5 &= [1,3,5,4,6,7] \\ C_6 &= [1,2,3,4]. \end{aligned}$$

Now we introduce the idea of extending a decomposition of K_m to a decomposition of K_n. Let $P_1, \ldots P_s, C_{s+1}, \ldots, C_{s+t}$ be an odd (s,t)-decomposition of K_m. For $n > m$, n odd, identify the vertices of K_m with m of the vertices of K_n. If K_n has a decompostion into cycles $C_1, \ldots, C_{s+t}$ such that, for $1 \leq i \leq s$, C_i is a supergraph of P_i, then we call this decomposition an *extension* of the decomposition of K_m. Similarly, for $n > m$, n even, an even (s,t)-decompostion of K_m, $P_0, \ldots P_s, C_{s+1}, \ldots, C_{s+t}$, can be extended to a decomposition of K_n less a one-factor I_n into cycles $C_1, \ldots, C_{s+t}$ if we have the additional property that I_n is a supergraph of P_0.

Theorem 2 *Let m, n, s and t be non-negative integers with $m < n$ and $s = \lfloor (n-1)/2 \rfloor$. Let $D = (P_0,)P_1, \ldots P_s, C_{s+1}, \ldots, C_{s+t}$ be an (s,t)-decomposition of K_m that is even or odd as the parity of n.*

Then D can be extended to a decomposition of K_n (less a one-factor I_n if n is even) into cycles $C_1, \ldots, C_{s+t}$ if and only if,

$$\begin{aligned} \textit{for } 1 \leq i \leq s, \quad n - m &\geq |V(P_i)| - |E(P_i)|, \textit{ and,} \qquad (1) \\ \textit{if } n \textit{ is even}, \quad |E(P_0)| &\geq m - n/2. \qquad (2) \end{aligned}$$

Notice that since each vertex of $V(K_n \setminus K_m)$ must be in s of the cycles, it must be in each C_i, $1 \leq i \leq s$, since the other cycles are subgraphs of K_m. Therefore C_i, $1 \leq i \leq s$, has length $|V(P_i)| + n - m$.

Before we prove Theorem 2, let us see how it can be used. We consider four examples.

Example 2. Let D be the $(4,2)$-decomposition of K_7 shown in Example 1. Apply Theorem 2 with $n = 9$, $m = 7$, $s = 4$ and $t = 2$. Checking that (1) is satisfied is easy if we notice that $|V(P_i)| - |E(P_i)|$ is equal to the number of paths in P_i (remember that we count an isolated vertex as a path). By Theorem 2, there exists a cycle decomposition $C_1, \ldots, C_6$ of K_9 where C_i, $1 \leq i \leq 4$ is a supergraph of P_i. As C_i has length $|V(P_i)| + n - m$, C_1 will be an 8-cycle and C_2, C_3 and C_4 will be 6-cycles. We display an example of a cycle decomposition obtained by extending D.

$$\begin{aligned} C_1 &= [1,5,2,4,9,7,3,8] \\ C_2 &= [1,6,2,7,8,9] \\ C_3 &= [2,9,3,6,5,8] \\ C_4 &= [4,7,5,9,6,8] \\ C_5 &= [1,3,5,4,6,7] \\ C_6 &= [1,2,3,4]. \end{aligned}$$

In the following three examples, we begin with a cycle decomposition of $K_m(-I_m)$. By making slight changes to this decomposition—we take the edges from one of the cycles, or from the one-factor I_m, and use them to create path-graphs—we obtain an (s,t)-decomposition of K_m. Then we apply Theorem 2 to obtain a cycle decomposition of K_n for some $n > m$. This method of obtaining a cycle decomposition of a complete graph from a cycle decomposition of a smaller complete graph will help us to give an inductive proof of Theorem 1 in the final section.

