A graph-theoretical generalization of Berge's analogue of the Erdős-Ko-Rado theorem.

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Abstract. A family $\mathcal{A}$ of r-subsets of the vertex set $V(G)$ of a graph G is *intersecting* if any two of the r-subsets have a non-empty intersection. The graph G is *r-EKR* if a largest intersecting family $\mathcal{A}$ of independent r-subsets of $V(G)$ may be obtained by taking all independent r-subsets containing some particular vertex.

In this paper, we show that if G consists of one path P raised to the power $k_0 \geq 1$, and s cycles ${}_1C, {}_2C, \ldots, {}_sC$ raised to the powers $k_1, k_2, \ldots, k_s$ respectively, with

$$\min\left(\omega({}_1C^{k_1}), \omega({}_2C^{k_2}), \ldots, \omega({}_sC^{k_s})\right) \geq \omega(P^{k_0}) \geq 2$$

where $\omega(H)$ denotes the clique number of H, and if G has an independent r-set (so r is not too large), then G is r-EKR. An intersecting family of the largest possible size may be found by taking all independent r-subsets of $V(G)$ containing one of the end-vertices of the path.

1. Introduction

We first discuss the Erdős-Ko-Rado theorem, Berge's analogue of it, and a recent further analogue due to Talbot. Then we show that all three can be presented in a unified way as being a property of some relevant graph. Then we give a much more general analogue, extending Berge's result.

1.1. The Erdős-Ko-Rado theorem

The Erdős-Ko-Rado (EKR) theorem [6] of 1961 states that if $\mathcal{A}$ is a family of r-subsets of $\{1, 2, \ldots, n\}$ with $r \leq n/2$ such that $\mathcal{A}$ is *intersecting* (that is $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cap A_2 \neq \emptyset$), then $|\mathcal{A}| \leq \binom{n-1}{r-1}$. From the Hilton-Milner theorem [9] it follows that, except if $n = r/2$, the only way of obtaining the equality $|\mathcal{A}| = \binom{n-1}{r-1}$ is by taking all r-sets containing a common element (but, as Claude Berge observed to the first author, this fact can also be determined by a close examination of the original proof of the EKR theorem).

1.2. Berge's analogue of the EKR theorem

Let $X_1, X_2, \ldots, X_s$ be finite sets with $|X_i| = k_i$ $(1 \leq i \leq s)$ and $2 \leq k_1 \leq k_2 \leq \ldots \leq k_s$. In 1972 Berge considered the hypergraph, say H_0, with vertex set $X_1 \cup X_2 \cup \ldots \cup X_s$ and (hyper)edge set all $k_1, k_2, \ldots, k_s$ subsets $\{x_1, x_2, \ldots, x_s\}$ with $x_i \in X_i$ $(1 \leq i \leq s)$. The *chromatic index* $q(H)$ of a hypergraph H is the smallest number of colours needed to colour the edges of H so that no two edges with a vertex in common have the same colour. Berge [1] showed that

$$q(H_0) = k_2 k_3 \cdots k_s.$$

A corollary of this is the analogue of the EKR theorem mentioned in the title of this paper. This is that the greatest number of pairwise intersecting hyperedges in H is the same number, namely $k_2 k_3 \cdots k_s$; this number is clearly the greatest number of hyperedges containing a common vertex, i.e. the maximum degree in H. This corollary can be expressed in terms of integer sequences (e.g. [4], [5], [7], [8], [12], [14]) and in other formulations as well (e.g. [2], [11], [15]) and is a special case of Theorem 1.3.

1.3. Talbot's analogue of the EKR theorem

Very recently, in 2003, Talbot [15], investigating a problem of Holroyd [10], produced a further analogue of the EKR theorem. Considering the numbers $1, 2, \ldots, n$ in cyclic fashion, so that i and $i+1$ are adjacent $(1 \leq i \leq n-1)$ and n and 1 are adjacent, Talbot treated r-subsets of $\{1, 2, \ldots, n\}$ which are *separated*, that is no adjacent pair is in any separated r-subset. Talbot showed that if $\mathcal{A}$ is an intersecting family of seperated r-subsets of $\{1, 2, \ldots, n\}$ then $|\mathcal{A}| \leq \binom{n-r-1}{r-1}$. He also characterized the families $\mathcal{A}$ for which there is equality here. Talbot's achievement in finding a proof of this was quite notable, as there seems to be no easy way of tackling this problem on the lines of the original proof [6], Katona's proof [13] or Daykin's proof [3], the three main proofs of the EKR theorem; Talbot's proof is more similar to the original proof than to the other two.

1.4. A unified viewpoint: r-EKR graphs

The EKR theorem, the corollary to Berge's theorem and Talbot's theorem can all be expressed in a very similar way in terms of graph theory. Let G be a given graph with n vertices and consider the *independent* (or stable) r-subsets of the vertex set $V(G)$ of G, that is, the r-subsets with no edge of G joining any pair of vertices. We look for an intersecting family of independent r-subsets. For the original EKR-theorem, we can take G to be the graph with $n \geq 2r$ vertices and no edges. For the corollary to Berge's theorem, we can take G to be the graph with r components, each a complete graph, the ith having order k_i. For Talbot's theorem we can take G to be an n-cycle.

We call a family $\mathcal{A}$ of independent r-subsets of $V(G)$, all containing the same vertex, say w, an *r-star*; the vertex w is called the *star centre*. We call a

FIGURE 1

graph G *r-EKR* if some largest intersecting family of independent r-subsets of $V(G)$ is an r-star. We call G *strictly r-EKR* if every largest intersecting family of independent r-sets is an r-star. The EKR theorem, the corollary to Berge's theorem, and Talbot's theorem may all be expressed by saying that the relevant graph is r-EKR.

We mention that Talbot actually proved more, namely that the k-th power of an n-cycle is r-EKR if $k \geq 1$, $r \geq 1$ and $n \geq r(k+1)$. He also showed exactly when C_n^k is strictly r-EKR.

Not all graphs need be r-EKR. For a much fuller discussion of this, see Holroyd and Talbot [12]. A simple example of a graph which is not r-EKR is provided, paradoxically, by the graph with n vertices and no edges when $n < 2r$. Then every r-set intersects every other r-set. Another simple example is provided by the graph G in Figure 1. This graph is not 3-EKR. A largest 3-star that can be obtained is clearly $\{acd, ace, acf, adf\}$, which has four members. Yet a largest intersecting family of independent 3-sets is $\{acd, ace, acf, adf, cdf\}$, which has five members.

We draw attention to the following interesting conjecture of Holroyd and Talbot. Let,

$$\mu(G) = \min\{|I| : I \text{ is a maximal independent subset of } V(G)\}.$$

Conjecture 1.1. *If $1 \leq r \leq \mu/2$, then G is r-EKR.*

1.5. Further extensions of Berge's analogue

Our main result, Theorem 1.3, generalizes Berge's theorem as well as a number of generalizations of Berge's theorem due to Gronau [8], Meyer [14], Deza and Frankl [4], Bollobás and Leader [2], culminating in the following theorem of Holroyd, Spencer and Talbot [11].

