Problems for Promoting Problem Solving Through Parabolas

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1. Prove that the tangents at the ends of a focal chord of a parabola are perpendicular.
2. Prove that the foot of the perpendicular from the focus of a parabola on to any tangent lies on the tangent at the vertex.
3. PQ is a variable chord of a parabola. If the chords joining the vertex O to P and Q are perpendicular, show that PQ meets the axis of the parabola in a fixed point R, and find the length of OR.
4. If the tangents at the end of a focal chord of a parabola meet the tangent at the vertex at C and D, prove that CD subtends a right angle at the focus.
5. The tangent at any point P of the parabola $y^2 = 4ax$ meets the tangent at the vertex at the point Q. S is the focus and SQ meets the line through P parallel to the tangent at the vertex at the point R. Find the locus of R.
6. Show that the portion of the tangent between the point of contact and the directrix subtends a right angle at the focus.
7. Show that the points of contact of tangents from a point on the directrix are at the ends of focal chord.
8. Show that the tangents at the ends of a focal chord intersect at right angles on the directrix.
9. P and Q are two points on the parabola with parameters p and q. O is the origin and OP is perpendicular to OQ. Show that $pq + 4 = 0$ and that the tangents to the curve at P and Q meet on the line $x + 4a = 0$.
10. Show that the normal to the parabola $y^2 = 4ax$ at the point $(ap^2, 2ap)$ meets the parabola again at the point whose parameter is $-(p^2 + 2)/p$.
11. A variable chord PQ of the parabola $y^2 = 4ax$ is parallel to a fixed line. If P and Q are the points $(ap^2, 2ap)$ and $(aq^2, 2aq)$, prove that the locus of the point of intersection of the tangents to the parabola at P and Q is a straight line.
12. Find the locus of the mid-points of focal chords of the parabola $y^2 = 4ax$.
13. The chord PQ of the parabola $y^2 = 4ax$ passes through the foot of the directrix. Find the locus of the intersection of the normals at P and Q.
14. A chord of the parabola $y^2 = 4ax$ subtends a right angle at the vertex. Find the locus of the mid-point of the chord.
15. Show that the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is
$$y + tx = 2at + at^3 \quad .$$
The normal at a point $\mathrm{P}(ap^2, 2ap)$ meets the x-axis at G. Find the coordinates of the point G. H is the point on PG produced, such that PG = GH. Find the coordinates of H in terms of p and show that H lies on the parabola $y^2 = 4a(x - 4a)$.

16. A variable tangent is drawn to the parabola $y^2 = 4ax$. If the perpendicular from the vertex meets the tangent at P, find the locus of P.

17. P is any point on the parabola $y^2 = 4ax$. The tangent at P meets the y-axis at Q, and R is the mid-point of PQ. Prove that the locus of R is a parabola.

18. Find the locus of the mid-points of chords of contact of tangents to the parabola $y^2 = 4ax$ drawn from points on the directrix.

19. H is a fixed point on the axis of the parabola $y^2 = 4ax$ whose focus is $\mathrm{S}(a, 0)$. A variable chord PQ passes through H, the tangents at P and Q meet in T, and R is the foot of the perpendicular from T to PQ. Find the locus of R.

20. If the normal to the parabola at P meets the axis of the parabola at G and GP is produced, beyond P, to Q so that P is the mid-point of GQ, show that the equation of the locus of Q is $y^2 = 16a(x + 2a)$.

21. Find the coordinates of N, the foot of the perpendicular drawn from the origin to the tangent to the parabola at a general point P. Show that, as P varies, the locus of N is the curve $x(x^2 + y^2) + ay^2 = 0$.

22. P is any point on the parabola $y^2 = 4ax$, and O is the origin; Q is the foot of the perpendicular from P to the y-axis, R is the foot of the perpendicular from Q to OP, and QR produced meets the x-axis at K. Prove that K is a fixed point, and find its coordinates. Prove also that the locus of R is a circle, and find its centre.