Example 3. Let Δ be a decomposition of K_{10} into p 4-cycles, q 6-cycles and r 8-cycles and a one-factor I_{10} where the vertices are labelled so that

$$I_{10} = (1,2),(3,4),(5,6),(7,8),(9,10).$$

Label the cycles $C_7 \ldots, C_{6+p+q+r}$ and let

$$\begin{aligned} P_1 &= (1,2),(3,4),5 \\ P_2 &= (5,6),7 \\ P_3 &= (7,8),9 \\ P_4 &= (9,10),1 \\ P_5 &= 2,3,4 \\ P_6 &= 6,8,10. \end{aligned}$$

Let $D = P_1, \ldots, P_6, C_7, \ldots, C_{p+q+r}$ and notice that it is a decomposition of K_{10}. As the cycles $C_7, \ldots, C_{6+p+q+r}$ form a decomposition of $K_{10} - I_{10}$, each vertex $v \in V(K_{10})$ will be in four of them (consider degrees). Each vertex is also in two of the path-graphs displayed above. Thus each vertex is in 6 of the graphs of D, and D is an odd $(6, p+q+r)$-decomposition of K_{10}. Apply Theorem 2 with $n = 13$, $m = 10$, $s = 6$ and $t = p+q+r$ (it is easy to check that (1) is satisfied). The decomposition of K_{13} obtained contains all the cycles of D and also cycles $C_1, \ldots, C_6$ that are supergraphs of the path-graphs $P_1, \ldots, P_6$. Thus C_1 has length 8 and C_i, $2 \le i \le 6$, has length 6, and the decomposition of K_{13} contains p 4-cycles, $q+5$ 6-cycles and $r+1$ 8-cycles.

Hence, if we require a decomposition of K_{13} into p' 4-cycles, q' 6-cycles and r' 8-cycles, we can obtain it from a decomposition of K_{10} into $p = p'$

4-cycles, $q = q' - 5$ 6-cycles and $r = r' - 1$ 8-cycles. Of course, we require that $q' \geq 5$ and $r' \geq 1$ so that p, q and r are non-negative.

Example 4. Let $m \equiv 1 \bmod 4$, $m \geq 9$. Suppose that we have a decomposition Δ of K_m into p 4-cycles, q 6-cycles and r 8-cycles, where $q \geq 1$. We are going to use this to find a decomposition of K_{m+4} so let $n = m+4$ and $s = (n-1)/2$. Let D be a decomposition of K_m that contains all the cycles of Δ except one of the 6-cycles which we may assume is $C = [1, 2, 3, 4, 5, 6]$. Label the other cycles $C_{s+1}, \ldots, C_{s+p+q+r-1}$. D also contains s path-graphs that contain the edges of C and also isolated vertices. If $m = 9$, then $s = 6$ and the path-graphs are

$$\begin{aligned} P_1 &= (1,2), 6, 7 \\ P_2 &= (2,3), 1, 7 \\ P_3 &= (3,4), 2, 8 \\ P_4 &= (4,5), 3, 8 \\ P_5 &= (5,6), 4, 9 \\ P_6 &= (1,6), 5, 9. \end{aligned}$$

If $m = 13$, then there are two further path-graphs

$$\begin{aligned} P_7 &= 10, 11, 12, 13 \\ P_8 &= 10, 11, 12, 13. \end{aligned}$$

For $m \geq 17$, there are further path-graphs $P_9, \ldots, P_s$, where, for $1 \leq i \leq (s-8)/2$,

$$P_{7+2i} = P_{8+2i} = 4i + 10, 4i + 11, 4i + 12, 4i + 13.$$

As the cycles of D form a decomposition of $K_m - C$, $v \in V(K_m) \setminus C$ will be in $s - 2$ of them; v is also in 2 of the path-graphs. If $v \in C$, then it is in only $s - 3$ of the cycles of D, but is in 3 of the path-graphs. As every vertex is in s of the graphs of D, it is an odd $(s, p + q + r - 1)$-decomposition of K_m. Use D to apply Theorem 2 with n, m and s as defined and $t = p + q + r - 1$. The decomposition of K_n obtained contains all the cycles of D and also cycles $C_1, \ldots, C_s$ that are supergraphs of the path-graphs $P_1, \ldots, P_s$, and C_i, $1 \leq i \leq s$, has length $|V(P_i)| + n - m = 8$. The decomposition of K_n that is obtained contains p 4-cycles, $q - 1$ 6-cycles and $r + s$ 8-cycles. Thus we can obtain a decomposition of K_n, $n \equiv 1 \bmod 4$, $n \geq 13$, into p' 4-cycles, q' 6-cycles and r' 8-cycles from a cycle decomposition of K_{n-4} whenever $r' \geq s$.

