On the chromatic index of graphs whose core has maximum degree two

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Abstract

Let G be a connected graph. The core of G, denoted by G_Δ, is the subgraph of G induced by the vertices of maximum degree. Hilton and Zhao [*On the edge-colouring of graphs whose core has maximum degree two*, JCMCC 21 (1996), 97-108] conjectured that, if $\Delta(G_\Delta) \leq 2$, then G is Class 2 if and only if G is overfull, with the sole exception of the Petersen graph with one vertex deleted. In this paper we prove this conjecture for all graphs G of even order such that $|V(G_\Delta)| > \max\{\frac{1}{2}|V(G)|, |V(G)| - 2\Delta(G) + 5\}$.

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1 Introduction

All graphs considered in this paper are finite and simple. The vertex and edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. If G is a graph and $S \subseteq V(G)$, by $N(S)$ we denote the set of vertices of G which are adjacent to at least one vertex in S. If H is a subgraph of G we denote this by $H \subseteq G$.

The *core* of G, denoted by G_Δ, is the subgraph of G induced by the set of vertices of maximum degree. For graph theoretic terminology not explicitly defined here, we refer the reader to [4].

An edge colouring of a graph G is a map $\varphi : E(G) \to \mathcal{C}$, where $\mathcal{C}$ is a set, called the *colour set*, and $\varphi(e_1) \neq \varphi(e_2)$ for any pair (e_1, e_2) of distinct mutually incident edges of G. The minimum cardinality of the colour set in an edge colouring of G is called the *chromatic index* of G and denoted by $\chi'(G)$.

Vizing [13] proved that, for any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. Accordingly, we say that G is *Class 1* if $\chi'(G) = \Delta(G)$ and *Class 2* if $\chi'(G) = \Delta(G) + 1$. G is called *critical* if it is Class 2, connected and, for every edge $e \in E(G)$, $\chi'(G - e) < \chi'(G)$.

A graph G is *overfull* if $|E(G)| > \Delta(G) \cdot \lfloor \frac{|V(G)|}{2} \rfloor$. It is easy to see that every overfull graph is Class 2. However the converse of this statement is not true, and it is very difficult in general to determine whether a given graph is Class 2 (or Class 1).

Fournier [5] proved that, if the core of G contains no cycles, then G is Class 1. It is natural to ask what can be said about G if G_Δ is indeed (isomorphic to) a cycle or, more generally, if it consists of vertex disjoint cycles and paths.

Let P^* denote the Petersen graph with one vertex deleted. Then P^* provides an example of a Class 2 graph whose core is a 6-cycle. Thus Fournier's result does not extend to graphs whose core is a cycle.

However Hilton and Zhao [9] have posed the following conjecture, which attributes to P^* an exceptional property among all connected graphs whose core has maximum degree at most two.

Conjecture 1 *Let G be a connected graph such that $\Delta(G_\Delta) \leq 2$. Then G is Class 2 if and only if G is overfull, unless $G \approx P^*$.*

Hilton and Zhao [7] proved the above conjecture for all graphs G such that $\Delta(G) \geq \frac{1}{2}(|V(G)| + 3)$. This bound has been recently improved to $\Delta(G) \geq \frac{1}{2}(|V(G)| - 1)$ by Koh and Song [10]. Hilton and Zhao [8] also proved Conjecture1 for all graphs G such that $\Delta(G) \geq |V(G)| - |V(G_\Delta)|$. Further progress was made by Hilton and Zhao in [9], where the conjecture was proved for all graphs G such that $|V(G)| \geq 2k^2 + \frac{32}{3}k + \frac{47}{3}$ if $|V(G)|$ is even and $|V(G)| \geq 3k^2 + 12k + 16$ if $|V(G)|$ is odd, where $\Delta(G) = |V(G)| - |V(G_\Delta)| - k \geq k + 5$.

G. Cariolaro and the present first author settled the case $\Delta(G) = 3$ of Conjecture 1 using a colour-exchange technique [1].