Theorem 1.2. *Let $t \geq r \geq 1$ and let G be a graph with t components, each being a complete graph of order at least two (the complete graphs not necessarily being of the same order). Then G is r-EKR, and a largest star may be found by taking the star centre to be a vertex in a complete graph of smallest order.*

The requirement in Theorem 1.2 that the components have order at least two is essential (apart from the fact (not observed by Holroyd, Spencer and Talbot) that we can permit one complete graph to be an isolated vertex.) In the extreme case, when all the components are isolated vertices, we are in the situation described in the EKR-theorem, where we needed the extra requirement that $r \leq t/2$ for G to be r-EKR.

1.6. Our further extension of Berge's Theorem

We let $\omega(G)$ be the clique number of a graph G, that is the largest order of a complete subgraph of G. Note that the formulae for the clique numbers of P_n^k and C_n^k, where P_n and C_n are the path and cycle respectively with n vertices, are

$$\omega(P_n^k) = \begin{cases} k+1 & \text{if } n \geq k+1, \\ n & \text{if } n \leq k, \end{cases}$$

and

$$\omega(C_n^k) = \begin{cases} k+1 & \text{if } n \geq 2k+2, \\ n & \text{if } n \leq 2k+1. \end{cases}$$

Our main result concerns a graph G consisting of cycles ${}_1C, {}_2C, \ldots, {}_sC$, raised to the powers $k_1, k_2, \ldots, k_s$ respectively and a path P raised to the power k_0. We let $c_i = |V({}_iC)|$ and $p = |V(P)|$ and we let

$$\kappa_i = \begin{cases} \lfloor c_i/(k_i+1) \rfloor & \text{if } c_i \geq k_i+1, \\ 1 & \text{if } 2 \leq c_i \leq k_i+1. \end{cases}$$

We shall denote this graph G by $G(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0})$. Our main result is:

Theorem 1.3. *Let $s \geq 0, p \geq 1$ and $c_i \geq 2$ $(1 \leq i \leq s)$. Let*

$$\min\left(\omega({}_1C^{k_1}), \omega({}_2C^{k_2}), \ldots, \omega({}_sC^{k_s})\right) \geq \max(\omega(P^{k_0}), 2), \tag{1}$$

and let

$$1 \leq r \leq \left(\sum_{i=1}^{s} \kappa_i\right) + \left\lceil \frac{p}{k_0+1} \right\rceil.$$

Then $G(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0})$ is r-EKR. An r-star of maximum size may be obtained by taking all independent r-subsets of $V(G)$ containing one of the end vertices of the path P.

It is not hard to verify that Condition (1) is equivalent to the following Condition (2).

$$\min\left(\min_{1 \leq i \leq s} (k_i+1, c_i)\right) \geq \max\left(\min\left(k_0+1, p\right), 2\right). \tag{2}$$

Thus we have:

Lemma 1.4. *Conditions (1) and (2) are equivalent.*

In Theorem 1.3, we include K_2's as cycles (degenerate cycles!), so that the equation $\omega({}_iC^{k_i}) = 2$ is permitted for any value of i, $1 \leq i \leq s$. The theorem remains true in this case, and the proof is considerably simplified. The theorem becomes untrue if we go further and include K_1's as (degenerate) cycles as well.

Graphs G consisting of powers of one path and several cycles may well be r-EKR even if $\omega(P^{k_0}) > \min_{1 \leq i \leq s} \omega({}_iC^{k_i})$, but it is not clear to the authors where the star centre of a largest star might be.

The curious term $\max(\omega(P^{k_0}), 2)$ in Theorem 1.3 is there simply to take account of the fact that the cycles ${}_iC$ $(1 \le i \le s)$ all have to have length at least two, whereas the path can just have length 1.

Our proof of Theorem 1.3 is inspired by Talbot's clever proof. It takes Theorem 1.2 as its starting point.

1.7. Notation

Given the path P and the cycles ${}_1C, {}_2C, \ldots, {}_sC$, we let $p = |V(P)|$, $c_i = |V({}_iC)|$, $\pi = c_1 + c_2 + \cdots + c_s$ and $n = p + c_1 + c_2 + \cdots + c_s (= p + \pi)$. We shall suppose that the vertices of ${}_iC$ are $c_1 + c_2 + \cdots + c_{i-1} + 1, \ldots, c_1 + c_2 + \cdots c_i$ and that the vertices $c_1 + c_2 + \cdots + c_{i-1} + j - 1$ and $c_1 + c_2 + \cdots + c_{i-1} + j$ are adjacent in ${}_iC$ $(1 \le i \le s, 2 \le j \le c_i)$ and that $c_1 + c_2 + \cdots + c_{i-1} + 1$ and $c_1 + c_2 + \cdots + c_i$ are adjacent in ${}_iC$. We shall suppose that the vertices of P are $\pi + 1, \pi + 2, \ldots, \pi + p (= n)$. The graph G described in Theorem 1.3 has cycles ${}_1C, {}_2C, \ldots {}_sC$ raised to the powers $k_1, k_2, \ldots, k_s$ respectively, and a path P raised to the power k_0; we shall suppose that G has vertex set $\{1, 2, \ldots, n\}$, and shall denote G by $G(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0})$.

We let $\mathcal{I}^{(r)}$ or $\mathcal{I}^{(r)}(G)$ denote the set of all independent r-sets of G, and let $\mathcal{I}_a^{(r)}$ or $\mathcal{I}_a^{(r)}(G)$ denote the set of all independent r-sets of G containing some vertex $a \in V(G)$.

We shall use the letter a for an end vertex of the path P.

2. Proof of Theorem 1.3

The proof proceeds through a number of lemmas and sublemmas. Throughout $\mathcal{A}$ will be an intersecting family of independent r-subsets of

$$V(G(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0})).$$

Lemma 2.1. *Theorem* 1.3 *is true for any graph which is the union of the k_0th power of a path and s cycles, where the ith cycle is raised to the power k_i, if*

1. *the length of the path is at least 2 and at most $k_0 + 1$,*
2. *for $1 \le i \le s$, the length of the ith cycle is at least 2 and at most $2k_i + 1$, and*
3. *the clique number of the power of the path is not more than the smallest clique number of the powers of the cycles.*

Proof. In this case, the power of the path and the various powers of the cycles are cliques, and then Theorem 1.3 reduces to Theorem 1.2. □

Lemma 2.2. *Theorem* 1.3 *is true if $p = k_0 + 1$, $r = \sum_{i=1}^{r} \kappa_i + \lceil p/(k_0 + 1) \rceil$, and condition* (1) *is satisfied.*

Before proving Lemma 2.2, let us introduce another piece of terminology. Consider a bijection $\mu : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ given by:

$$\begin{aligned}
\mu(c_1 + c_2 + \cdots + c_{i-1} + j) &= c_1 + c_2 + \cdots + c_{i-1} + j + 1 \quad (1 \le j < c_i,\ 1 \le i \le s),\\
\mu(c_1 + c_2 + \cdots + c_i) &= c_1 + c_2 + \cdots + c_{i-1} + 1 \quad (1 \le i \le s),\\
\mu(\pi + j) &= \pi + j + 1 \quad (1 \le j < p),\\
\mu(n) &= \pi + 1.
\end{aligned}$$

We call μ a *clockwise rotation.*

Proof of Lemma 2.2 Since $p = k_0 + 1$, $\lceil p/(k_0 + 1) \rceil = 1$. Let $r = \sum_{i=1}^{s} \kappa_i + 1$. Then r is the largest possible cardinality that an independent set can have; moreover any independent r-set must contain exactly one vertex of P.

