Problems and selected solutions for Promoting Problem Solving Through Parabolas

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1. Prove that the tangents at the ends of a focal chord of a parabola are perpendicular.

   Solution

   Equation of chord: $x - \frac{1}{2}(p+q)y + apq = 0$ (end points P$(ap^2, 2ap)$, Q$(aq^2, 2aq)$, as usual.)

   Focal chord passes through $(x, y) = (a, 0)$, i.e.

$$a - \frac{1}{2}(p+q) \times 0 + apq = 0$$

   i.e.

$$a(1 + pq) = 0$$

   i.e.

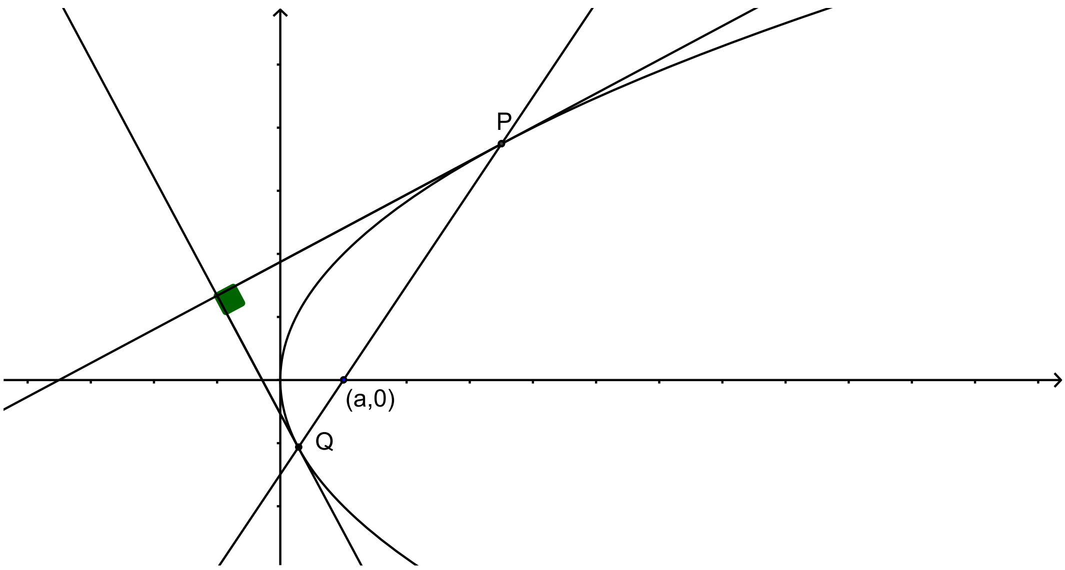
$$pq = -1$$

   Tangent at p has slope $1/p$ and the tangent at q has slope $1/q$.

   Therefore the product of the slopes of the tangents is

$$\frac{1}{p} \times \frac{1}{q} = \frac{1}{pq} = \frac{1}{-1} = -1$$

   Hence the tangents are perpendicular.



Problem 1 (Geogebra file)

2. Prove that the foot of the perpendicular from the focus of a parabola on to any tangent lies on the tangent at the vertex.

   Solution

   Let P$(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$. The equation of the tangent at P is

$$x - ty + at^2 = 0 \qquad (i)$$

   with gradient $1/t$.

The equation of the line through the focus $\mathrm{S}(a,0)$ which is perpendicular to the tangent (so has slope $-t$) is

$$\frac{y-0}{x-a} = -t$$

i.e.

$$y + xt - at = 0 \qquad (ii)$$

The point of intersection of (i) and (ii), $\mathrm{R}(X,Y)$ (which is the foot of the perpendicular) satisfies:

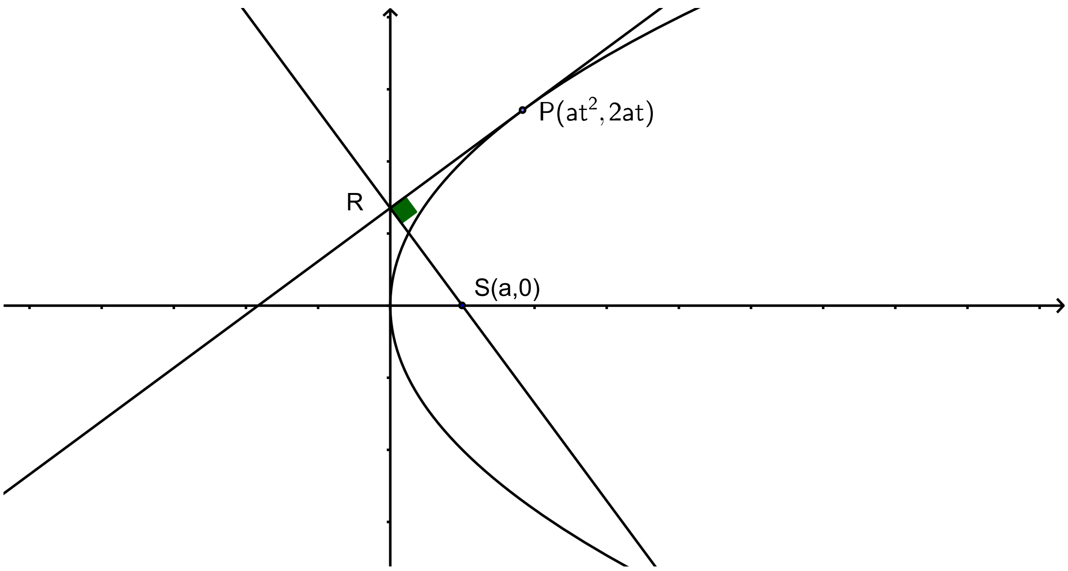
$$X - tY + at^2 = 0 \qquad (iii)$$

$$Y + Xt - at = 0 \qquad (iv)$$

Forming $(iii) + t \times (iv)$:

$$X(1+t^2) = 0$$

i.e. $X = 0$, and hence R lies on the y-axis, which is the tangent to the parabola at the vertex.



Problem 2 (Geogebra file)

3. PQ is a variable chord of a parabola. If the chords joining the vertex O to P and Q are perpendicular, show that PQ meets the axis of the parabola in a fixed point R, and find the length of OR.

Solution

Let $\mathrm{P}(ap^2, 2ap), \mathrm{Q}(aq^2, 2aq)$ be two points on the parabola $y^2 = 4ax$. The equation of the chord through P and Q is

$$x - \frac{1}{2}(p+q)y + apq = 0 \qquad (i)$$

The chord joining the vertex O to P has slope

$$\frac{2ap-0}{ap^2-0} = \frac{2}{p}$$

and similarly the chord joining O to Q has slope $\dfrac{2}{q}$.

If these two chords are perpendicular then

$$\frac{2}{p} \times \frac{2}{q} = -1$$

i.e. $pq = -4$, and the chord (i) has equation

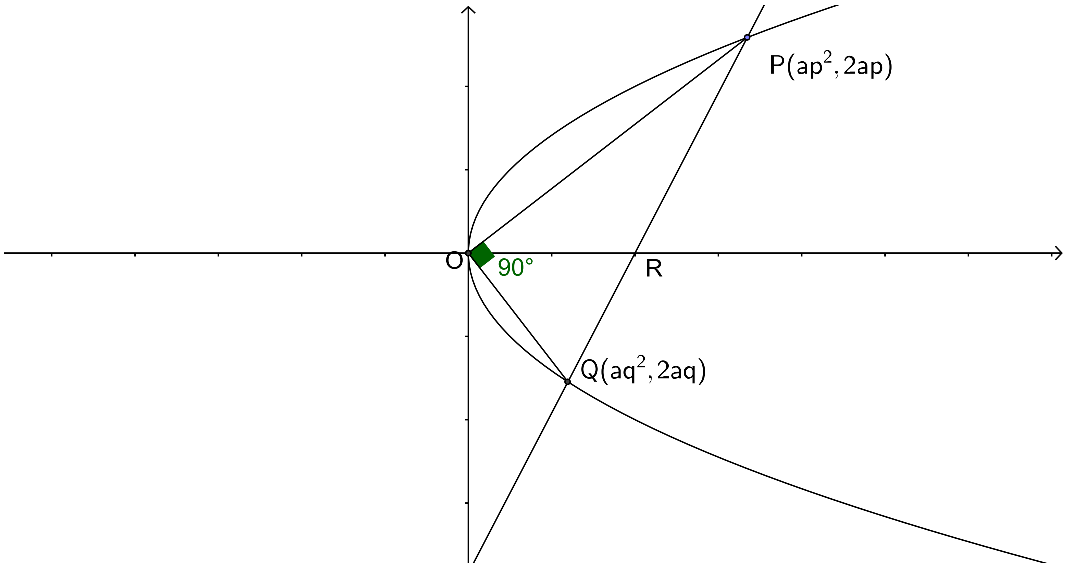
$$x - \frac{1}{2}(p+q)y - 4a = 0 \qquad (ii)$$

where $pq = -4$. Now (ii) meets the axis of the parabola (the x-axis) where $y = 0$, i.e. from (ii)

$$x = 4a$$

which is independent of p and q, and hence a fixed point whose coordinates are $\mathrm{R}(4a, 0)$.

The length of $\mathrm{OR} = 4a$.



Problem 3 (Geogebra file)

4. If the tangents at the end of a focal chord of a parabola meet the tangent at the vertex at C and D, prove that CD subtends a right angle at the focus.