From the classification of all critical graphs with at most five vertices of maximum degree due to Chetwynd and Hilton [2, 3], Song [11] and Song and

Yap [12], and Lemma 1 below, it also follows that Conjecture 1 holds for all graphs with $|V(G_\Delta)| \leq 5$.

The purpose of the present paper is to prove the following result:

Theorem 1 *Let G be a connected graph of even order such that $\Delta(G_\Delta) \leq 2$ and $|V(G_\Delta)| > \max\{\frac{1}{2}|V(G)|, |V(G)| - 2\Delta(G) + 5\}$. Then G is Class 2 if and only if G is overfull.*

This improves, for graphs of even order such that $|V(G_\Delta)| > \frac{1}{2}|V(G)|$, the Hilton and Zhao [8] and Koh and Song [10] bounds by almost a factor of 2.

2 Preliminary lemmas

The following two important lemmas were established by Hilton and Zhao in [7].

Lemma 1 *Let G be a connected Class 2 graph with $\Delta(G_\Delta) \leq 2$. Then:*

1. *G is critical;*
2. *$\delta(G_\Delta) = 2$;*
3. *$\delta(G) = \Delta(G) - 1$, unless G is an odd cycle;*
4. *$N(G_\Delta) = V(G)$.*

Lemma 2 *Let G be a connected overfull graph, which is not an odd cycle, such that $\Delta(G_\Delta) \leq 2$. Then*

$$\Delta(G) \geq \frac{1}{2}(|V(G)| + 3).$$

By Lemma 2 and the fact that Conjecture 1 has been settled for $\Delta(G) \geq \frac{1}{2}(|V(G)| - 1)$ and for $\Delta(G) \leq 3$, Conjecture 1 is reduced to the following conjecture.

Conjecture 2 *Let G be a connected graph such that $\Delta(G_\Delta) \leq 2$ and $3 < \Delta(G) < \frac{1}{2}(|V(G)| - 1)$. Then G is Class 1.*

We shall make use, in the proof of Theorem 1, of the following well known result of P. Hall [6]. Let G be a bipartite graph with bipartition (V_1, V_2). A matching M from V_1 to V_2 will be called *complete* if each vertex of V_1 is incident with an edge in M.

Lemma 3 *A bipartite graph with bipartition (V_1, V_2) contains a complete matching from V_1 to V_2 if and only if*

$$|N(S)| \geq |S| \text{ for every } S \subseteq V_1. \qquad (1)$$

We shall refer to the condition (1) above as to *Hall's Condition*.

Finally, we shall need the following consequence of the well known Vizing's Adjacency Lemma [14].

Lemma 4 *Let G be a critical graph and let $u \in V(G)$. Then u is adjacent to at least two vertices of maximum degree of G.*

3 Proof of Theorem 1

We now prove Theorem 1.

Proof of Theorem 1: Let $\Delta = \Delta(G)$, let $p = |V(G_\Delta)|$ and let $q = |V(G)| - |V(G_\Delta)|$. Since Conjecture 1 has been reduced to Conjecture 2, we may assume that

$$4 \leq \Delta \leq \tfrac{1}{2}(q + p - 2) \; . \tag{2}$$

We argue by contradiction, so suppose that G is Class 2. We shall show that G has a 1-factor F, and then derive a contradiction. Notice that $p + q = |V(G)|$ is even by assumption, so that

$$q \equiv p \pmod 2. \tag{3}$$

From the hypothesis that $|V(G_\Delta)| > \max\{\frac{1}{2}|V(G)|, |V(G)| - 2\Delta(G) + 5\}$ it follows that

$$p > q \tag{4}$$

and

$$\Delta \geq \frac{1}{2}(q + 6). \tag{5}$$

Let $\partial(G_\Delta)$ denote the set of edges of G with exactly one end in G_Δ. By Lemma 1, G_Δ is 2-regular, so that

$$|\partial(G_\Delta)| = (\Delta - 2)p. \tag{6}$$

Moreover, by Lemma 1, every non-core vertex has degree $\Delta - 1$, so that

$$|\partial(G_\Delta)| \leq (\Delta - 1)q, \tag{7}$$

and comparing (7) with (6), we see that

$$q(\Delta - 1) \geq p(\Delta - 2). \tag{8}$$

Let $\beta_1(G_\Delta)$ denote the *edge-independence number* of G_Δ, i.e. the maximum number of independent edges in G_Δ. We show that $\beta_1(G_\Delta) \geq \frac{1}{2}(p - q)$.