23. Find the coordinates of R, the point of intersection of the tangents to the parabola at P and Q, the points with parameters p and q. Show that, if PQ touches the parabola $y^2 = 2ax$, the point R lies on the parabola $y^2 = 8ax$.

24. A tangent to the parabola $y^2 = 4ax$ at the point P meets the parabola $y^2 = 4bx$ at the points Q and R, and the tangents at P and Q meet at U. Find the locus of U as P varies.

25. Find the locus of the mid-points of chords which subtend a right angle at the vertex of the parabola.

26. The chord PQ of a parabola passes through the focus, and the normals at P and Q meet in R. If P and Q have parameters p and q show that $pq + 1 = 0$. Find the coordinates of R, and hence find its locus.

27. If two perpendicular tangents meet at T and the corresponding normals meet at N, show that, for all such pairs of tangents, TN is parallel to the axis of the parabola, and find the locus of N.

28. Find the locus of the point of intersection of the normals at two points on a parabola when the chord joining them subtends a right angle at the vertex.

29. P and Q are two points on a parabola whose focus is S. The tangents at P and Q meet at T, and the normals at these points meet at N. R is the mid-point of TN. Prove that the angle TSR is a right angle. If the chord PQ passes through the focus of the parabola, prove that the locus of R is a parabola with its axis coinciding with that of the first parabola, and vertex S.

30. If OP, OQ are a variable pair of perpendicular chords through the vertex O of the parabola $y^2 = 4ax$, prove that the chord PQ cuts the axis in a fixed point. Find the equation of the locus of the point of intersection of the normals at P and Q to the parabola.

31. Find in each case the equation of the locus of the point of intersection of tangents to a parabola $y^2 = 4ax$ at P and Q, when (a) PQ is perpendicular to the tangent at P; (b) the tangents or normals at P and Q are perpendicular; (c) PQ passes through a fixed point on the directrix; (d) PQ is a focal chord.

32. Find in each case the equation of the locus of the point of intersection of normals to a parabola $y^2 = 4ax$ at P and Q, when (a) PQ is perpendicular to the tangent at P; (b) the tangents or normals at P and Q are perpendicular; (c) PQ passes through a fixed point on the directrix; (d) PQ is a focal chord.

33. If the normals at two points on a parabola meet on the same parabola, show that the chord joining the two points meets the axis in a fixed point.

34. Consider the parabola $y^2 = 4ax$, where $a > 0$, and a variable focal chord PQ. The tangents at P and Q meet at T, and the normals at P and Q meet at N. Determine the locus of the mid-point of TN.

35. P and Q are two points on the parabola $y^2 = 4ax$ and PQ passes through the foot of the directrix on the x-axis, i.e. the point $(-a, 0)$. Determine the locus of the mid-point of PQ.

36. P and Q are two points on the parabola $y^2 = 4ax$ and PQ passes through the foot of the directrix on the x-axis, i.e. the point $(-a, 0)$. Determine the locus of the point of intersection of the tangents at P and Q.

37. P and Q are two points on the parabola $y^2 = 4ax$ and PQ passes through the foot of the directrix on the x-axis, i.e. the point $(-a, 0)$. Determine the locus of the point of intersection of the normals at P and Q.

38. A tangent to the parabola $y^2 = 4ax$ at the point P meets the parabola $y^2 = 4bx$ at the points Q and R, where $2b > a > 0$. Determine the equation of the locus of the mid-point of QR.

39. The normal at a point P of a parabola, focus S, meets the axis in G. Prove that there are two positions of P such that the triangle SPG is equilateral, and that the sides then have length $4a$.

40. A variable chord PQ of the parabola $y^2 = 4ax$ is parallel to the line $y = x$. If P and Q are the points $(ap^2, 2ap)$ and $(aq^2, 2aq)$, show that $p + q = 2$, and prove that the locus of the point of intersection of the normals to the parabola at P and Q is a straight line.