   Solution

   Let $\mathrm{P}(ap^2, 2ap), \mathrm{Q}(aq^2, 2aq)$ be two points on the parabola $y^2 = 4ax$. The equation of the chord through P and Q is

$$x - \frac{1}{2}(p+q)y + apq = 0 \qquad (i)$$

   Focal chord passes through $(x, y) = (a, 0)$, i.e.

$$a - \frac{1}{2}(p+q) \times 0 + apq = 0$$

   i.e.

$$a(1 + pq) = 0$$

   i.e.

$$pq = -1$$

The tangents at P and Q have equations

$$x - py + ap^2 = 0$$

$$x - qy + aq^2 = 0$$

These meet the tangent at the vertex (the y-axis) where $x = 0$, i.e.

$$-py + ap^2 = 0$$

and

$$-qy + aq^2 = 0$$

giving

$$y = \frac{ap^2}{p} = ap \quad \text{for} \;\; \text{P}$$

$$y = aq \;\; \text{for} \;\; \text{Q}$$

(neither p nor q is zero since $pq = -1$.

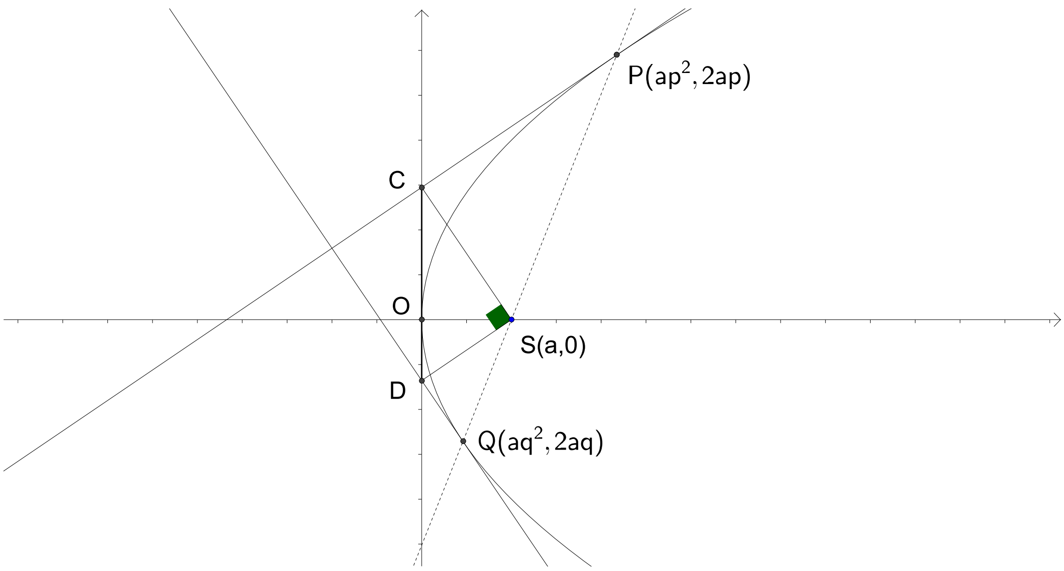
Thus the coordinates of C and D are C$(0, ap)$ and D$(0, aq)$. The slope of the line joining the focus S$(a, 0)$ to C is

$$\frac{ap - 0}{0 - a} = -p$$

and the slope of the line joining the focus to D is, similarly, $-q$, and the product of these is

$$(-p) \times (-q) = pq = -1$$

and hence CD subtends a right angle at the focus.



Problem 4 (Geogebra file)

5. The tangent at any point P of the parabola $y^2 = 4ax$ meets the tangent at the vertex at the point Q. S is the focus and SQ meets the line through P parallel to the tangent at the vertex at the point R. Find the locus of R.

   $\boxed{\textbf{Solution}}$

   The tangent at the point P$(at^2, 2at)$ has equation

$$x - ty + at^2 = 0$$

This meets the tangent at the vertex (the y-axis) where $x = 0$, i.e.

$$-ty + at^2 = 0 \quad \text{and hence} \quad y = at$$

so that Q has coordinates $\mathrm{Q}(0, at)$.

The equation of the line through $\mathrm{S}(a, 0)$ and Q is

$$\frac{y-0}{x-a} = \frac{at-0}{0-a}$$

i.e.

$$y + tx = at \qquad (i)$$

The line through $\mathrm{P}(at^2, 2at)$ parallel to the tangent at the vertex (the y-axis) has equation

$$x = at^2 \qquad (ii)$$

From (i) and (ii) the point $\mathrm{R}(X, Y)$ of intersection satisfies

$$Y + tX = at \qquad (iii)$$

$$X = at^2 \qquad (iv)$$

From (iii) $Y = (a - X)t$ and hence

$$Y^2 = (a-X)^2 t^2 \qquad (v)$$

and substituting for $t^2 = \dfrac{X}{a}$ from (iv) into (v), gives

$$Y^2 = (a-X)^2 \frac{X}{a}$$

i.e.

$$\boxed{aY^2 = X(a-X)^2}$$

(Note also that by substituting for $X = at^2$ from (iv) into (iii) gives $Y = at - at^3$, so that R has coordinates $\mathrm{R}(at^2, at - at^3)$, i.e. $Y^2 = a^2t^2 - 2a^2t^4 + a^2t^6 = aX - 2X^2 + X^3/a$, i.e. $aY^2 = a^2X - 2aX^2 + X^3 = X(a^2 - 2aX + X^2) = X(a-X)^2$, as above.)

6. Show that the portion of the tangent between the point of contact and the directrix subtends a right angle at the focus.

Solution

Let $\mathrm{P}(at^2, 2at)$ be a general point on the parabola $y^2 = 4ax$.

The equation of the tangent at P is

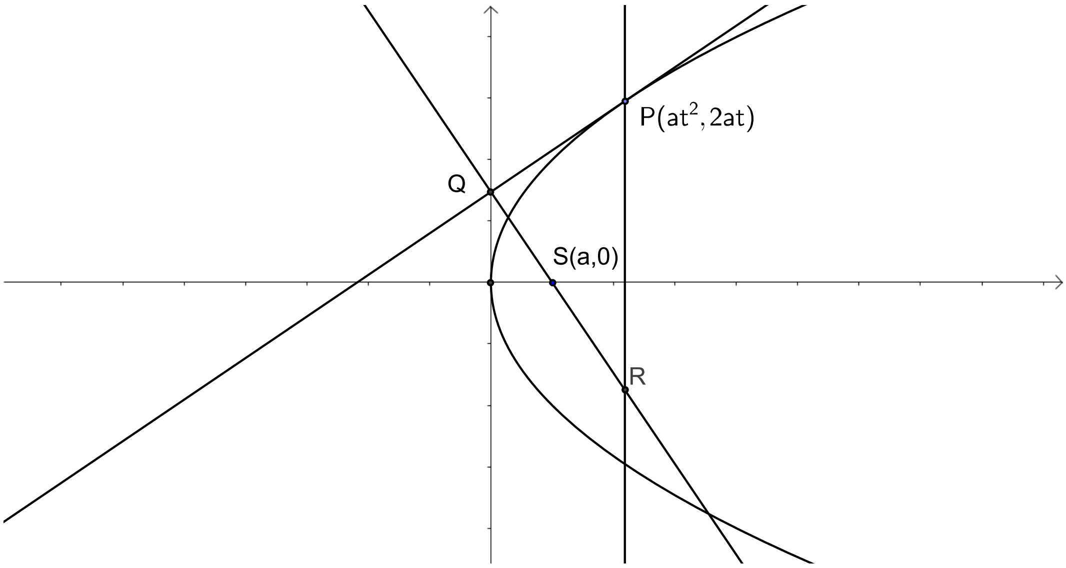
$$x - ty + at^2 = 0$$

which meets the directrix $x = -a$ where

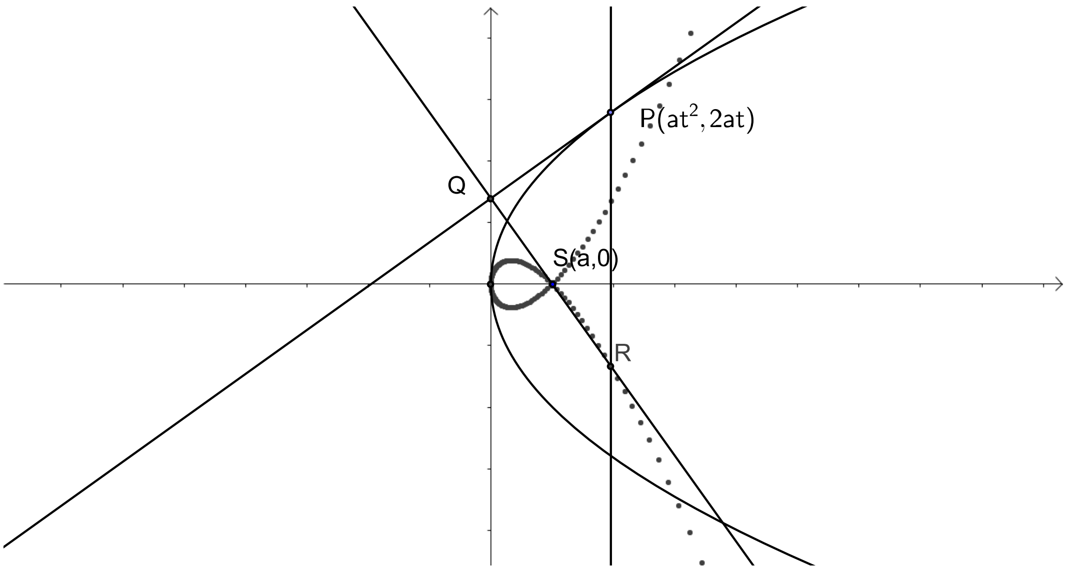
$$-a + ty + at^2 = 0$$

i.e. where

$$y = a\left(\frac{t^2 - 1}{t}\right)$$



Problem 5 (Geogebra file)



Problem 5 - locus (Geogebra file)

at the point Q $\left(-a, a\left(\frac{t^2-1}{t}\right)\right)$.