By condition (2), $c_i \ge p = k_0 + 1$ $(1 \le i \le s)$, so for any independent r-set A, the intersecting family $\mathcal{A}$ will contain at most one of $A, \mu(A), \mu^2(A), \ldots, \mu^{k_0}(A)$. Therefore $|\mathcal{A}| \le |\mathcal{I}^{(r)}|/(k_0+1)$. But since $\mathcal{I}^{(r)} = \mathcal{I}^{(r)}_{\pi+1} \cup \cdots \cup \mathcal{I}^{(r)}_{\pi+p}$ and $\mathcal{I}^{(r)}_k \cap \mathcal{I}^{(r)}_l = \emptyset$ $(\pi + 1 \le k < l \le \pi + p)$, it follows that $|\mathcal{I}^{(r)}| = p|\mathcal{I}^{(r)}_{\pi+1}| = (k_0 + 1)|\mathcal{I}^{(r)}_{\pi+1}|$, so that $|\mathcal{A}| \le |\mathcal{I}^{(r)}_{\pi+1}|$, which proves Lemma 2.2. □

Lemma 2.3. *Theorem* 1.3 *is true if* $|V(P)| = k_0 + 1$, $1 \le r \le \sum_{i=1}^{s} \kappa_i + 1$, *and condition* (1) *is satisfied.*

Proof. In view of Lemma 2.1, we may assume that ${}_iC^{k_i}$ is not a complete graph for at least one i, $1 \le i \le s$. Without loss of generality, assume that $c_1 > \max(3, \omega(P^{k_0})) = \max(3, k_0 + 1)$. In particular, this implies that $c_1 \ge 4$ if $k_1 = 1$ and $c_1 \ge 2k_1 + 2$ if $k_1 \ge 2$ (since ${}_1C^{k_1}$ is a complete graph if $c_1 \le 2k_1 + 1$). It also implies that $C^{k_1}_{c_1}$ contains a K_p. Notice that $C^{k_1}_{c_1-1}$ and $C^{k_1}_{c_1-k_1-1}$ also contain a K_p; this is obvious if neither of these is a complete graph, but if, for example, $c_1 = 2k_1 + 2$, then $K^{k_1}_{c_1-k_1-1}$ is a $K_{k_1+1} \supset K_{k_0+1} = K_p$.

We use induction on c_1 and, in particular, we shall assume that Lemma 2.3 is true for $c_1 - 1$ and $c_1 - 2$. Lemma 2.1 provides the base step for our induction hypothesis. In view of Lemma 2.2, we may assume that $r < \sum_{i=1}^{s} \kappa_i + 1$.

Define the function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n - 1\}$ by

$$f(j) = \begin{cases} 1 & \text{if } j = 1,\\ j - 1 & \text{if } 2 \le j \le n. \end{cases}$$

We shall need the following very easy sublemmas:

Sublemma 2.3.1. *If* $\mathcal{G}$ *is an intersecting family, then so is* $f(\mathcal{G})$.

Sublemma 2.3.2. *If* A *and* B *are independent* r*-subsets of* $G(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0})$ *and* $A \ne B$, *then, for* $1 \le j \le k_s$, $f^j(A) = f^j(B) \Rightarrow A \,\Delta\, B = \{c, d\}$ *for some* c, d *with* $1 \le c < d \le j + 1$.

Consider the following partition of our intersecting family $\mathcal{A}$ of independent r-subsets of G:

$$\mathcal{A} = \mathcal{B} \cup \mathcal{C} \cup \left(\bigcup_{i=0}^{k_1} \mathcal{D}_i \right),$$

where

$$\mathcal{B} = \left\{ A \in \mathcal{A} \,:\, 1 \notin A \text{ and } f(A) \in \mathcal{I}^{(r)} \left(G \left((c_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0} \right) \right) \right\},$$

$$\mathcal{C} = \left\{ A \in \mathcal{A} \,:\, 1 \in A \text{ and } f(A) \in \mathcal{I}^{(r)} \left(G \left((c_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0} \right) \right) \right\},$$

$$\mathcal{D}_0 = \{ A \in \mathcal{A} \,:\, 1, k_1+2 \in A \}$$

$$\mathcal{D}_i = \{ A \in \mathcal{A} \,:\, c_1+1-i, k_1+2-i \in \mathcal{B} \} \quad (1 \le i \le k_1).$$

Since $f(\mathcal{B}) \cup f(\mathcal{C}) = f(\mathcal{B} \cup \mathcal{C})$, and since, by Sublemma 2.3.1, $f(\mathcal{B} \cup \mathcal{C})$ is an intersecting family of independent r-subsets of $\mathcal{I}^{(r)} \left(G \left((c_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0} \right) \right)$, it follows by induction that

$$|f(\mathcal{B}) \cup f(\mathcal{C})| \le \left| \mathcal{I}_a^{(r)} \left(G \left((c_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0} \right) \right) \right|. \tag{3}$$

It follows from Sublemma 2.3.2 with $j=1$ that $|f(\mathcal{B})| = |\mathcal{B}|$ (as no set in $\mathcal{B}$ contains 1) and $|f(\mathcal{C})| = |\mathcal{C}|$ (as no set in $\mathcal{C}$ contains 2). Therefore $|\mathcal{B}| + |\mathcal{C}| = |f(\mathcal{B})| + |f(\mathcal{C})|$. Consequently

$$|\mathcal{B}| + |\mathcal{C}| = |f(\mathcal{B}) \cup f(\mathcal{C})| + |f(\mathcal{B}) \cap f(\mathcal{C})|.$$

Let

$$\mathcal{E} = f(\mathcal{B}) \cap f(\mathcal{C}).$$

Then, by (3),

$$|\mathcal{B}| + |\mathcal{C}| \le |\mathcal{E}| + \left| \mathcal{I}_a^{(r)} \left(G \left((c_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0} \right) \right) \right|. \tag{4}$$

For any family $\mathcal{G}$ of sets, let $\mathcal{G} - \{1\} = \{ G \backslash \{1\} \,:\, G \in \mathcal{G} \}$. Define,

$$\mathcal{F} = \left(f^{k_1-1}(\mathcal{E} - \{1\}) \right) \cup \left(\bigcup_{i=0}^{k_1} \left(f^{k_1}(\mathcal{D}_i) - \{1\} \right) \right).$$

Note that if $E \in \mathcal{E}$ then $E = f(C)$ for some $C \in \mathcal{C}$, so $1 \in E$. Also note that if $D \in \mathcal{D}_i$ for some i, $0 \le i \le k_1$, then $1 \in f^{k_1}(D)$. Therefore $\mathcal{F}$ is a family of $(r-1)$-subsets. The family $\mathcal{F}$ has many further properties given in the following sublemmas.

Sublemma 2.3.3.