Example 5. An example for complete graphs of even order. Let Δ be a decomposition of $K_m - I_m$, $m \geq 8$ even, into p 4-cycles, q 6-cycles and r 8-cycles, where $p \geq 1$. We are going to find a decomposition of K_{m+4} so let $n = m+4$, $s = (n-2)/2$ and $t = p+q+r-1$. Let one of the 4-cycles be

$C = [1, 2, 3, 4]$. Let D be a decomposition of K_m that contains the cycles of $\Delta - C$ (labelled $C_{s+1}, \ldots, C_{s+t}$), a matching $P_0 = I_m$ and s path-graphs that contain the edges of C. If $m = 8$, the path-graphs are

$$\begin{aligned} P_1 &= (1,2), 5, 6 \\ P_2 &= (2,3), 5, 6 \\ P_3 &= (3,4), 7, 8 \\ P_4 &= (1,4), 7, 8 \\ P_5 &= 1, 2, 3, 4. \end{aligned}$$

If $m = 10$, then $P_2, \ldots, P_5$ are as above and

$$\begin{aligned} P_1 &= (1,2), 9, 10 \\ P_6 &= 5, 6, 9, 10. \end{aligned}$$

For $m \geq 12$, let $P_1 = (1,2), m-1, m$, $P_2, \ldots, P_5$ be as above and

$$\begin{aligned} P_6 &= 5, 6, 9, 10 \\ P_7 &= 9, 10, 11, 12 \\ P_8 &= 11, 12, 13, 14 \\ \vdots &\ \ \vdots \ \ \vdots \\ P_s &= m-3, m-2, m-1, m. \end{aligned}$$

Notice that D is an even (s,t)-decomposition of K_m (it is easy to check that every vertex is in P_0 and s of the other graphs in D). Thus from D, a cycle decomposition of $K_n - I_n$ is obtained by applying Theorem 2 with n, m, s and t as defined. The decomposition of $K_n - I_n$ obtained contains cycles $C_1, \ldots, C_s$ of length, for $1 \leq i \leq s$, $|V(P_i)| + n - m = 8$. Therefore it contains $p-1$ 4-cycles, q 6-cycles and $r+s$ 8-cycles, and we note that we can obtain a decomposition of $K_n - I_n$, $n \geq 12$ even, into p' 4-cycles, q' 6-cycles and r' 8-cycles from a cycle decomposition of $K_{n-4} - I_{n-4}$ whenever $r \geq s$.

Proof of Theorem 2: Necessity: for $1 \leq i \leq s$, C_i contains the edges of P_i plus at most $2(n-m)$ edges from $E(K_n) \setminus E(K_m)$. As it has length $|V(P_i)| + n - m$, we have

$$|E(P_i)| + 2(n-m) \geq |V(P_i)| + n - m.$$

Rearranging, (1) is obtained. Similarly, I_n contains the edges of P_0 plus at most $n - m$ edges from $E(K_n) \setminus E(K_m)$. As I_n has $n/2$ edges,

$$|E(P_0)| + n - m \geq n/2.$$

Rearranging, (2) is obtained.

Sufficiency: to simplify the presentation we will prove only the (slightly trickier) case where n is even. Thus $s = (n-2)/2$. Let the vertices of K_m be $v_1, \ldots, v_m$.

First consider the case $m = n-1$. From (1) and (2) we find that, for $1 \le i \le s$,

$$\begin{aligned} |E(P_i)| &\geq |V(P_i)| - 1, \text{ and} \\ |E(P_0)| &\geq n/2 - 1. \end{aligned}$$