The slope of the line from Q to the focus S$(a,0)$ is

$$\frac{0-a\left(\frac{t^2-1}{t}\right)}{a--a} = -\left(\frac{t^2-1}{2t}\right) \qquad (i)$$

The slope of the line from the focus S$(a,0)$ to the point of contact P$(at^2, 2at)$ is

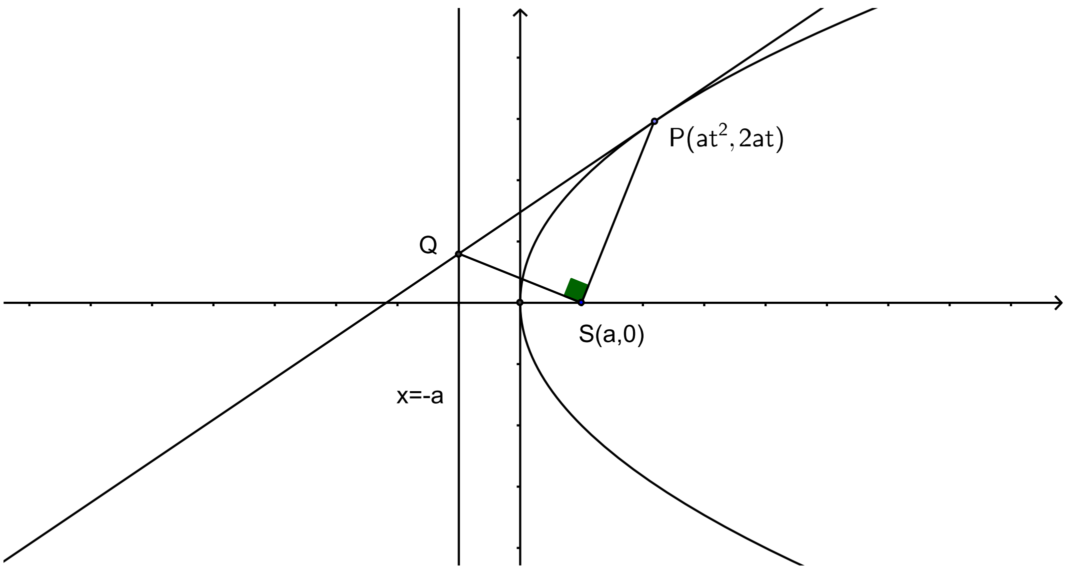
$$\frac{2at-0}{at^2-a} = \frac{2t}{t^2-1} \qquad (ii)$$

The product of the slopes in (i) and (ii) is

$$-\frac{t^2-1}{2t} \times \frac{2t}{t^2-1} = -1$$

i.e. these lines are perpendicular, and the portion of the tangent between the point of contact and the directrix subtends a right angle at the focus.

(Note that if $t = 1$ then (ii) is not defined. In this case the slope in (i) is zero since Q$(-a, 0)$ and QS is horizontal. P has coordinates P$(a, 2a)$ so that P is on the vertical line through S so that SP is perpendicular to QS again, and the result is true in this case too.)



Problem 6 (Geogebra file)

7. Show that the points of contact of tangents from a point on the directrix are at the ends of focal chord.

   $\boxed{\textbf{Solution}}$

   Let P$(ap^2, 2ap)$, Q$(aq^2, 2aq)$ be two points on the parabola $y^2 = 4ax$ whose tangents meet on the directrix.

   The equation of the chord through P and Q is

$$x - \frac{1}{2}(p+q)y + apq = 0 \qquad (i)$$

   However, the point of intersection of the tangents is T$(apq, a(p+q)$ which lies on the directrix $x = -a$, and hence $apq = -a$, i.e. $pq = -1$.

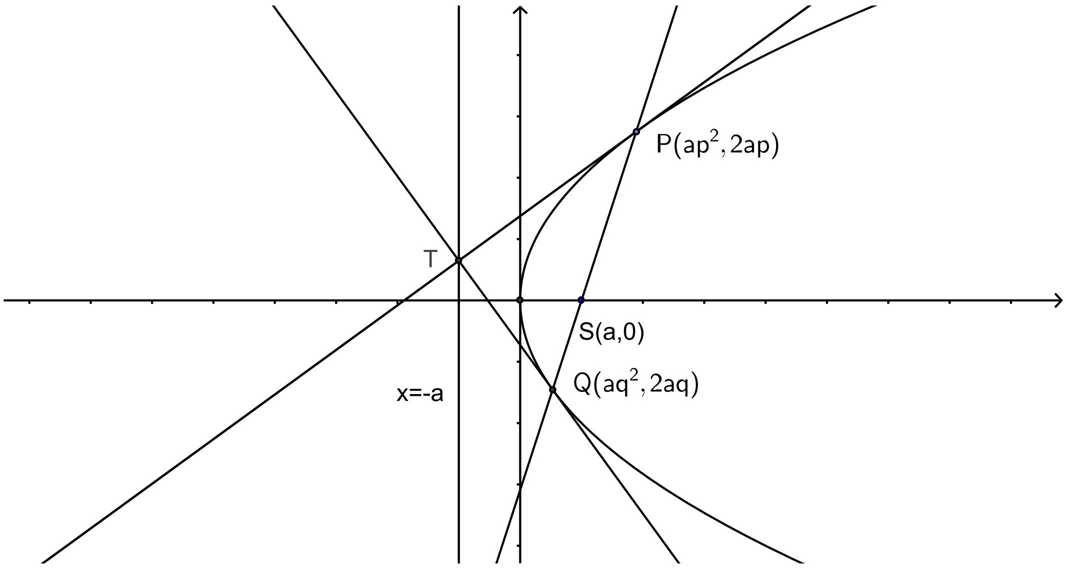
Thus the chord in (i) has equation

$$x - \frac{1}{2}(p+q)y - a = 0$$

which crosses the x-axis where $y = 0$, i.e.

$$x - \frac{1}{2}(p+q) \times 0 - a = 0$$

i.e. $x = a$, i.e. at the focus $\mathrm{S}(a, 0)$, and the chord is a focal chord.



Problem 7 (Geogebra file)

8. Show that the tangents at the ends of a focal chord intersect at right angles on the directrix.

Solution

Let $\mathrm{P}(ap^2, 2ap), \mathrm{Q}(aq^2, 2aq)$ be two points on the parabola $y^2 = 4ax$. The equation of the chord through P and Q is

$$x - \frac{1}{2}(p+q)y + apq = 0$$

A focal chord passes through $(x, y) = (a, 0)$, i.e.

$$a - \frac{1}{2}(p+q) \times 0 + apq = 0$$

i.e.

$$a(1 + pq) = 0$$

i.e.

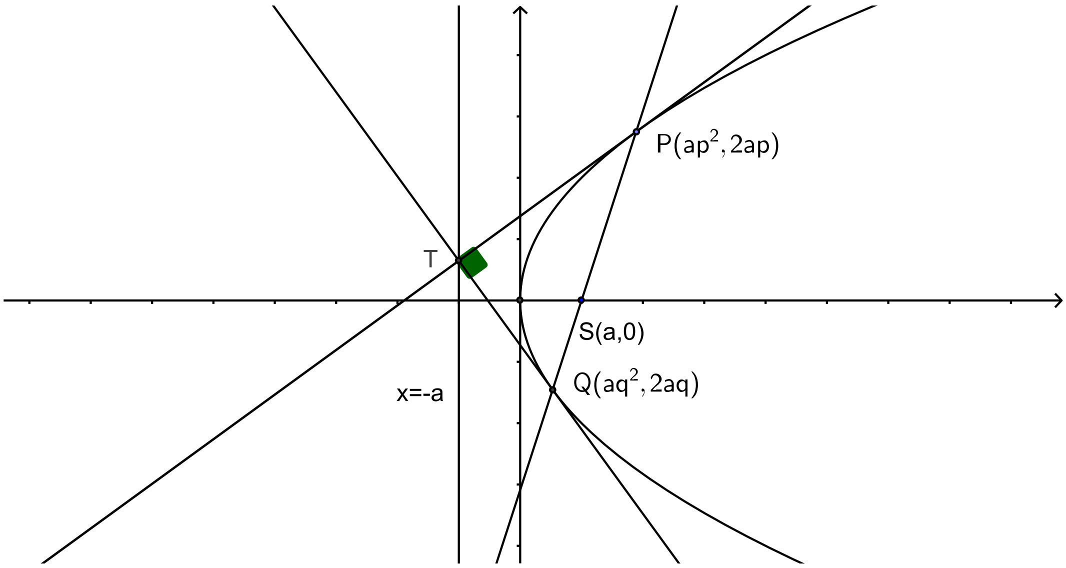
$$pq = -1$$

The point of intersection of the tangents is $\mathrm{T}(apq, a(p+q))$, i.e. $\mathrm{T}(-a, a(p+q))$, and hence the tangents meet on the directrix $x = -a$.

The tangents at P and Q have slopes $1/p$ and $1/q$, respectively, and since the product of these slopes is

$$\frac{1}{p} \times \frac{1}{q} = \frac{1}{pq} = \frac{1}{-1} = -1$$

the tangents meet at right angles on the directrix.



Problem 8 (Geogebra file)

9. P and Q are two points on the parabola with parameters p and q. O is the origin and OP is perpendicular to OQ. Show that $pq + 4 = 0$ and that the tangents to the curve at P and Q meet on the line $x + 4a = 0$.

   Solution

   Let $\mathrm{P}(ap^2, 2ap), \mathrm{Q}(aq^2, 2aq)$ be two points on the parabola $y^2 = 4ax$.