1. *$\mathcal{F}$ is a family of independent $(r-1)$-subsets of*

$$V\Big(G \left((c_1-k_1)^{k_1}, c_2^{k_2}, c_3^{k_3}, \ldots, c_s^{k_s}, p^{k_0} \right) \Big).$$

2. $f^{k_1}(\mathcal{D}_0 - \{1\}), f^{k_1}(\mathcal{D}_1 - \{1\}), \ldots, f^{k_1}(\mathcal{D}_k - \{1\})$ *and* $f^{k_1-1}(\mathcal{E} - \{1\})$ *are pairwise disjoint families of sets.*
3. *$\mathcal{F}$ is intersecting.*

4. $f(\mathcal{F})$ *is a family of independent* $(r-1)$*-subsets of*

$$V\Big(G\left((c_1-k_1-1)^{k_1}, c_2^{k_2}, c_3^{k_3}, \dots, c_s^{k_s}, p^{k_0}\right)\Big).$$

Sublemma 2.3.4. *With* $p = k_0+1$*,*

$$\left|\mathcal{I}_a^{(r)}\left(G(c_1^{k_1}, c_2^{k_2}, \dots, c_s^{k_s}, p^{k_0})\right)\right| = \left|\mathcal{I}_a^{(r)}\left(G\left((c_1-1)^{k_1}, c_2^{k_2}, \dots, c_s^{k_s}, p^{k_0}\right)\right)\right| + \left|\mathcal{I}_a^{(r-1)}\left(G\left((c_1-k_1-1)^{k_1}, c_2^{k_2}, \dots, c_s^{k_s}, p^{k_0}\right)\right)\right|$$

By Sublemma 2.3.2, f^{k_1} acts as an injective mapping on $\mathcal{D}_i$ $(0 \le i \le k_1)$, so

$$|\mathcal{D}_i| = |f^{k_1}(\mathcal{D}_i)| \quad (0 \le i \le k_1). \tag{5}$$

By Sublemma 2.3.2 again, f^{k_1} also acts as an injective mapping on $\mathcal{C}$, so f^{k_1-1} acts injectively on $\mathcal{E}$, and so

$$|\mathcal{E}| = |f^{k_1-1}(\mathcal{E})|. \tag{6}$$

By Sublemma 2.3.3(2) it follows that $|\mathcal{F}| = |f^{k_1-1}(\mathcal{E})| + \sum_{i=0}^{k_1} |f^{k_1}(\mathcal{D}_i)|$. Therefore, by (5) and (6),

$$|\mathcal{F}| = |\mathcal{E}| + \sum_{i=0}^{k_1} |\mathcal{D}_i|. \tag{7}$$

As no set in $\mathcal{F}$ contains the vertex 1, the map $f : \mathcal{F} \to f(\mathcal{F})$ is bijective, so

$$|\mathcal{F}| = |f(\mathcal{F})|. \tag{8}$$

By Sublemma 2.3.3(3) $\mathcal{F}$ is intersecting, so by Sublemma 2.3.2, $f(\mathcal{F})$ is also intersecting. By Sublemma 2.3.3(4), $f(\mathcal{F})$ is a family of independent $(r-1)$-subsets of $V\Big(G\left((c_1-k_1-1)^{k_1}, c_2^{k_2}, \dots, c_s^{k_s}, p^{k_0}\right)\Big)$. Therefore, by induction,

$$|f(\mathcal{F})| \le \left|\mathcal{I}_a^{(r-1)}\left(G\left((c_1-k_1-1)^{k_1}, c_2^{k_2}, \dots, c_s^{k_s}, p^{k_0}\right)\right)\right|. \tag{9}$$

Therefore, using (3), (7), (8) and (9),

$$\begin{aligned}
|\mathcal{A}| &= |\mathcal{B}| + |\mathcal{C}| + \sum_{i=0}^{k_1} |\mathcal{D}_i| \\
&= |f(\mathcal{B}) \cup f(\mathcal{C})| + |\mathcal{E}| + \sum_{i=0}^{k_1} |\mathcal{D}_i| \\
&= |f(\mathcal{B}) \cup f(\mathcal{C})| + |\mathcal{F}| \\
&= |f(\mathcal{B}) \cup f(\mathcal{C})| + |f(\mathcal{F})| \\
&\le \left|\mathcal{I}_a^{(r)}\left(G\left((c_1-1)^{k_1}, c_2^{k_2}, \dots, c_s^{k_s}, p^{k_0}\right)\right)\right| \\
&\quad + \left|\mathcal{I}_a^{(r-1)}\left(G\left((c_1-k_1-1)^{k_1}, c_2^{k_2}, \dots, c_s^{k_s}, p^{k_0}\right)\right)\right|.
\end{aligned}$$

Therefore, by Sublemma 2.3.4,

$$|\mathcal{A}| \le \left| \mathcal{I}_a^{(r)} \left(G \left((c_1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0} \right) \right) \right| .$$

Lemma 2.3 now follows by induction on c_1. □

Lemma 2.4. *Theorem* 1.3 *is true if* $1 \le |V(P)| \le k_0 + 1$, $1 \le r \le \sum_{i=1}^{s} \kappa_i + \lceil p/(k_0+1) \rceil$, *and Condition* 1 *is satisfied.*

Proof. From Lemma 2.3 we know that $G(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0})$ is r-EKR if $|V(P)| = k_0 + 1$, $1 \le r \le (\sum_{i=1}^{s} \kappa_i) + 1$ and Condition (1) is satisfied, and that an r-star of maximum size can be found by taking all independent r-sets containing an end-point a of P (in fact, since P^{k_0} is a complete graph, any vertex of P could be the centre of a suitable r-star).

If several vertices of P^{k_0} are removed leaving at least one vertex, say w, the number of independent r-sets centred on w remains unaltered. Lemma 2.4 follows. □

The rest of the proof of Theorem 1.3 bears a close resemblance to the proof of Lemma 2.3, and is similarly modelled on Talbot's proof of his separated sets result in [15]. We still need to show that, with the cycle powers fixed, we can "grow" the length of the path, P, to the required value p.

We argue by induction on p. The basis for the induction is provided by Lemma 2.4 which established Theorem 1.3 whenever $1 \le r \le (\sum_{i=1}^{s} \kappa_i) + 1$, Condition (1) is satisfied, and $1 \le |V(P)| \le k_0 + 1$. Recall that the vertices of P are labelled $\pi + 1, \pi + 2, \ldots, \pi + p$.

Consider the following partition of our intersecting family of $\mathcal{A}$ independent r-sets:

$$\mathcal{A} = \mathcal{Q} \cup \mathcal{R} \cup \mathcal{S}_0,$$

where

$$\mathcal{Q} = \left\{ A \in \mathcal{A} : \pi + 1 \notin A \text{ and } g(A) \in \mathcal{I}_a^{(r)} \left(G(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, (p-1)^{k_0}) \right) \right\},$$

$$\mathcal{R} = \left\{ A \in \mathcal{A} : \pi + 1 \in A \text{ and } g(A) \in \mathcal{I}_a^{(r)} \left(G(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, (p-1)^{k_0}) \right) \right\},$$

$$\mathcal{S}_0 = \{ A \in \mathcal{A} : \pi + 1, \pi + k_0 + 2 \in A \}.$$

Define the function $g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n-1\}$ by

$$g(j) = \begin{cases} j & \text{if } 1 \le j \le \pi + 1, \\ j - 1 & \text{if } \pi + 2 \le j \le \pi + p. \end{cases}$$

The analogues of Sublemmas 2.3.1 and 2.3.2 are:

Sublemma 2.4.1. *If* $\mathcal{G}$ *is an intersecting family, then so is* $g(\mathcal{G})$.