41. A variable chord PQ of the parabola $y^2 = 4ax$ is parallel to a fixed line. If P and Q are the points $(ap^2, 2ap)$ and $(aq^2, 2aq)$, prove that the locus of the point of intersection of the tangents to the parabola at P and Q is a straight line.

42. A variable straight line with constant gradient m meets the parabola

$$y^2 = 4ax$$

at Q, R. Find the locus of P, the mid-point of QR.

43. A variable straight line with constant intercept c meets the parabola

$$y^2 = 4ax$$

at Q, R. Find the locus of P, the mid-point of QR.

44. Find the equations of the tangents to the parabola $y^2 = 4ax$ from the point $(-a/2, a/2)$.

45. A variable chord of the parabola $y^2 = 4ax$ has a fixed gradient k. Find the locus of the mid-point.

46. A chord of the parabola $y^2 = 4ax$ is drawn to pass through the point $(-a, 0)$. Find the locus of the point of intersection of the tangents at the ends of the chord.

47. The normal at the point P of the parabola $y^2 = 4ax$ meets the curve again at Q. The circle on PQ as diameter goes through the vertex. Find the x-coordinate of P.

48. A tangent to the parabola $y^2 = 4ax$ meets the parabola $y^2 = 8ax$ at P, Q. Find the locus of the mid-point of PQ.

49. Find the locus of the mid-point of a variable chord through the point $(a, 2a)$ of the parabola $y^2 = 4ax$.

50. If the chord PQ of the parabola $y^2 = 4ax$ is normal at P$(a, 2a)$ find the length of PQ, the equations of the tangents PT and QT, and the area of the triangle PQT, where T is the point of intersection of the tangents at P and Q.

51. The straight line through the vertex of the parabola perpendicular to the tangent at $P(4a, 4a)$ meets this tangent at Q. Find the equation of the other tangent from Q.

52. Prove that the line $x - 2y + 4a = 0$ touches the parabola, and find the coordinates of P, the point of contact. If the line $x - 2y + 2a = 0$ meets the parabola in Q, R and M is the mid-point of QR, prove that PM is parallel to the axis of x, and that this axis and the line through M perpendicular to it meet on the normal at P to the parabola.

53. The chord PQ of the parabola cuts the x-axis in R, and the tangent at P cuts the y-axis in S. If P and Q are variable points on the parabola, and if RS passes through the fixed point $(2a, -a)$ prove that the equation of the locus of the mid-point of RS is $2xy + ax - 2ay = 0$.

54. $H(h, k)$ is a fixed point, and R is the foot of the perpendicular from H to its polar line with respect to the parabola $y^2 = 4ax$. Prove that, as a varies, the locus of R is a circle through H.

55. Prove that, in general, three normals can be drawn to the parabola $y^2 = 4ax$ to pass through a given point P(h, k). If two of these three normals are perpendicular, prove that P lies on the parabola

$$y^2 = a(x - 3a),$$

and that the length of the chord joining the feet of the perpendicular normals is $h + a$.

Prove also that the length of the third normal is

$$3\sqrt{a(h - 2a)}$$

56. The normals at the point P, Q and R to a parabola meet in the point T. Prove that SP+SQ+SR+SO=2TM, where S is the focus of the parabola, O is the vertex and M is the foot of the perpendicular from T to the tangent at the vertex.

57. A tangent to the parabola $y^2 = 4ax$ at the point P meets the parabola $y^2 = 4bx$ at the points Q and R, where $2b > a > 0$. The normals at Q and R meet at U. Determine the equation of the locus of U.

58. P is any point on the parabola $y^2 = 4ax$. The tangent at P meets the x-axis at Q, and R is the mid-point of PQ. Detemine the locus of R.

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60. P is any point on the parabola $y^2 = 4ax$. The normal at P meets the x-axis at Q, and R is the mid-point of PQ. Determine the locus of R.

61. P is any point on the parabola $y^2 = 4ax$. The normal at P meets the y-axis at Q, and R is the mid-point of PQ. Determine the locus of R.