   The slope of the line from O to P is

   $$\frac{2ap - 0}{ap^2 - 0} = \frac{2}{p}$$

   and from O to Q is similarly $\dfrac{2}{q}$. Since these are perpendicular we have

   $$\frac{2}{p} \times \frac{2}{q} = -1$$

   i.e. $pq = -4$, and hence $pq + 4 = 0$.

   The tangents at P and Q meet at the point $\mathrm{T}(apq, a(p+q))$, and since $pq = -4$ this point has coordinates $\mathrm{T}(-4a, a(p+q))$, which lies on the line $x = -4a$.

10. Show that the normal to the parabola $y^2 = 4ax$ at the point $(ap^2, 2ap)$ meets the parabola again at the point whose parameter is $-(p^2 + 2)/p$.

    Solution The equation of the normal, as found above, is

    $$px + y = ap^3 + 2ap \ .$$

    This meets the parabola, whose general point is $(at^2, 2at)$, where

    $$pat^2 + \ 2at = ap^3 + 2ap \ .$$

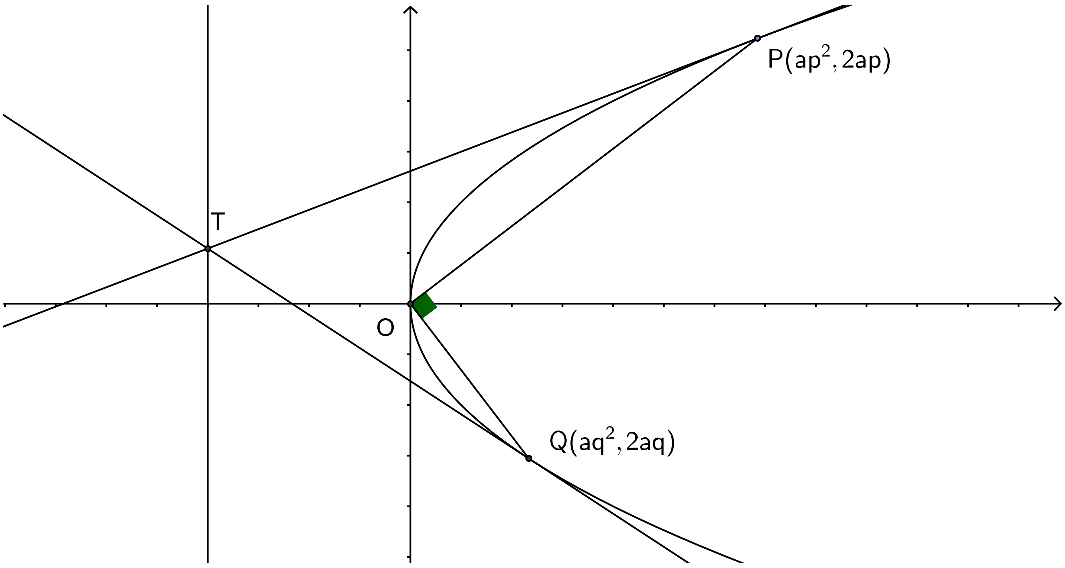
    This is a quadratic equation in t:

    $$(ap)t^2 + (2a)t - (ap^3 + 2ap) = 0$$

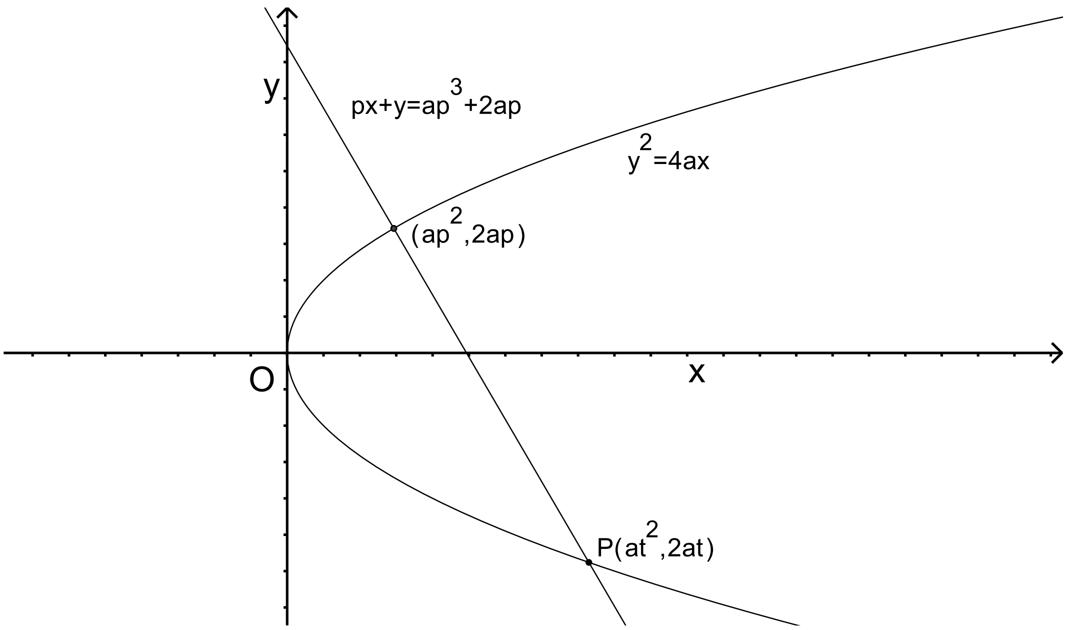
    whose sum of roots (see §2.1) is

    $$t_1 + t_2 = -2a/ap = -2/p \ .$$

    One root is known to be $t_1 = p$; so the other must be $t_2 = -2/p - t_1 = -2/p - p = -(p^2 + 2)/p$.



Problem 9 (Geogebra file)



Problem 9 (Geogebra file)

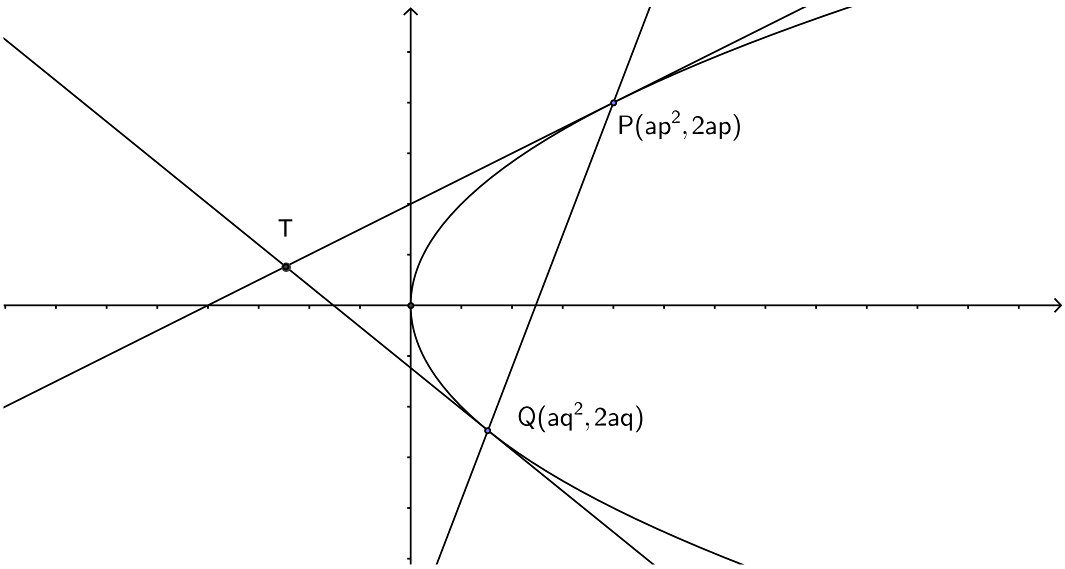
11. A variable chord PQ of the parabola $y^2 = 4ax$ is parallel to a fixed line. If P and Q are the points $(ap^2, 2ap)$ and $(aq^2, 2aq)$, prove that the locus of the point of intersection of the tangents to the parabola at P and Q is a straight line.

$\boxed{\textbf{Solution}}$

With $\mathrm{P}(ap^2, 2ap), \mathrm{Q}(aq^2, 2aq)$ the slope of the chord PQ is given by

$$\frac{2aq - 2ap}{aq^2 - ap^2} = \frac{2}{p+q}$$

If this chord is parallel to a fixed line then the slope has a fixed value, say k, where $k = \dfrac{2}{p+q}$. However, the point of intesection of the tangents at P and Q has coordinates $\mathrm{T}(apq, a(p+q))$, i.e. $\mathrm{T}(apq, 2/k)$, which lies on the straight line $y = \dfrac{2}{k}$.