Sublemma 2.4.2. *If $\mathcal{A}$ and $\mathcal{B}$ are independent r-subsets of $G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)$ and $A \neq B$, then, for $1 \leq j \leq k_0$,*

$$g^j(A) = g^j(B) \Rightarrow A \,\Delta\, B = \{c, d\}$$

for some c, d with $\pi + 1 \leq c < d \leq \pi + 1 + j$.

Since $g(\mathcal{Q}) \cup g(\mathcal{R}) = g(\mathcal{Q} \cup \mathcal{R})$ and since, by Sublemma 2.4.1, $g(\mathcal{Q} \cup \mathcal{R})$ is an intersecting family of independent r-subsets of $\mathcal{I}_a^{(r)}\left(G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, (p-1)^{k_0}\right)\right)$, it follows by induction that

$$|g(\mathcal{Q}) \cup g(\mathcal{R})| \leq \left|\mathcal{I}_a^{(r)}\left(G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, (p-1)^{k_0}\right)\right)\right|. \tag{10}$$

It follows by Sublemma 2.4.2 with $j = 1$ that $|g(\mathcal{Q})| = |\mathcal{Q}|$ (as no set in $\mathcal{Q}$ contains $\pi + 1$) and that $|g(\mathcal{R})| = |\mathcal{R}|$ (as no set in $\mathcal{R}$ contains $\pi + 2$). Therefore $|\mathcal{Q}| + |\mathcal{R}| = |g(\mathcal{Q})| + |g(\mathcal{R})|$. Consequently

$$|\mathcal{Q}| + |\mathcal{R}| = |g(\mathcal{Q}) \cup g(\mathcal{R})| + |g(\mathcal{Q}) \cap g(\mathcal{R})|.$$

Let $\mathcal{T} = g(\mathcal{Q}) \cap g(\mathcal{R})$. Then, by (10),

$$|\mathcal{Q}| + |\mathcal{R}| \leq |\mathcal{T}| + \left|\mathcal{I}_a^{(r)}\left(G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, (p-1)^{k_0}\right)\right)\right|. \tag{11}$$

Define,

$$\mathcal{U} = \left(g^{k_0 - 1}\left(\mathcal{T} - \{\pi + 1\}\right)\right) \cup \left(g^{k_0}\left(\mathcal{S}_0\right) - \{\pi + 1\}\right).$$

Note that if $T \in \mathcal{T}$ then $T = g(R)$ for some $R \in \mathcal{R}$, so $\pi + 1 \in T$. Also note that if $S \in \mathcal{S}_0$ then $\pi + 1 \in g^{k_0}(S)$. Therefore $\mathcal{U}$ is a family of $(r-1)$-subsets. The family $\mathcal{U}$ has the following further properties:

Sublemma 2.4.3.

1. *$\mathcal{U}$ is a family of independent $(r-1)$-subsets of*
$$V\left(G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, (p-1)^{k_0}\right)\right),$$
2. *$g^{k_0}\left(\mathcal{S}_0 - \{\pi + 1\}\right)$ and $g^{k_0 - 1}\left(\mathcal{T} - \{\pi + 1\}\right)$ are disjoint families of sets,*
3. *$\mathcal{U}$ is intersecting,*
4. *$g(\mathcal{U})$ is a family of intersecting $(r-1)$-subsets of*
$$V\left(G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, (p - k_0 - 1)^{k_0}\right)\right).$$

By Sublemma 2.4.2, g^{k_0} acts injectively on $\mathcal{S}_0$, so

$$|\mathcal{S}_0| = \left|g^{k_0}\left(\mathcal{S}_0\right)\right|. \tag{12}$$

Again, by Sublemma 2.4.2, g^{k_0} acts injectively on $\mathcal{R}$, and so $g^{k_0 - 1}$ acts injectively on $\mathcal{T}$, and so

$$|\mathcal{T}| = \left|g^{k_0 - 1}\left(\mathcal{T}\right)\right| \tag{13}$$

By Sublemma 2.4.3(2) it follows that

$$|\mathcal{U}| = \left|g^{k_0 - 1}\left(\mathcal{T}\right)\right| + \left|g^{k_0}(\mathcal{S}_0)\right|.$$

Therefore, by (12) and (13),

$$|\mathcal{U}| = |\mathcal{T}| + |\mathcal{S}_0| . \tag{14}$$

As no set in $\mathcal{U}$ contains 1, the map $g : \mathcal{U} \to g(\mathcal{U})$ is injective, so

$$|\mathcal{U}| = |g(\mathcal{U})|. \tag{15}$$

By Sublemma 2.4.3(3), $\mathcal{U}$ is intersecting, so by Sublemma 2.4.1, $g(\mathcal{U})$ is also intersecting. By Sublemma 2.4.3(4), $g(\mathcal{U})$ is a family of independent $(r-1)$-subsets of $V\Big(G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, (p-k_0-1)^{k_0}\right)\Big)$. Therefore, by induction,

$$|g(\mathcal{U})| \le \left|\mathcal{I}_a^{(r-1)}\left(G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, (p-k_0-1)^{k_0}\right)\right)\right|. \tag{16}$$

Therefore, using (10),(14),(15) and (16),

$$\begin{aligned}
|\mathcal{A}| &= |\mathcal{Q}| + |\mathcal{R}| + |\mathcal{S}_0| \\
&= |g(\mathcal{Q}) \cup g(\mathcal{R})| + |\mathcal{T}| + |\mathcal{S}_0| \\
&= |g(\mathcal{Q}) \cup g(\mathcal{R})| + |\mathcal{U}| \\
&= |g(\mathcal{Q}) \cup g(\mathcal{R})| + |g(\mathcal{U})| \\
&\le \left|\mathcal{I}_a^{(r)}\left(G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, (p-1)^{k_0}\right)\right)\right| \\
&\quad + \left|\mathcal{I}_a^{(r-1)}\left(G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, (p-k_0-1)^{k_0}\right)\right)\right|.
\end{aligned}$$

We now need the following sublemma.

Sublemma 2.4.4. *If $p \ge k_0 + 2$ then*

$$\begin{aligned}
\left|\mathcal{I}_a^{(r)}\left(G(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0})\right)\right| &= \left|\mathcal{I}_a^{(r)}\left(G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, (p-1)^{k_0}\right)\right)\right| \\
&\quad + \left|\mathcal{I}_a^{(r-1)}\left(G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, (p-k_0-1)^{k_0}\right)\right)\right|.
\end{aligned}$$

Using this, it now follows that

$$|\mathcal{A}| \le \left|\mathcal{I}_a^{(r)}\left(G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right)\right|.$$

Thus $\mathcal{G}$ is r-EKR. Theorem 1.3 now follows by induction on p.

3. Proofs of the lemmas

In this section we prove those lemmas used in the proof of Theorem 1.3 which still await a proof. We only give a proof of Sublemmas 2.3.1, 2.3.2, 2.3.3 and 2.3.4 because, for $1 \le x \le 4$, the proof of Sublemma 2.4.x is either virtually the same, or is a considerable simplification of the proof of Sublemma 2.3.x.