By (2), $\Delta \geq 4$. Hence $\Delta \geq 5/2$, so that

$$3 \geq \frac{\Delta - 1}{\Delta - 2} \; . \tag{9}$$

By (8) and (9), we have

$$3q \geq p. \tag{10}$$

Hence

$$\tfrac{1}{2}q \geq \tfrac{1}{6}p.$$

Therefore,

$$\tfrac{1}{3}p = \tfrac{1}{2}p - \tfrac{1}{6}p \geq \tfrac{1}{2}(p - q). \tag{11}$$

Since G_Δ consists of disjoint cycles, and since any cycle of length k has at least $k/3$ independent edges, we have

$$\beta_1(G_\Delta) \geq \tfrac{1}{3}p. \tag{12}$$

By (11) and (12), we have

$$\beta_1(G_\Delta) \geq \tfrac{1}{2}(p - q). \tag{13}$$

Let S be a set of exactly $\frac{1}{2}(p-q)$ independent edges in G_Δ, which exists by (13). We say that a vertex is *missed* by S if it is not incident with any edge in S. There are obviously exactly q core and q non-core vertices which are missed by S. Let $W = \{w_1, w_2, \ldots, w_q\}$ and $X = \{v_1, v_2, \ldots, v_q\}$ be, respectively, the core and non-core vertices missed by S. We show, applying Hall's Theorem, that there exists a complete matching[1] from W to X. Let $\Gamma(w_i)$, for $1 \leq i \leq q$, be the set of non-core neighbours of the vertex $w_i \in W$. Notice that, by the 2-regularity of the core,

$$|\Gamma(w_i)| = \Delta - 2 \ \text{ for all } i = 1, 2, \ldots, q. \tag{14}$$

Let $A \subseteq W$ and let $\Gamma = \bigcup_{w_i \in A} \Gamma(w_i)$. By (14), we may assume, in verifying Hall's condition, that $|A| \geq \Delta - 1$. Suppose that $|A| = \Delta - 1$. Without loss of generality, assume $A = \{w_1, w_2, \ldots, w_{\Delta-1}\}$. If Hall's condition is not satisfied, then $|\Gamma| = \Delta - 2$. In that case all vertices of A are adjacent to all vertices in Γ. By (14), there are exactly $(\Delta-2)(\Delta-1) = \Delta^2 - 3\Delta + 2$ edges from A to Γ. By Lemma 1, each non-core vertex has degree $\Delta - 1$, so that there are certainly no more than $(\Delta-2)(\Delta-1)$ edges incident with vertices of Γ in G. Therefore all the edges incident with vertices of Γ in G join Γ to A. However, since $p > q$ by (4), we have $V(G_\Delta) \setminus A \neq \emptyset$.

Let $v \in V(G_\Delta) \setminus A$. Since there are $\Delta - 2$ edges joining v to X, this implies that there are at least $\Delta - 2$ vertices in $X \setminus \Gamma$, implying

$$q \geq 2(\Delta - 2),$$

which contradicts (5).

Hence we are left only with the case that $|A| = \Delta + t$, where t is a nonnegative integer. Arguing by contradiction, assume that $|\Gamma| < |A|$. Let j be the positive integer defined by the equation

$$|\Gamma| = \Delta + t - j.$$

[1] Notice that we are using the same idea used by Hilton and Zhao in [7], except that Hall's condition is applied to W instead of X.