Problem 10 (Geogebra file)

12. Find the locus of the mid-points of focal chords of the parabola $y^2 = 4ax$.

$\boxed{\textbf{Solution}}$

Let $\mathrm{P}(ap^2, 2ap), \mathrm{Q}(aq^2, 2aq)$ be two points on the parabola $y^2 = 4ax$. The equation of the chord through P and Q is

$$x - \frac{1}{2}(p+q)y + apq = 0$$

A focal chord passes through $(x, y) = (a, 0)$, i.e.

$$a - \frac{1}{2}(p+q) \times 0 + apq = 0$$

i.e.

$$a(1 + pq) = 0$$

i.e.

$$pq = -1$$

The mid-point $\mathrm{M}(X, Y)$ of the chord PQ has coordinates given by

$$X = \tfrac{1}{2}(ap^2 + aq^2) \quad Y = \tfrac{1}{2}(2ap + 2aq)$$

i.e.

$$X = \tfrac{1}{2}a(p^2 + q^2) \quad Y = a(p+q)$$

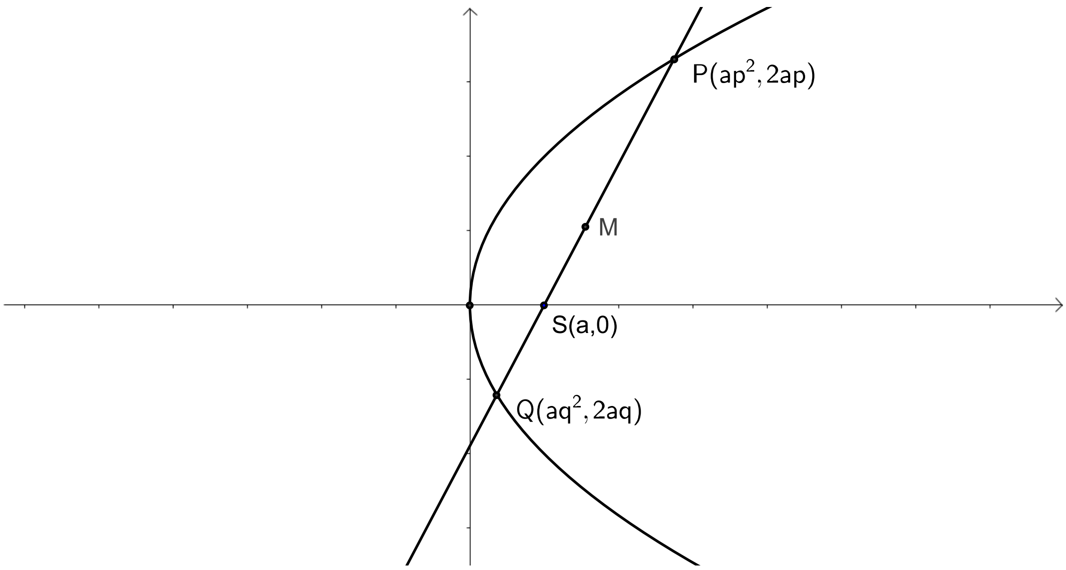
where $pq = -1$.

Using $(p+q)^2 \equiv p^2 + q^2 + 2pq$, we have the locus of the mid-points given by

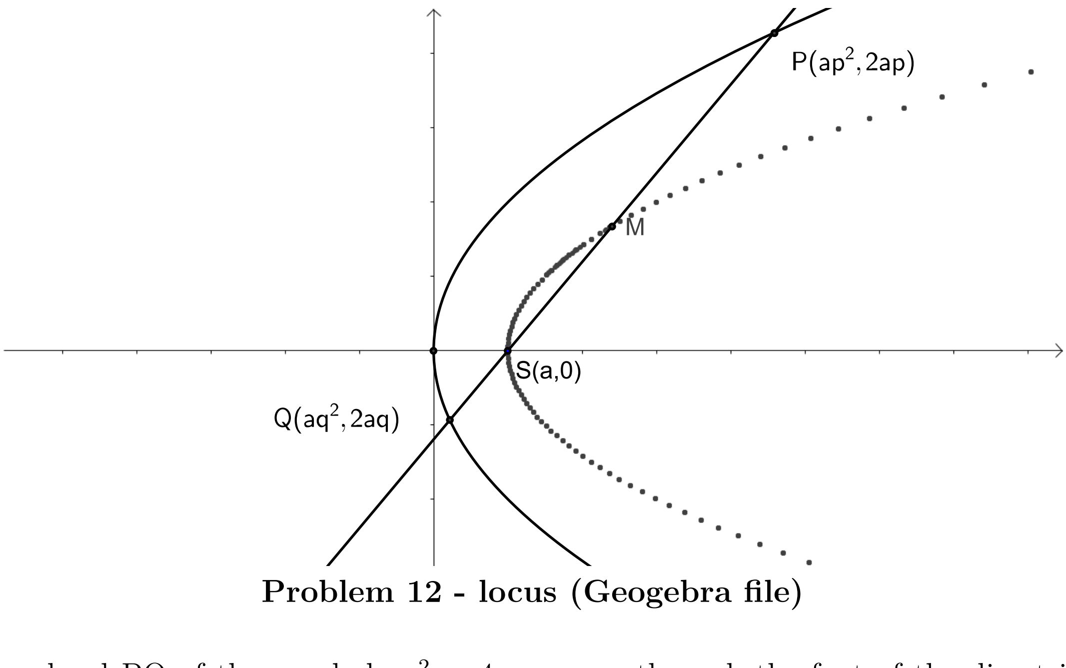
$$\left(\frac{Y}{a}\right)^2 = \frac{2X}{a} + 2 \times -1$$

i.e.

$$\boxed{Y^2 = 2a(X - a)}$$



Problem 12 (Geogebra file)



Problem 12 - locus (Geogebra file)

13. The chord PQ of the parabola $y^2 = 4ax$ passes through the foot of the directrix. Find the locus of the intersection of the normals at P and Q.
Solution Here there are two variable points, P and Q, connected by the condition that PQ passes through $(-a, 0)$. If P and Q have parameters p and q, the equation of the chord PQ is

$$x - \tfrac{1}{2}(p+q)y + apq = 0 \ .$$

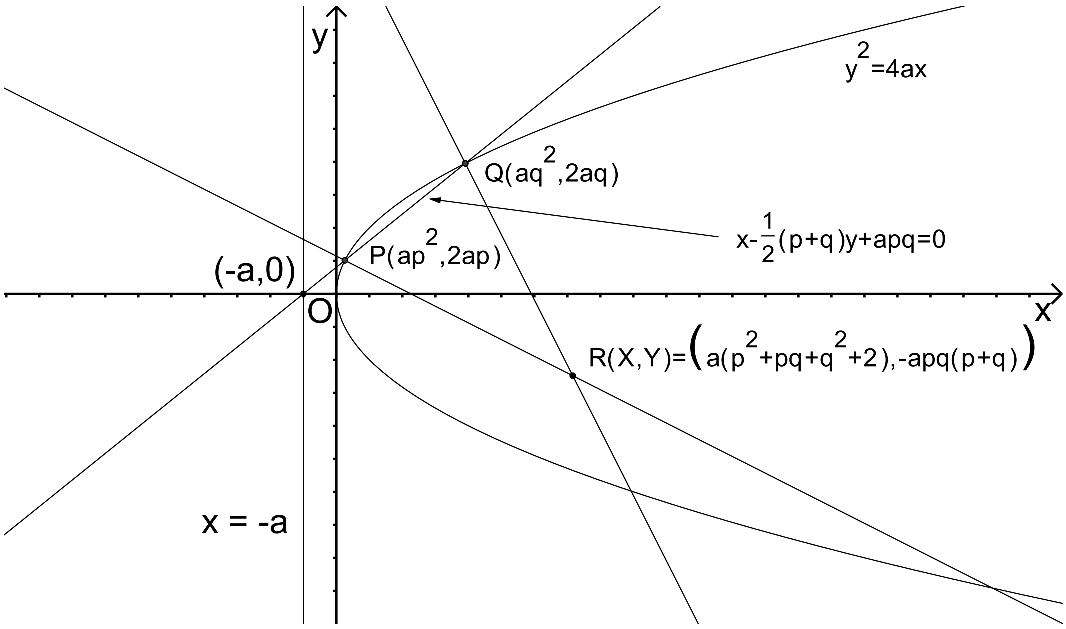
This passes through $(-a, 0)$, and so

$$pq = 1 \ \ .$$

(We could now write $q = 1/p$ throughout, but it is neater to keep both p and q in the working, and remember the connection between them.)

We now find R(X, Y). The equations of normals at P and Q are

$$px + y = ap^3 + 2ap \ \text{ and } \ qx + y = aq^3 + 2aq \ \ .$$



Problem 13 (Geogebra file)

Where these meet

$$X = a(p^2 + pq + q^2 + 2) \ \ , \ \ Y = -apq(p + q) \ \ . \tag{1}$$

To eliminate p and q we will need to remember that $pq = 1$.

However, before using this it is often better in loci problems like this to first rewrite the expression $p^2 + q^2$ in X in equation (1) in terms of $p + q$ and pq using the identity $(p + q)^2 \equiv p^2 + q^2 + 2pq$, i.e. $p^2 + q^2 \equiv (p + q)^2 - 2pq$. This means that X can be written as

$$\begin{aligned} X &= a(p^2 + pq + q^2 + 2) \\ &= a\left((p+q)^2 - 2pq + pq + 2\right) \\ &= a\left((p+q)^2 - pq + 2\right) \ \ . \end{aligned}$$

Hence

$$X = a\left((p+q)^2 - pq + 2\right) \ \text{ and } \ Y = -apq(p + q) \ \ . \tag{2}$$