Proof of Sublemma 2.3.1 If $\mathcal{G}$ is intersecting family and $A, B \in f(\mathcal{G})$, then there exist $C, D \in \mathcal{G}$ such that $A = f(C)$ and $B = f(D)$. Then $\emptyset \neq f(C \cap D) \subseteq f(C) \cap f(D) = A \cap B$. Thus $f(\mathcal{G})$ is intersecting. □

Proof of Sublemma 2.3.2 Let $A, B \in \mathcal{I}_a^{(r)}\Big(G(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0})\Big)$ with $A \neq B$ but $f^j(A) = f^j(B)$ for some j, $1 \leq j \leq k_1$. If $2 \leq a \leq c_1 - j$ then $a \in f^j(A) \Leftrightarrow a + j \in A$. Hence

$$A \cap \{j+2, j+3, \ldots, c_1\} = B \cap \{j+2, j+3, \ldots, c_1\}.$$

So as $f^j(A) = f^j(B)$ but $A \neq B$, there exist $c, d \in \{1, 2, \ldots, j+1\}$ such that $c \in A$ and $d \in B$, say. But since $j \leq k_1$ we have that $A \cap \{1, 2, \ldots, j+1\} = \{c\}$ and $B \cap \{1, 2, \ldots, j+1\} = \{d\}$. Thus $A \,\Delta\, B = \{c, d\}$. □

Proof of Sublemma 2.3.3*(*1*)* We have already that

$$\mathcal{F} = \left(f^{k_1-1}\left(\mathcal{E} - \{1\}\right)\right) \cup \left(\bigcup_{i=0}^{k_1}\left(f^{k_i}\left(\mathcal{D}_i\right) - \{1\}\right)\right)$$

is a family of $(r-1)$-sets. We have to show that the sets are independent. There are three cases.

First consider the sets in $f^{k_1-1}(\mathcal{E} - \{1\})$. Let $H \in f^{k_1-1}(\mathcal{E})$. Then there exists $E \in \mathcal{E}$ such that $f^{k_1-1}(E) = H$, and, as $\mathcal{E} = f(\mathcal{B}) \cap f(\mathcal{C})$, there also exists $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $f(B) = f(C) = E$. By Sublemma 2.3.2 with $j = 1$, we know that one of the sets B, C contains 1 and the other 2, and, by the definition of $\mathcal{B}$ and $\mathcal{C}$, we have that $1 \in C$, so $1 \in H$ and $2 \in B$. Moreover, $C \cap \{c_1 - k_1 + 1, \ldots, c_1\} = \emptyset$ so $E \cap \{c_1 - k_1, \ldots, c_1\} = \emptyset$. Therefore $H \cap \{c_1 - 2k_1 + 1, \ldots, c_1\} = \emptyset$. Since $2 \in B$ it follows that $E \cap \{2, \ldots, k_1 + 1\} = \emptyset$. It now follows that

$$H \cap (\{c_1 - 2k_1 + 1, \ldots, c_1 - k_1\} \cup \{1, 2\}) = \{1\}, \tag{17}$$

and thence that $H - \{1\} \in f^{k_1-1}(\mathcal{E} - \{1\})$ is an independent $(r-1)$-subset of

$$V\Big(H\left((c_1 - k_1^{k_1}), c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\Big).$$

Next suppose that $H \in f^{k_1}(\mathcal{D}_0)$. Then there exists $D \in \mathcal{D}_0$ such that $H = f^{k_1}(D)$, and, as $D \in \mathcal{D}_0$, $1, k_1 + 2 \in D$. Thus $1 \in H$ and

$$H \cap (\{c_1 - 2k_1 + 1, \ldots, c_1 - k_1\} \cup \{1, \ldots, k_1 + 2\}) = \{1, 2\} \tag{18}$$

and so $H - \{1\}$ is an independent $(r-1)$-subset of

$$V\Big(H\left((c_1 - k_1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\Big).$$

Finally let $H \in f^{k_1}(\mathcal{D}_i)$ for some i, $1 \leq i \leq k_1$. Then there exists a $D \in \mathcal{D}_i$ with $H = f^{k_1}(D)$, and, as $D \in \mathcal{D}_i$, $c_1 + 1 - i, k_1 + 2 - i \in D$. Since $k_1 \geq i \geq 1$, $f^{k_1}(k_1 + 2 - i) = 1$, so $1 \in H$. Therefore

$$H \cap (\{c_1 - i - 2k_1 + 1, \ldots, c_1 - k_1\} \cup \{1, \ldots, k_1 + 2 - i\}) = \{c_1 - i - k_1 + 1, 1\}. \tag{19}$$

Hence $H - \{1\}$ is an independent $(r-1)$-subset of

$$V\Big(H\left((c_1 - k_1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\Big).$$

Sublemma 2.3.3 now follows from (17), (18) and (19). □

Proof of Sublemma 2.3.3*(*2*)* To show that these families are pairwise disjoint, we consider how the members of each family intersect the set $\{c_1 - 2k_1 + 1, \dots c - k_1\} \cup \{2\}$. Let $H \in f^{k_1-1}(\mathcal{E}) - \{1\}$. From (17) we have that

$$H \cap (\{c_1 - 2k_1 + 1, \dots, c_1 - k_1\} \cup \{2\}) = \emptyset.$$

Next let $H \in f^{k_1}(\mathcal{D}_0)$. From (18) it follows that

$$H \cap (\{c_1 - 2k_1 + 1, \dots, c_1 - k_1\} \cup \{2\}) = \{2\}.$$

Finally let $H \in f^{k_1}(\mathcal{D}_i) - \{1\}$ for some i, $1 \leq i \leq k_1$. From (19) we have

$$H \cap (\{c_1 - 2k_1 + 1, \dots, c_1 - k_1\} \cup \{2\}) = \{c_1 - i - k_1 + 1\}.$$

Hence the families are pairwise disjoint. □

Proof of Sublemma 2.3.3*(*3*)* Let $A, B \in \mathcal{F}$. First suppose that $A, B \in f^{k_1}(\mathcal{D}_i) - \{1\}$ for some i, $0 \leq i \leq k_1$. Then there exist $D', D'' \in \mathcal{D}_i$ with $A = f^{k_1}(D') - \{1\}$ and $B = f^{k_1}(D'') - \{1\}$. If $i = 0$ then $k + 2 \in D' \cap D''$ and so $2 = f^{k_1}(k_1 + 2) \in A \cap B$. If $1 \leq i \leq k_1$ then $c_1 + 1 - i \in D' \cap D''$, so $c_1 + 1 - i - k_1 \in A \cap B$.

Now suppose that $A, B \in f^{k_1}(\mathcal{E}) - \{1\}$. Then there exist $E', E'' \in \mathcal{E}$ with $A = f^{k_1-1}(E') - \{1\}$ and $B = f^{k_1-1}(E'') - \{1\}$. As $E', E'' \in \mathcal{E} = f(\mathcal{B}) \cap f(\mathcal{C})$, it follows that there are $B_1 \in \mathcal{B}$ and $C_1 \in \mathcal{C}$ such that $f(B_1) = E'$ and $f(C_1) = E''$. As $B_1, C_1 \in \mathcal{A}$, we have that $B_1 \cap C_1 \neq \emptyset$. By the definitions of $\mathcal{B}$ and $\mathcal{C}$, and by Sublemma 2.3.2 with $j = 1$, we have that $1 \in C_1$ and $2 \in B_1$. It follows that, for some $j \geq k_1 + 3$, $j \in B_1 \cap C_1$. Therefore $3 \leq f^{k_1}(j) \in A \cap B$.