In fact, we must have equality in each case since P_0 is a matching on $n-1$ vertices and, for $1 \le i \le s$, P_i is acyclic. Thus each P_i, $1 \le i \le s$, must be a single path and P_0 contains $n/2-1$ independent edges and an isolated vertex. Each vertex has degree $n-2$ in K_{n-1}, is in $s = (n-2)/2$ of the subgraphs $P_1, \ldots, P_s, C_{s+1}, \ldots, C_{s+t}$, and has degree at most 2 in each of these subgraphs. Thus the vertex that has degree 0 in P_0 must have degree 2 in each of the other subgraphs that contain it, and each vertex of degree 1 in P_0, must have degree 1 in one of the other subgraphs that contain it and degree 2 in the rest; that is, it must be the endvertex of precisely one of the paths P_i, $1 \le i \le s$. Therefore we obtain the cycle decomposition of K_n from D, the (s,t)-decomposition of K_{n-1}, by adding edges (v_j, v_n), $1 \le j \le n-1$, to the subgraphs in the following way. If v_j is an endvertex in P_i, then the new edge (v_j, v_n) is placed in the subgraph P_i. Hence P_i becomes a cycle of length $|V(P_i)|+1$. Finally, if v_j is the isolated vertex in P_0, then (v_j, v_n) is the additional edge required to form the one-factor I_n.

Now we show that if $m < n-1$, then D can be extended to D', an even (s,t)-decomposition of K_{m+1}, so that (1) and (2) are satisfied with m replaced by $m+1$. By repeating this argument a finite number of times an even (s,t)-decomposition of K_{n-1} that satisfies (1) and (2) with m replaced by $n-1$ can be found.

To obtain D', a new vertex v_{m+1} is added to K_m. It must be joined to each vertex of K_m by one edge and each of these m additional edges must be added to exactly one of the path-graphs or the matching of D. Note that as $s = \lfloor (n-1)/2 \rfloor$ we require that v_{m+1} is in all s of the path-graphs, so it must be added as an isolated vertex to any path-graph that has been given no new edges.

We need a way to decide which subgraph each new edge should be placed in. Construct a bipartite multigraph B with vertex sets $\{P_0', \ldots, P_s'\}$ and $\{v_1', \ldots, v_m'\}$. For $1 \le i \le s$, $1 \le j \le m$, if $v_j \in P_i$, then join P_i' to v_j' by $2 - d_{P_i}(v_j)$ edges. Also join P_0' to v_j by $1 - d_{P_0}(v_j)$ edges. In fact, we think of B as being constructed as follows: for $1 \le i \le s$, join P_i' to v_j' by two edges if $v_j \in P_i$ and join P_0' to v_j by one edge; then for each edge (v_j, v_k) in P_i, $0 \le i \le s$, delete the edges (P_i', v_j') and (P_i', v_k').

If v_j is in x of the cycles in D, then it is in $s-x$ of the path-graphs; it is also in the matching P_0. As it is incident with $2x$ edges in the cycles, it is incident with $m-1-2x$ edges in the matching and path-graphs. When B is constructed, we begin by placing $2(s-x)+1$ edges at v_j'. For $0 \le i \le s$, for each edge incident at v_j in P_i, we delete an edge (P_i', v_j') in B. Therefore

$$d_B(v_j') = 2(s-x) + 1 - (m-1-2x) = n - m. \tag{3}$$

When we construct B, we first place $2|V(P_i)|$ edges at P_i', $1 \le i \le s$. Then for each edge (v_j, v_k) in P_i, we delete two of these edges: (P_i', v_j') and (P_i', v_k'). Thus, by (1), for $1 \le i \le s$,

$$d_B(P_i') = 2(|V(P_i)| - |E(P_i)|) \le 2(n-m). \tag{4}$$

Using a similar argument, by (2),

$$d_B(P_0') = |V(P_0)| - |E(P_0)| \le n - m. \tag{5}$$

We need the following: a set $\mathcal{F}$ of sets is a *laminar* set if, for all $X, Y \in \mathcal{F}$, either $X \subseteq Y$, or $Y \subseteq X$ or $X \cap Y = \emptyset$; we say $x \approx y$ if $\lfloor y \rfloor \le x \le \lceil y \rceil$ (note that the relation is not symmetric).