There are exactly

$$(\Delta + t)(\Delta - 2) = \Delta^2 + (t-2)\Delta - 2t \tag{15}$$

edges joining A to Γ. But, by summing the degrees of the vertices of Γ, we see that the number of edges of G incident with the vertices of Γ cannot exceed

$$(\Delta + t - j)(\Delta - 1) = \Delta^2 + (t - 1 - j)\Delta + j - t. \tag{16}$$

Therefore it is obvious that the quantity in (15) cannot be larger than the quantity in (16), i.e. that

$$\Delta^2 + (t - 1 - j)\Delta + j - t \geq \Delta^2 + (t-2)\Delta - 2t.$$

Cancelling out and simplifying, we obtain

$$j(\Delta - 1) \leq \Delta + t. \tag{17}$$

We cannot have $\Delta - 2 \leq t$, otherwise

$$\Delta + t \geq 2\Delta - 2 \geq q + 4,$$

contradicting the fact that $\Delta + t = |A| \leq q$. Thus

$$t < \Delta - 2, \tag{18}$$

and hence

$$\Delta + t < 2\Delta - 2 = 2(\Delta - 1). \tag{19}$$

Comparing (17) and (19), and recalling that j is positive, we conclude that $j = 1$, so that

$$|\Gamma| = \Delta + t - 1. \tag{20}$$

By (16) and the fact that $j = 1$, there are no more than

$$\Delta^2 + (t-2)\Delta + 1 - t \tag{21}$$

edges incident with Γ in G. Subtracting (15) from (21), we conclude that there are at most $t+1$ edges joining Γ to $V(G_\Delta) \setminus A$. Let w^* be a vertex in $V(G_\Delta) \setminus A$, which exists because $p > q$ by (4).

The vertex w^* is adjacent to exactly $\Delta - 2$ non-core vertices, at most $t + 1$ of which are in Γ. Therefore w^* is adjacent to at least

$$\Delta - 2 - (t+1) = \Delta - t - 3 \tag{22}$$

non-core vertices, none of which is in Γ. From (20) and (22) it follows that

$$q \geq (\Delta + t - 1) + (\Delta - t - 3) = 2\Delta - 4,$$

contradicting inequality (5). Therefore Hall's condition is satisfied. By Hall's Theorem, there exists a complete matching from W to X. Adding S to the edges of this matching, we obtain the desired 1-factor F of G.

We now prove that $G - F$ satisfies all the conditions of Lemma 1. Since G is Class 2, $G - F$ is Class 2, too. Notice that, trivially,

$$(G - F)_\Delta \subseteq G_\Delta, \tag{23}$$

so that $\Delta((G-F)_\Delta) \leq 2$. Notice also that the inclusion in (23) is strict, since some edges of F (namely those in S) were specifically chosen to be in $E(G_\Delta)$.

We now prove that $G - F$ is connected. Notice that

$$V((G - F)_\Delta) = V(G_\Delta). \tag{24}$$

Recall that W is the set of core vertices missed by S. Let $v_1 \in V(G_\Delta) \setminus W$, and suppose that there exists $v_2 \in V(G_\Delta)$ such that v_2 lies in a different connected component of $G - F$ than v_1. Since $v_1 \notin W$ and by the identity (24), v_1 has exactly $(\Delta - 2)$ non-core neighbours in $G - F$. By (24) and since v_2 is in the core of $G - F$, v_2 has at least $\Delta - 3$ non-core neighbours in $G - F$. Since v_1 and v_2 are in distinct connected components of $G - F$, their corresponding sets of non-core neighbours must be disjoint. By (24), this implies that

$$(\Delta - 2) + (\Delta - 3) \leq q,$$

contradicting inequality (5). Hence all core-vertices are in the same connected component of $G - F$. By Lemma 1, G is critical. By Lemma 4, every vertex of G is joined by at least two edges to G_Δ, and since at most one of these edges is in F, we conclude that every vertex in $G - F$ is joined by at least one edge to $(G - F)_\Delta$. This, added to the fact that all vertices in the core of $G - F$ are in the same connected component of $G - F$, proves that $G - F$ is connected. Thus $G - F$ satisfies all the hypotheses of Lemma 1. By Lemma 1, the core of $G - F$ is 2-regular. But this contradicts the fact that the inclusion (23) is strict, as observed above, which excludes the core of $G - F$ from being 2-regular. Hence we have a contradiction, and this contradiction proves that G is Class 1. $\square$

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