Next suppose that $0 \leq i < j \leq k_1$ and $A \in f^{k_1}(D_i) - \{1\}$ and $B \in f^{k_1}(D_j) - \{1\}$. In this case there exist $D' \in \mathcal{D}_i$ and $D'' \in \mathcal{D}_j$ with $A = f^{k_1}(D') - \{1\}$ and $B = f^{k_1}(D'') - \{1\}$. This implies that $D' \cap \{1, 2, \dots, k_1 + 2\} = \{k_1 + 2 - j\}$ and $D'' \cap \{1, 2, \dots, k_1 + 2\} = \{k_1 + 2 - i\}$, where $2 \leq k_1 + 2 - j < k_1 + 2 - i$. But $D', D'' \in \mathcal{A}$ which is intersecting, so $D' \cap D'' \neq \emptyset$. Therefore there is some $l \geq k_1 + 3$ with $l \in D' \cap D''$. Then $3 \leq f^{k_1}(l) \in A \cap B$.

Finally suppose that $A \in f^{k_1-1} - \{1\}$ and $B \in f^{k_1}(\mathcal{D}_i) - \{1\}$. In this case there exist $D \in \mathcal{D}_i$ and $E \in \mathcal{E}$ with $B = f^{k_1}(D) - \{1\}$ and $A = f^{k_1-1}(E) - \{1\}$, and, since $\mathcal{E} = f(\mathcal{B}) \cap f(\mathcal{C})$, there exist $B_1 \in \mathcal{B}$ and $C_1 \in \mathcal{C}$ such that $E = f(B_1) = f(C_1)$. From Sublemma 2.3.2 with $j = 1$ it follows that $1 \in C_1$ and $2 \in B_1$, so $B_1 \cap \{1, 2, \dots, k_1 + 2\} = \{2\}$ and $C_1 \cap \{1, 2, \dots, k_1 + 1\} = \{1\}$. Also, from the definition of $\mathcal{D}_i$ we have that

$$D \cap \{1, 2, \dots, k_1 + 2\} = \begin{cases} \{1, k_1 + 2\} & \text{if } i = 0, \\ \{k_1 + 2 - i\} & \text{if } 1 \leq i \leq k_1. \end{cases}$$

As B_1, C_1, D are all elements of $\mathcal{A}$, we know that $B_1 \cap D$ and $C_1 \cap D$ are both non-empty. If $i = 0$ then there exists some $j \geq 2k_1 + 3$ such that $j \in B_1 \cap D$. Otherwise we have that $1 \leq i \leq k_1$, and in this case there exists some $j_1 \geq 2k_1 + 2 - i + 1 \geq k_1 + 3$ with $j \in C_1 \cap D$. Hence $3 \leq f^{k_1}(j) \in A \cap B$. □

Proof of Sublemma 2.3.3*(*4*)* From Sublemma 2.3.3(1) we know that $\mathcal{F}$ is a family of independent $(r-1)$-subsets of $V\Big(G\left((c_1 - k_1)^{k_1}, c_2^{k_2}, \dots, c_s^{k_s}, p^{k_0}\right)\Big)$. Let $F \in \mathcal{F}$

and consider $f(\mathcal{F})$. Clearly $f(F)$ is an $(r-1)$-subset of

$$V\Big(G\left((c_1-k_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\Big).$$

We just need to check that $f(\mathcal{F})$ is an independent set. Since F was an independent $(r-1)$-subset of $V\Big(G\left((c_1-k_1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\Big)$, the only way that $f(\mathcal{F})$ could fail to be an independent $(r-1)$-subset of

$$V\Big(G\left((c_1-k_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\Big)$$

is if F contains one of the following pairs of elements:

$$(c_1-2k_1+1, 2), (c_1-2k_1+2, 3), \ldots, (c_1-k_1, k_1+1), (1, k_1+2).$$

The vertex 1 has been removed from every set in $\mathcal{F}$ so the last pair $(1, k_1+2)$ cannot be contained in $\mathcal{F}$. If $F \in f^{k_1-1}(\mathcal{E}) - \{1\}$ then by (17) (as the H there is in $f^{k-1}(\mathcal{E})$) it follows that

$$F \cap \{c_1-2k_1+1, \ldots, c_1-k_1\} = \emptyset.$$

This also follows from (18) if $F \in f^{k_1}(\mathcal{D}_0) - \{1\}$ (as the H in (19) is in $f^{k_1}(\mathcal{D}_0)$). It remains to check what happens if $F \in f^{k_1}(\mathcal{D}_i) - 1$ for some i, $1 \le i \le k_1$. In this case it follows from (18) that

$$F \cap (\{c_1-2k_1+1, \ldots, c_1-k_1\} \cup \{1, \ldots, k_1+2-i\}) = \{c_1-i-k_1+1\} \qquad (20)$$

(as the G in (18) is in $f^{k_1}(\mathcal{D}_i)$). Note that all pairs of vertices in the list other than the excluded pair $(1, k_1+2)$ are of the form $(c_1-2k_1+j, j+1)$. Since we have $c_1-k_1-i+1 = c_1-2k_1+(k_1-i+1) \in F$ it follows from (20) that $(k_1-i+1)+1 = k_1-i+2 \notin F$. Thus F cannot contain any of the pairs of vertices in the list. □

Proof of Sublemma 2.3.4 To prove this we let

$$\mathcal{A} = \mathcal{I}_a^{(r)}\left(G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right)$$

and follow the line of reasoning in the induction step in the proof of Lemma 2.3. We may suppose here that the end-vertex a of the path is the vertex n.

From the definitions of $\mathcal{B}$ and $\mathcal{C}$ it follows that

$$f(\mathcal{B}) \cup f(\mathcal{C}) = \mathcal{I}_a^{(r)}\left(G\left((c_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right),$$

so that (3) holds with equality. We therefore have that

$$|\mathcal{B}| + |\mathcal{C}| = \left|\mathcal{I}_a^{(r)}\left(G\left((c_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right)\right| + |\mathcal{E}|$$

so that (4) holds with equality.

Since $\mathcal{A}$ is partitioned into $\mathcal{B}, \mathcal{C}, \mathcal{D}_0, \ldots, \mathcal{D}_k$, it follows that

$$\begin{aligned} & \left|\mathcal{I}_a^{(r)}\left(G\left(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right)\right| \\ & -\left|\mathcal{I}_a^{(r)}\left(G\left((c_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right)\right| &= |\mathcal{E}| + \textstyle\sum_{i=0}^{k_1} |\mathcal{D}_i| \\ & &= |\mathcal{F}|, \text{ by (7)}, \\ & &= |f(\mathcal{F})|, \text{ by (8)}. \end{aligned} \tag{21}$$

From Sublemma 2.3.3(4) we know that $f(\mathcal{F})$ is a family of independent $(r-1)$-subsets of $V\left(G\left((c_1-k_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right)$. Since $f(\mathcal{F})$ is a subfamily of $f^{k_1+1}(\mathcal{A})$, it follows that every independent $(r-1)$-set in $f(\mathcal{F})$ contains the end-vertex of P^{k_0}, and thus in $\mathcal{I}_a^{(r-1)}\left(G\left((c_1-k_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right)$ it follows that

$$|f(\mathcal{F})| \leq \mathcal{I}_a^{(r-1)}\left(G\left((c_1-k_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right). \tag{22}$$

For a family $\mathcal{G}$ of subsets of $\{1, 2, \ldots, n\}$, let $\mathcal{G}+\{i\}$ denote the family $\{G \cup \{i\} : G \in \mathcal{G}\}$.