Lemma 3 [9] *If $\mathcal{F}$ and $\mathcal{G}$ are laminar sets of subsets of a finite set M and h is a positive integer, then there exists a set $L \subseteq M$ such that*

$$|L \cap X| \approx |X|/h \textit{ for every } X \in \mathcal{F} \cup \mathcal{G}.$$

We construct two laminar sets $\mathcal{F}$ and $\mathcal{G}$ which contain subsets of $E(B)$. Let $\mathcal{F}$ contain sets $P_0^*, \ldots, P_s^*$, where P_i^*, $0 \le i \le s$, contains the set of all edges incident with P_i' in B. Also if v_{j_1} and v_{j_2} are endvertices of a path in P_i, then let $\{(P_i', v_{j_1}'), (P_i', v_{j_2}')\}$ be a set in $\mathcal{F}$ (call these endvertex-sets). Let $\mathcal{G}$ contain sets $v_1^*, \ldots, v_m^*$, where v_j^*, $1 \le j \le m$, contains the set of all edges incident with v_j' in B.

Apply Lemma 3 with $M = E(B)$ and $h = n - m$ to obtain a set of edges L that, by (3), (4) and (5), contains exactly one edge incident with v_j', $1 \le j \le m$, at most two edges incident with P_i', $1 \le i \le s$, and at most one edge incident with P_0. Also L contains at most one edge from each endvertex-set.

Now we extend D to D'. For $1 \le j \le n$, if (P_i', v_j') is in L, then (v_{m+1}, v_j) is placed in P_i. Then v_{m+1} is added as an isolated vertex to any P_i to which no new edges have been added. Since L contains exactly one edge incident with each v_j, each new edge is placed in exactly one subgraph. There is only an edge (P_i', v_j'), $1 \le i \le s$, $1 \le j \le m$, in B if v_j has degree less than 2, so after the new edges are added v_j has degree at most 2 in P_i. Since L contains at most two edges incident with P_i, $1 \le i \le s$, v_{m+1} has degree at most 2 in each P_i, $1 \le i \le s$. As L contains at most one edge from each endvertex-set, v_{m+1} cannot have been joined to both ends of a path in P_i (thus creating a cycle). Therefore P_i, $1 \le i \le s$, is still a path-graph. By a similar argument, P_0 is still a matching.

We must check that (1) and (2) remain satisfied with m replaced by $m+1$. First (1): note that $|V(P_i)|$ increases by one (as the new vertex is adjoined to every path-graph) and $E(P_i)$ increases by at most two. If initially we have

$$n - m - 2 \ge |V(P_i)| - |E(P_i)|,$$

then clearly (1) remains satisfied. If

$$n - m - 1 = |V(P_i)| - |E(P_i)|,$$

then, arguing as for (4), $d_B(P_i') = 2(|V(P_i)| - |E(P_i)|) = 2(n-m) - 2 \geq n - m$ (since $n - m \geq 2$). So L contains at least one edge incident with P_i' and at least one edge is added to P_i and (1) remains satisfied. If

$$n - m = |V(P_i)| - |E(P_i)|,$$

then $d_B(P_i') = 2(n-m)$, and L contains two edges incident with P_i' and hence two edges are added to P_i and (1) remains satisfied.

Finally, if initially we have

$$|E(P_0)| - 1 \geq m - n/2,$$

then (2) remains satisfied. If

$$|E(P_0)| = m - n/2,$$

then $d_B(P_0') = n - m$, and L contains an edge incident with P_0' and hence an edge is added to P_0 and (2) remains satisfied. □

3 Proof of Theorem 1

The necessity of the conditions is clear.

Sufficiency: as we remarked in the Introduction, all possible cycle decompositions of K_n have been found for $n \leq 10$ [10]. For $n > 10$, we assume that cycle decompositions for $K_{n'}$, $n' < n$, are known. Then we use two techniques to find decompositions of K_n. Some cases are found using Theorem 2 to extend a decomposition of $K_{n'}$, $n' < n$. The second method is to consider K_n as the union of several edge-disjoint subgraphs. Each cycle required in the decomposition of K_n is assigned to one of the subgraphs in such a way that the total number of edges in the cycles assigned to a subgraph is equal to the number of edges in that subgraph. Thus the problem of decomposing K_n becomes the problem of decomposing its subgraphs. If a subgraph is a smaller complete graph, then we can assume the decomposition exists, and if it is a complete bipartite graph, we can use the following result of Chou, Fu and Huang.