Now using $pq = 1$ (2) can be written as

$$X = a\left((p+q)^2 + 1\right) \ \text{ and } \ Y = -a(p + q) \tag{3}$$

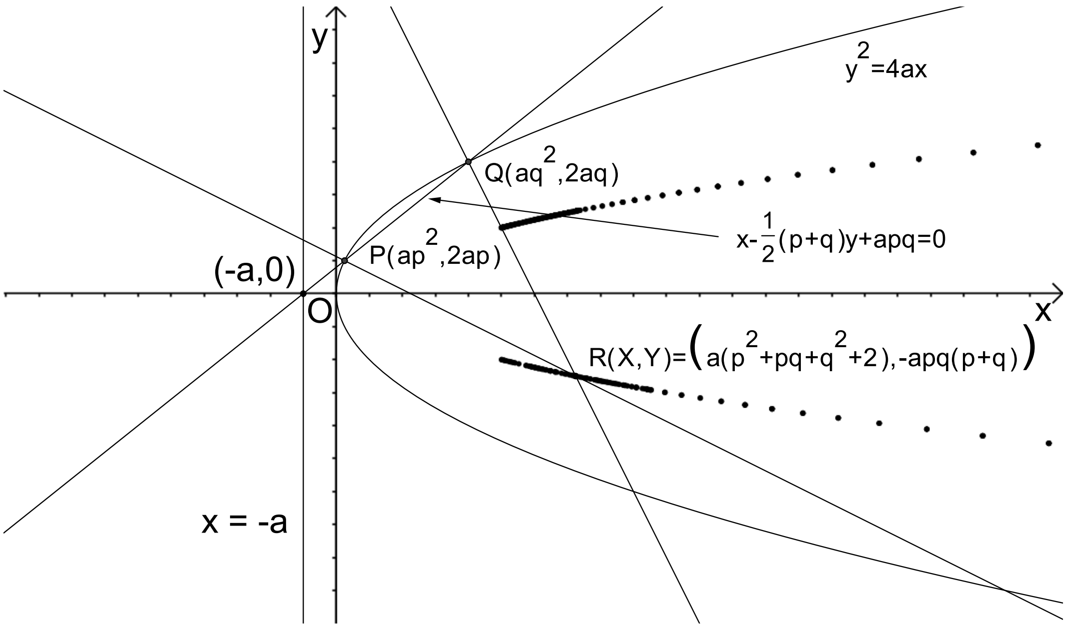
and hence

$$\begin{aligned} X &= a(p+q)^2 + a \\ &= a\left(\frac{-Y}{a}\right)^2 + a \\ &= \frac{Y^2}{a} + a \ . \end{aligned}$$

It follows that $Y^2 = a(X-a)$. The locus of R is therefore given by the equation

$$y^2 = a(x-a) \ .$$

This is the equation of another parabola whose vertex is at $(a,0)$, the focus of the given parabola.



Problem 13 - locus (Geogebra file)

14. A chord of the parabola $y^2 = 4ax$ subtends a right angle at the vertex. Find the locus of the mid-point of the chord.
$\boxed{\textbf{Solution}}$ Let the ends of the chord be $\mathrm{P}_1(at_1^2, 2at_1), \mathrm{P}_2(at_2^2, 2at_2)$. Then the gradient of the line joining the vertex $\mathrm{O}(0,0)$ to P_1 is

$$\frac{2at_1}{at_1^2} = \frac{2}{t_1} \ .$$

Similarly the gradient of OP_2 is $2/t_2$.

$\mathrm{P}_1\mathrm{P}_2$ subtends a right angle at O if $\mathrm{OP}_1, \mathrm{OP}_2$ are perpendicular, i.e.

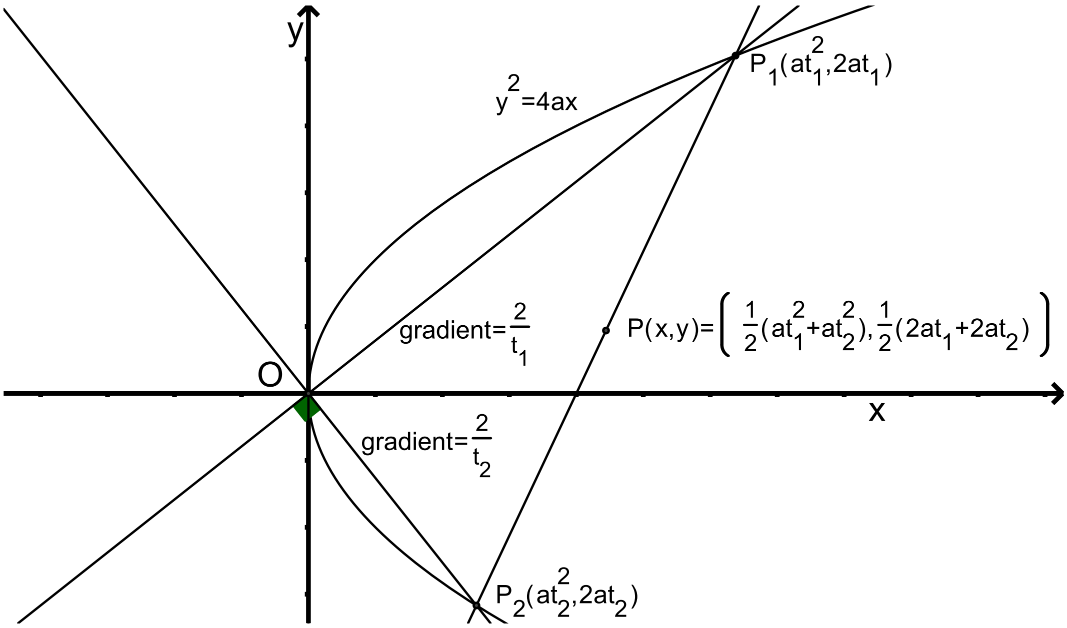
$$\frac{2}{t_1} \times \frac{2}{t_2} = -1$$

and hence

$$t_1 t_2 = -4 \ . \tag{1}$$

The mid-point of $\mathrm{P}_1\mathrm{P}_2$ is given by (see §5.6)

$$x = \frac{a(t_1^2 + t_2^2)}{2} \tag{2}$$



Problem 14 (Geogebra file)

$$y = a(t_1 + t_2) \quad . \tag{3}$$

Note that we have three equations, (1), (2), (3), from which to eliminate the two parameters t_1, t_2. Note, also, that these equations are symmetrical in t_1, t_2. Here, as is often the case, we use the following identity:

$$(t_1 + t_2)^2 \equiv t_1^2 + t_2^2 + 2t_1t_2 \quad .$$

Substituting from equations (3), (2), (1):

$$\begin{aligned} \left(\frac{y}{a}\right)^2 &= \frac{2x}{a} + 2 \times (-4) \\ &= \frac{2x}{a} - 8 \quad . \end{aligned}$$

Therefore the locus is $y^2 = 2a(x - 4a)$.

15. Show that the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$$y + tx = 2at + at^3 \quad .$$

The normal at a point P$(ap^2, 2ap)$ meets the x-axis at G. Find the coordinates of the point G. H is the point on PG produced, such that PG = GH. Find the coordinates of H in terms of p and show that H lies on the parabola $y^2 = 4a(x - 4a)$.

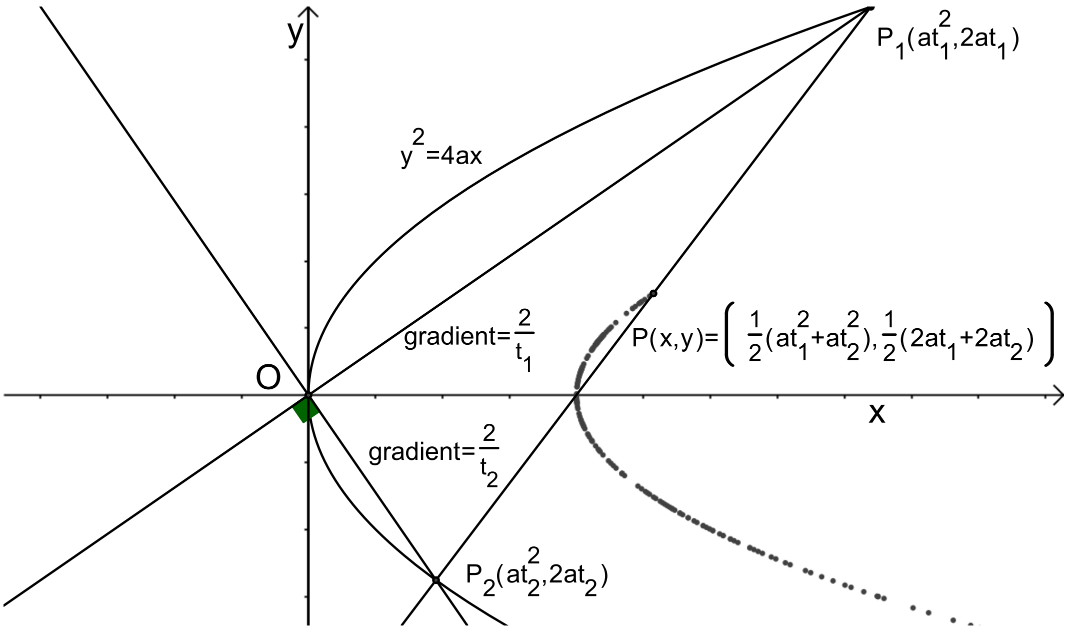
Solution

The gradient of the parabola at the point $(at^2, 2at)$ is given by

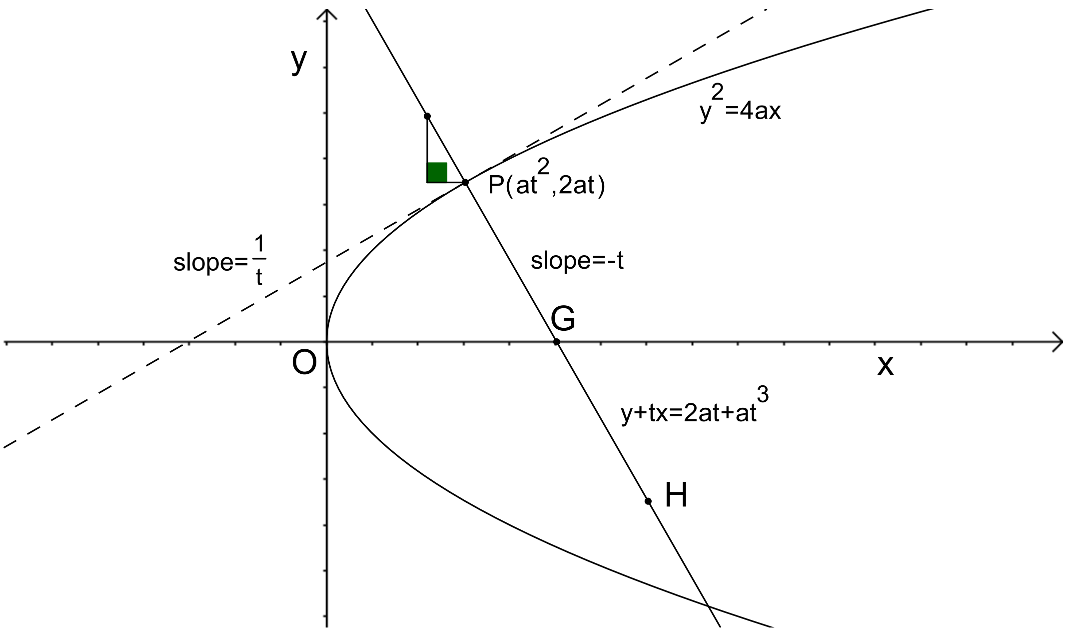
$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t} \quad .$$