Now consider the "reverse" map $f^{-(k_1+1)} : \{1, 2, \ldots, n-k_1-1\} \to \{k_1+2, \ldots, n\}$ given by $f^{-(k_1+1)}(j) = k_1+1+j$. Under this map the independent $(r-1)$-sets in $\mathcal{I}_a^{(r-1)}\left(G\left((c_1-k_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right)$ are taken to $\mathcal{E}' \cup \sum_{i=0}^{k_1} \mathcal{D}'_i$, where $\mathcal{E}' \cup \{2\} \subseteq \mathcal{B}$, $\mathcal{E}' \cup \{1\} \subseteq \mathcal{C}$ (and $f(\mathcal{E}') \subseteq \mathcal{E} - \{1\}$) and $\mathcal{D}'_i + \{k_1+2-i\} \subseteq \mathcal{D}_i$ $(1 \leq i \leq k_1)$ and $\mathcal{D}'_0 + \{1\} \subseteq \mathcal{D}_0$. Let us describe this in more detail. Consider an independent $(r-1)$-set S in $\mathcal{I}_a^{(r-1)}\left(G\left((c_1-k_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right)$.

(a) If S contains the vertex 1 (so does not contain any vertex in $\{c_1-2k_1, c_1-2k_1+1, \ldots, c_1-k_1-1\}$), then $f^{-(k_1+1)}(S)$ contains the vertex k_1+2 and does not contain any of the vertices in $\{c_1-k_1+1, \ldots, c_1\} \cup \{1, \ldots, k_1+1\}$. We let

$$\mathcal{D}'_0 = \Big\{f^{-(k_1+1)}(S) : 1 \in S \text{ and } S \in \mathcal{I}_a^{(r-1)}\left(G\left((c_1-k_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right)\Big\};$$

then $\mathcal{D}'_0 \cup \{1\} \subseteq \mathcal{D}_0$.

(b) If S contains a vertex c_1-k_1-i for some i, $1 \leq i \leq k_1$ (and so does not contain any vertex in the set $\{c_1-k_1-i+1, \ldots, c_1-k_1-1\} \cup \{1, 2, \ldots, k_1+1-i\}$, then $f^{-(k_1+1)}(S)$ contains the vertex c_1-i+1 and does not contain any of the vertices in $\{c_1-i+2, \ldots, c_1\} \cup \{1, 2, \ldots, 2k_1+2-i\}$. For $1 \leq i \leq k_1$, we let

$$\mathcal{D}'_i = \Big\{f^{-(k_1+1)}(S) : c_1-k_1-i \in S \text{ and } S \in \mathcal{I}_a^{(r-1)}\left(G\left((c_1-k_1-1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right)\Big\};$$

then $\mathcal{D}'_i + \{k_1+2-i\} \subseteq \mathcal{D}_i$.

(c) If S contains no vertex from the set $\{c_1 - 2k_1, \ldots, c_1 - k_1 - 1\} \cup \{1\}$, then $f^{-(k_1+1)}(S)$ contains no vertex from the set $\{c_1 - k_1 + 1, \ldots, c_1\} \cup \{1, 2, \ldots, k_1 + 2\}$. We let

$$\mathcal{E}' = \Big\{ f^{-(k_1+1)}(S) : S \cap (\{c_1 - 2k_1, \ldots, c_1 - k_1 - 1\} \cup \{1\}) = \emptyset \text{ and } S \in \mathcal{I}_a^{(r-1)}\left(G\left((c_1 - k_1 - 1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right) \Big\}.$$

Then $\mathcal{E}' \cup \{2\} \subset \mathcal{B}$ and $\mathcal{E}' \cup \{1\} \subseteq \mathcal{C}$, so $\mathcal{E}' \subseteq \mathcal{E}$. It follows that

$$\left| \mathcal{I}_a^{(r-1)}\left(G\left((c_1 - k_1 - 1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right) \right| \leq |\mathcal{E}'| + \sum_{i=0}^{k_1} |\mathcal{D}_i'|. \tag{23}$$

The family $\mathcal{E}' \cup \sum_{i=0}^{k_1} \mathcal{D}_i'$ has the properties that

$$|\mathcal{E}'| + \sum_{i=0}^{k_1} |\mathcal{D}_i'| \leq |\mathcal{E}| + \sum_{i=0}^{k_1} |\mathcal{D}_i| = |\mathcal{F}| = |f(\mathcal{F})|.$$

It therefore follows from (22) and (23) that

$$|f(\mathcal{F})| = \left| \mathcal{I}_a^{(r-1)}\left(G\left((c_1 - k_1 - 1)^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0}\right)\right) \right|. \tag{24}$$

The equality we wish for now follows from (21) and (24). □

4. Final remarks

There are a number of operations which can be used to obtain new r-EKR graphs from old. The first is described by the following lemma of Holroyd, Spencer and Talbot [11]. Let $N(v)$ denote the neighbourhood of v, that is $N(v) = \{w : w \in V(G) \text{ and } vw \in E(G)\}$.

Lemma 4.1. *Let G be an r-EKR graph with a star centre v. If $S \subset N(v)$ then $G - S$ is also r-EKR with a star centre v.*

By applying this to the graph $G(c_1^{k_1}, c_2^{k_2}, \ldots, c_s^{k_s}, p^{k_0})$ in the case where $p \geq k_0 + 2$ and $S = N(n)$ (n being the end vertex of the path P), we obtain the following theorem.

Theorem 4.2. *Let $s \geq 0$, $p \geq 1$ and $c_i \geq 2$ ($1 \leq i \leq s$). Let G be a graph consisting of cycles ${}_1C, {}_2C, \ldots, {}_sC$ raised to the powers $k_1, k_2 \ldots, k_s$ respectively, a path P raised to the power k_0, and an isolated vertex. Let $c_1 = |V({}_iC)|$ ($1 \leq i \leq s$) and $p = |V(P)| + k_0 + 1$. Also let*

$$1 \leq r \leq \sum_{i=1}^{s} \left\lfloor \frac{c_i}{k_i + 1} \right\rfloor + \left\lceil \frac{p}{k_0 + 1} \right\rceil.$$

Let

$$\min\left(\omega({}_1C^{k_1}), \omega({}_2C^{k_2}), \ldots, \omega({}_sC^{k_s})\right) \geq k_0 + 1.$$

Then G is r-EKR with the isolated vertex w as star centre.

It is worth remarking that in a similar vein Holroyd, Spencer and Talbot in [11] showed that if G is a graph with q components being paths, cycles, complete graphs, and at least one isolated vertex, and if $q \geq 2r$, then G is r-EKR.

Finally we make two further comments.

1. If $w \in N(v^\star)$, where $v^\star$ is a star centre of an r-EKR graph G, then it is clear that the addition of any edge wv produces a further graph that is r-EKR with star centre $v^\star$.
2. If G is an r-EKR graph, then we can introduce a further vertex w and join it to each vertex of G, and by this means produce a further r-EKR graph. Conversely, if G is an r-EKR graph, and G contains a vertex w which is joined to all other vertices, then $G - w$ is also r-EKR.

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