Theorem 4 [7] *The complete bipartite graph $K_{m,n}$ can be decomposed into p 4-cycles, q 6-cycles and r 8-cycles if and only if*

1. *m and n are even,*
2. *no cycle has length greater than $2\min\{m,n\}$,*
3. *$mn = 4p + 6q + 8r$, and*

4. if $m = n = 4$, then $r \neq 1$.

First suppose that n is odd. Note that if $n \equiv 3 \bmod 4$, K_n has an odd number of edges and cannot have a decompisition into cycles of even length.

Case 1. $n = 13$. We require a decomposition of K_{13} into p 4-cycles, q 6-cycles and r 8-cycles. Note that

$$4p + 6q + 8r = |E(K_{13}| = 78.$$

Thus q is odd since $78 \not\equiv 0 \bmod 4$. In Section 2, we saw that we could find a decomposition of K_{13} by extending a decomposition of K_9 if $r \geq 6$ (Example 4) or by extending a decomposition of K_{10} if $q \geq 5$ (Example 3). We may now assume that $q \in \{1, 3\}$ and $r \leq 5$, which implies that $4p \geq 78 - (3 \times 6) - (5 \times 8) = 20$, that is, $p \geq 5$.

Consider K_{13} as the union of K_5, K_9 and $K_{8,4}$ where K_5 is defined on the vertex set $\{1, \ldots, 5\}$, K_9 on the vertex set $\{5, \ldots, 13\}$, and $K_{4,8}$ on the vertex sets $\{1, \ldots, 4\}$ and $\{6, \ldots, 13\}$. Let $C = [1, 6, 2, 7]$. Let $H_1 = K_5 \cup C$ and $H_2 = K_{8,4} - C$. Note that H_1 is the union of two 4-cycles and a 6-cycle. K_{13} is the union of K_9, H_1 and H_2. For the remaining cases, we assign the cycles required in the decomposition of K_{13} to these three subgraphs. (A decomposition of H_2 is found by finding a decomposition of $K_{8,4}$ that contains, as well as the required cycles, a further 4-cycle which can be labelled $[1, 6, 2, 7]$ and discarded.) Two 4-cycles and a 6-cycle are assigned to H_1. There are at most two further 6-cycles which are assigned to K_9. Up to three 8-cycles are assigned to H_2; any remaining 8-cycles (there are at most two more) are assigned to K_9. This only leaves some 4-cycles to be assigned, and clearly the number of edges not accounted for in H_2 and K_9 is, in both cases, positive and equal to 0 mod 4.

Case 2. $n = 17$. We require a decomposition of K_{17} into p 4-cycles, q 6-cycles and r 8-cycles. Note that

$$4p + 6q + 8r = |E(K_{17})| = 136.$$

If $r \geq (n - 1)/2 = 8$, then, as seen in Example 4, we can apply Theorem 2 to obtain the decomposition of K_{17} from a decomposition of K_{13}.

For the remaining cases, let $K_{17} = K_9 \cup K_9 \cup K_{8,8}$. We will assign the required cycles to these three subgraphs. Suppose that $4p + 8r \geq 64$. Then we can assign all of the 8-cycles (since $r < 8$) and some 4-cycles to $K_{8,8}$ so that the assigned cycles have precisely 64 edges. There remain to be assigned some 4-cycles and 6-cycles which have a total of 72 edges. Thus either $4p \geq 36$ or $6q \geq 36$ and we can assign cycles all of the same length to a K_9. We assign the remaining cycles to the other K_9.

If $4p + 8r < 64$, then $6q > 136 - 64 = 72$. We can assign six 6-cycles to each K_9 and the remaining cycles to $K_{8,8}$.

Case 3. $n \geq 21$ odd. We require a decomposition of K_n into p 4-cycles, q 6-cycles and r 8-cycles. If $r \geq (n-1)/2$, then, as seen in Example 4, we can apply Theorem 2 to obtain the decomposition of K_n from a decomposition of K_{n-4}. Otherwise $r < (n-1)/2$. Let $K_n = K_{n-12} \cup K_{13} \cup K_{n-13,12}$. We assign the required cycles to these subgraphs.