Hence the gradient of the normal is $-t$, and consequently the equation of the normal at the point $(at^2, 2at)$ is

$$\frac{y - 2at}{x - at^2} = -t$$



Problem 14 - locus (Geogebra file)



Problem 15 (Geogebra file)

i.e.

$$y + tx = 2at + at^3 \quad .$$

The equation of the normal at the point $(at^2, 2at)$ is therefore

$$y + tx = 2at + at^3$$

and we obtain the x-coordinate of G by putting $y = 0$. Therefore at G

$$tx = 2at + at^3$$

i.e.

$$x = 2a + at^2 \quad .$$

Hence G is the point $(2a+at^2, 0)$. Let H be the point (X, Y), then since G is the mid-point of PH

$$2a + at^2 = \tfrac{1}{2}(at^2 + X)$$

i.e.

$$X = 4a + at^2 \tag{1}$$

and

$$0 = \tfrac{1}{2}(2at + Y)$$

i.e.

$$Y = -2at \quad . \tag{2}$$

Hence H is the point $(4a + at^2, -2at)$. Eliminating t from equations (1) and (2) gives

$$\begin{aligned} X &= 4a + a\left(-\frac{Y}{2a}\right)^2 \\ &= 4a + \frac{Y^2}{4a} \end{aligned}$$

i.e.

$$Y^2 = 4a(X - 4a) \quad .$$

Hence the point H lies on the curve $y^2 = 4a(x - 4a)$. This is the equation of a parabola, with its vertex at $(4a, 0)$.

16. A variable tangent is drawn to the parabola $y^2 = 4ax$. If the perpendicular from the vertex meets the tangent at P, find the locus of P.

Solution Let P$(at^2, 2at)$ so that the variable tangent is

$$x - ty + at^2 = 0 \tag{1}$$

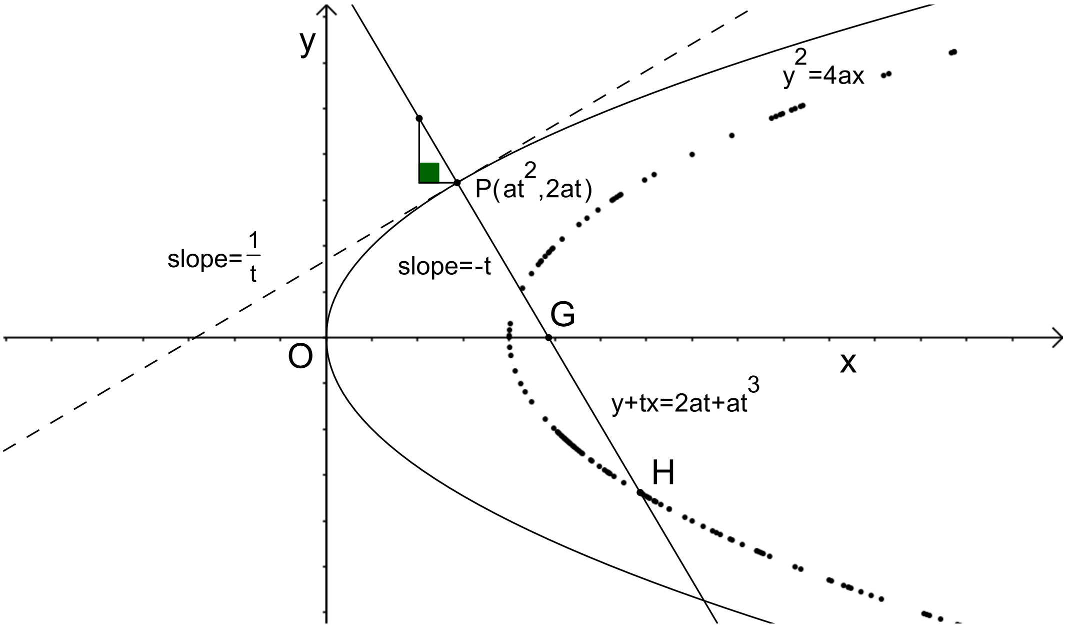
for points (x, y) on the tangent).

The gradient at P is $1/t$ and so the perpendicular through P has slope $-t$ and hence the equation of the perpendicular from the vertex $(0, 0)$ is

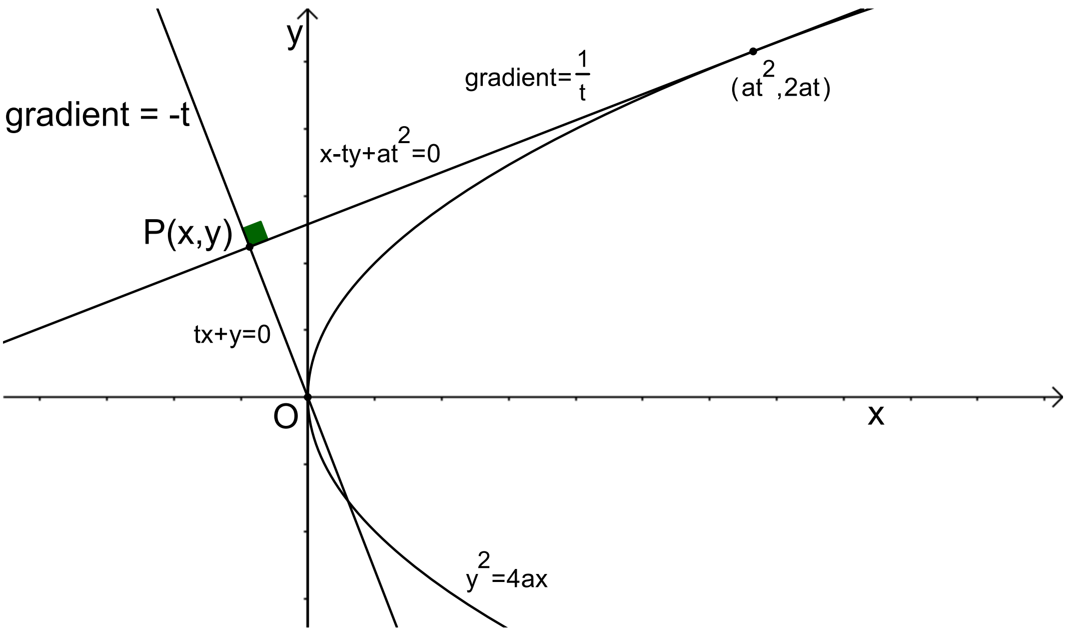
$$\frac{y}{x} = -t$$

i.e.

$$tx + y = 0 \quad , \tag{2}$$



Problem 15 - locus (Geogebra file)

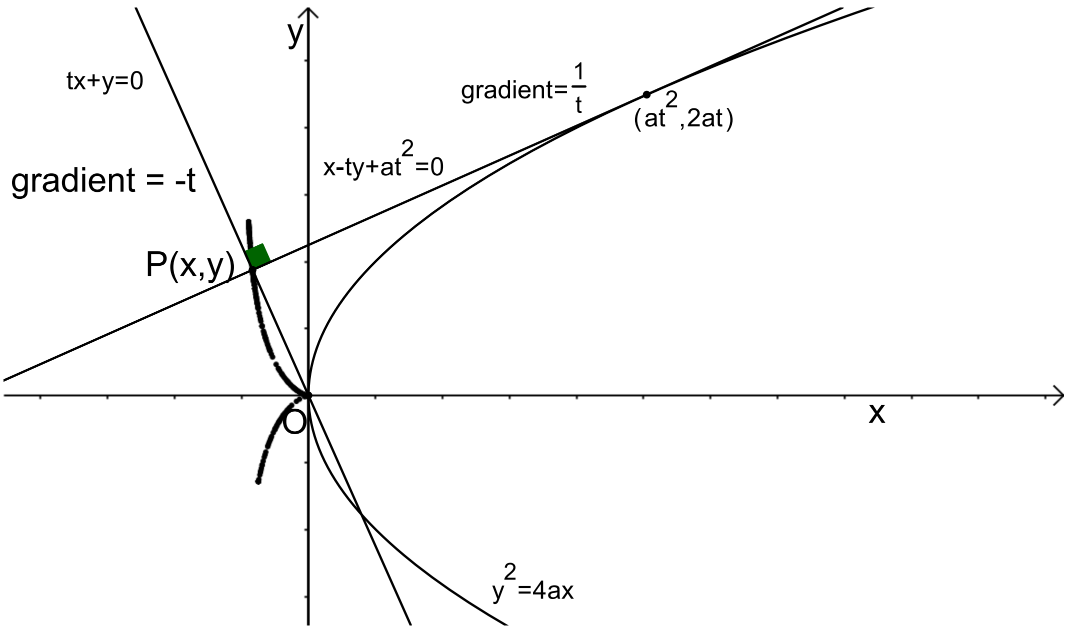


Problem 16 (Geogebra file)

P(x, y) satisfies both equations (1) and (2) so that the locus of P may be found by eliminating t from these equations. From (2), $t = -y/x$, and substituting in (1),

$$x + \frac{y^2}{x} + a\frac{y^2}{x^2} = 0 \quad .$$

So the locus of P is $x^3 + xy^2 + ay^2 = 0$. [Note that it is not necessary to solve (1) and (2) to find the coordinates of P in terms of t. However, if we do so we note that these are linear equations for x and y, and so solving gives $x = -\dfrac{at^2}{1+t^2}$ and $y = \dfrac{at^3}{1+t^2}$ and these are parametric equations for the locus and can be used to draw it.]