Suppose that $4p + 8r \geq |E(K_{n-13,12})|$. Then we can assign all of the 8-cycles and some 4-cycles to $K_{n-13,12}$ (since $8r < 4(n-1) < 12(n-13)$ for $n \geq 21$). We are left with 4-cycles and 6-cycles to assign to K_{n-12} and K_{13}. We can assign cycles all of the same length to the smaller of these graphs (which is either K_9 or K_{13}—both have a number of edges equal to 0 mod 4 and 0 mod 6) and the remaining cycles to the other.

If $4p+8r < |E(K_{n-13,12})|$, then $6q \geq |E(K_{n-12})|+|E(K_{13})|$. We cannot simply assign 6-cycles to K_{n-12} and K_{13} however, since $|E(K_{n-12})|$ is not equal to 0 mod 6 for all $n \equiv 1 \bmod 4$. If $6q \geq |E(K_{n-13,12})| + |E(K_{13})|$, then we can assign 6-cycles to $K_{n-13,12}$ and K_{13} and any remaining cycles to K_{n-12}. If $6q < |E(K_{n-13,12})|+|E(K_{13})|$, then $4p+8r > |E(K_{n-12})| > 16$ (as $n \geq 21$). One of the following must be true.

$$\begin{aligned} |E(K_{n-12})| &\equiv 0 \bmod 6 \\ |E(K_{n-12})| + 8 &\equiv 0 \bmod 6 \\ |E(K_{n-12})| + 16 &\equiv 0 \bmod 6 \end{aligned}$$

We assign 6-cycles to K_{13} and to K_{n-12}, except that to K_{n-12} we also assign 4-cycles and 8-cycles with a total of 8 or 16 edges if the number of edges in K_{n-12} is 2 mod 6 or 4 mod 6, respectively. The remaining edges are assigned to $K_{n-13,12}$.

Case 4. $n = 12$. We require a decomposition of $K_{12} - I_{12}$ into p 4-cycles, q 6-cycles and r 8-cycles. Then

$$4p + 6q + 8r = |E(K_{12} - I_{12})| = 60,$$

and so q is even. In Example 5, we saw that if $r \geq (n-2)/2 = 5$, we can use Theorem 2 to extend a decomposition of K_8 to obtain the required cycle decomposition of $K_{12} - I_{12}$. Otherwise let $K_{12} - I_{12} = (K_6 - I_6) \cup (K_6 - I_6) \cup K_{6,6}$. We will assign the required cycles to these subgraphs.

Note that $4p + 6q \geq 60 - 32 = 28$ (since $r \leq 4$). If $q = 0$, then $p \geq 7$ and we can assign 4-cycles to each K_6. If $q = 2$, we assign the two 6-cycles to one K_6 and 4-cycles to the other K_6. If $q \geq 4$, we assign 6-cycles to each K_6. In each case the remaining cycles are assigned to $K_{6,6}$.

Case 5. $n \geq 14$ even. We require a decomposition of $K_n - I_n$ into p 4-cycles, q 6-cycles and r 8-cycles. In Example 5, we saw that if $r \geq (n-2)/2$, we can use Theorem 2 to extend a decomposition of K_{n-4} to find the required cycle decomposition of $K_n - I_n$. Otherwise let $K_n - I_n = (K_6 - I_6) \cup (K_{n-6} - I_{n-6}) \cup K_{6,n-6}$. We will assign the required cycles to these subgraphs.

We have

$$\begin{aligned} 4p+6q &= |E(K_n-I_n)|-8r \\ &\geq n(n-2)/2-4(n-4) \\ &= (n^2-10n+32)/2 \\ &\geq 44, \end{aligned}$$

as $n \geq 14$. Therefore $4p \geq 22$ or $6q \geq 22$ and we can assign cycles all of length 4 or all of length 6 to $K_6 - I_6$. Note that the number of edges in $K_{6,n-6}$ is 0 mod 12. If $q \geq n-6$, we assign only 6-cycles to $K_{6,n-6}$. Otherwise we assign all the 6-cycles to $K_{6,n-6}$ if q is even, or all but one of them if q is odd. Then the number of remaining edges is also 0 mod 12 so we can assign as many 4-cycles as necessary. All remaining cycles are assigned to K_{n-6}. □

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