Problem 16 - locus (Geogebra file)

17. P is any point on the parabola $y^2 = 4ax$. The tangent at P meets the y-axis at Q, and R is the mid-point of PQ. Prove that the locus of R is a parabola.
$\boxed{\textbf{Solution}}$ The equation of the tangent at P$(ap^2, 2ap)$ is

$$x - py + ap^2 = 0$$

and it meets the y-axis, $x = 0$, in Q$(0, ap)$. R is the mid-point of PQ, and so if R is (X, Y),

$$X = \tfrac{1}{2}(ap^2 + 0) = \tfrac{1}{2}\,ap^2 \quad , \quad Y = \tfrac{1}{2}\,(2ap + ap) = \tfrac{3}{2}\,ap \quad .$$

To eliminate p we write these equations as

$$p^2 = 2X/a \quad , \quad p = 2Y/3a$$

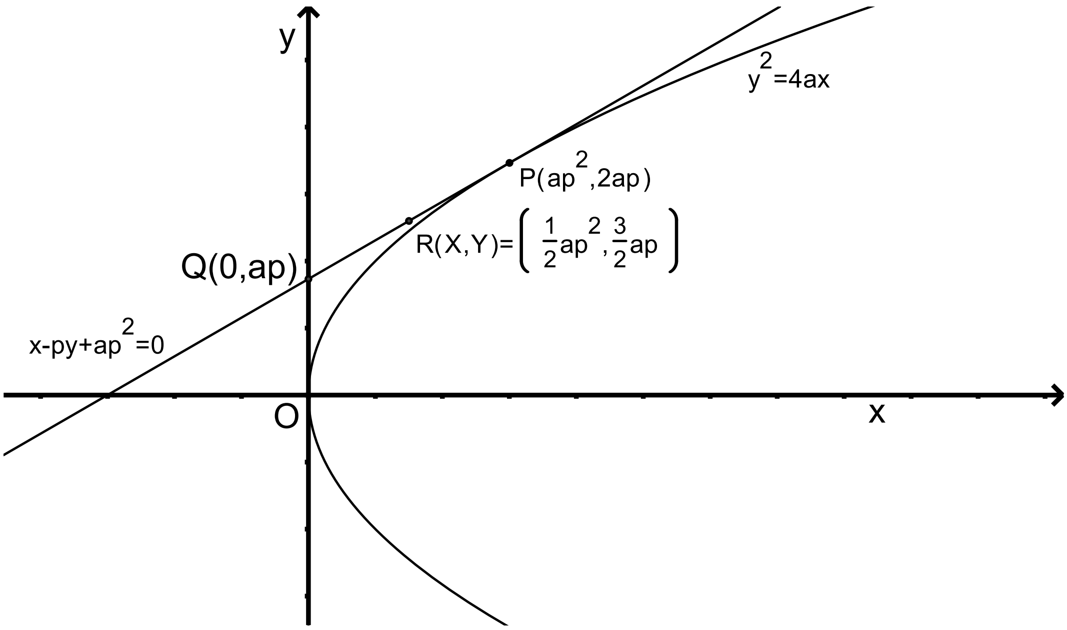
and so

$$(2Y/3a)^2 = (2X/a) \quad .$$

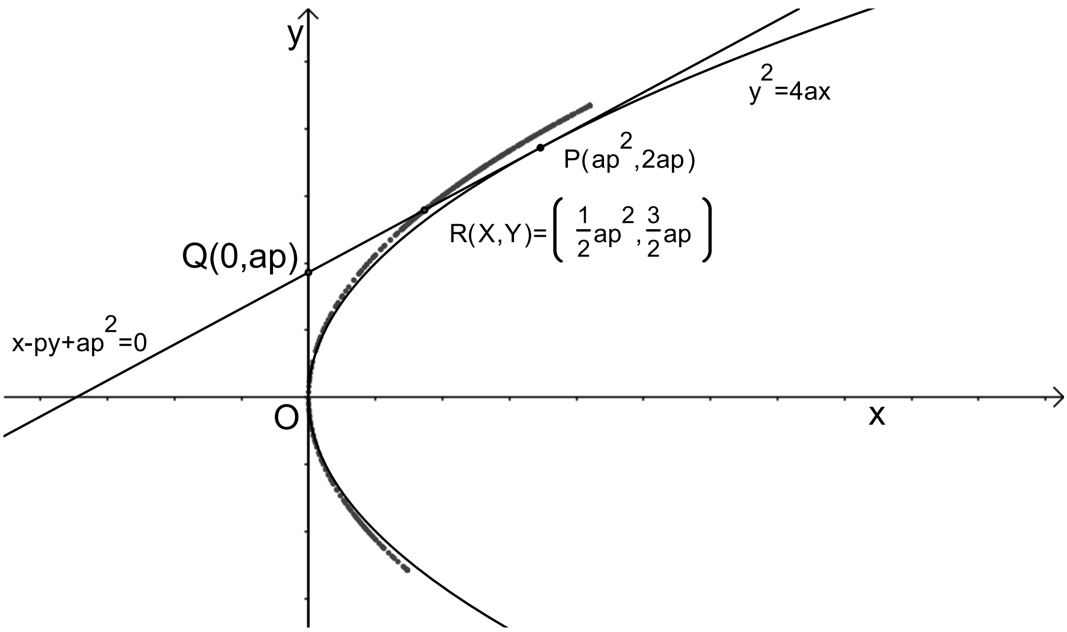
This is the equation between X and Y, which does not involve the varying p.

Now replace X by x and Y by y, and simplify. The equation of the locus of R is thus

$$2y^2 = 9ax \quad .$$



Problem 17 (Geogebra file)



Problem 17 - locus (Geogebra file)

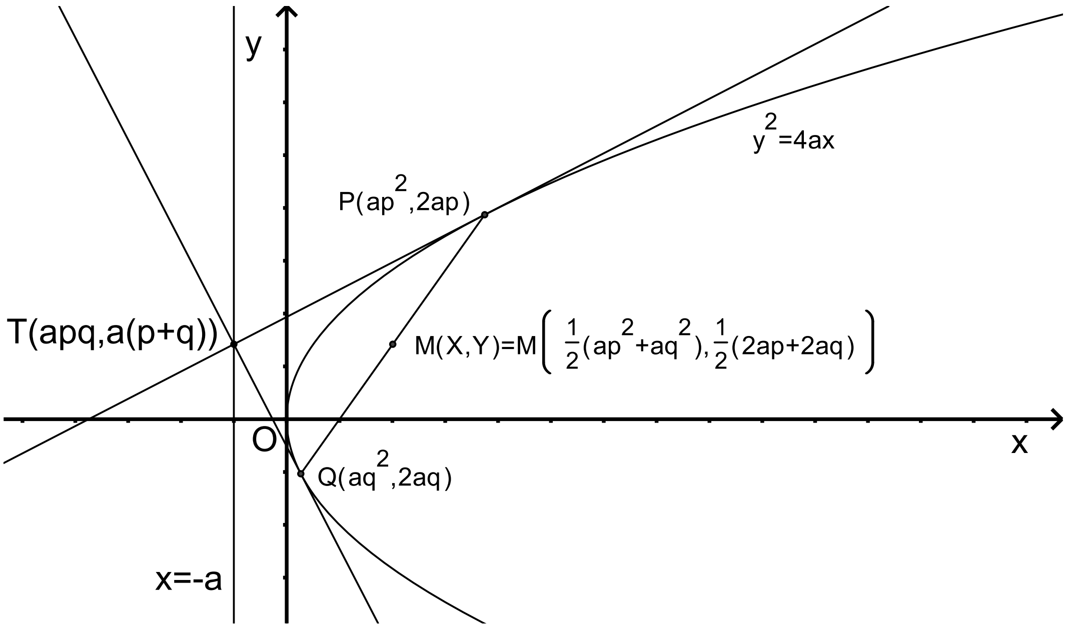
This, we notice, is the equation of a parabola which has the same axis and vertex as the one we started with, but its focus is at $(9a/8, 0)$.

18. Find the locus of the mid-points of chords of contact of tangents to the parabola $y^2 = 4ax$ drawn from points on the directrix.

Solution

Let the general point on the directrix ($x = -a$) be T and the points of contact drawn from this point to the parabola be P and Q with parameters p and q. Now the coordinates of the point of intersection of these tangents, which meet at T, are $(apq, a(p+q))$. However T is on the directrix, $x = -a$, and hence $apq = -a$, i.e.

$$pq = -1 \ .$$



Problem 18 (Geogebra file)

Now if M(X, Y) is the mid-point of the chord,

$$X = \tfrac{1}{2}(ap^2 + aq^2) \ \text{ and } \ Y = \tfrac{1}{2}(2ap + 2aq)$$

i.e.

$$p^2 + q^2 = \frac{2X}{a} \ \text{ and } \ p + q = \frac{Y}{a} \ .$$

To find the equation of the locus of (X, Y), we eliminate p and q, using the identity $(p + q)^2 \equiv p^2 + q^2 + 2pq$ and the fact that $pq = -1$, i.e.

$$\left(\frac{Y}{a}\right)^2 = \frac{2X}{a} - 2$$

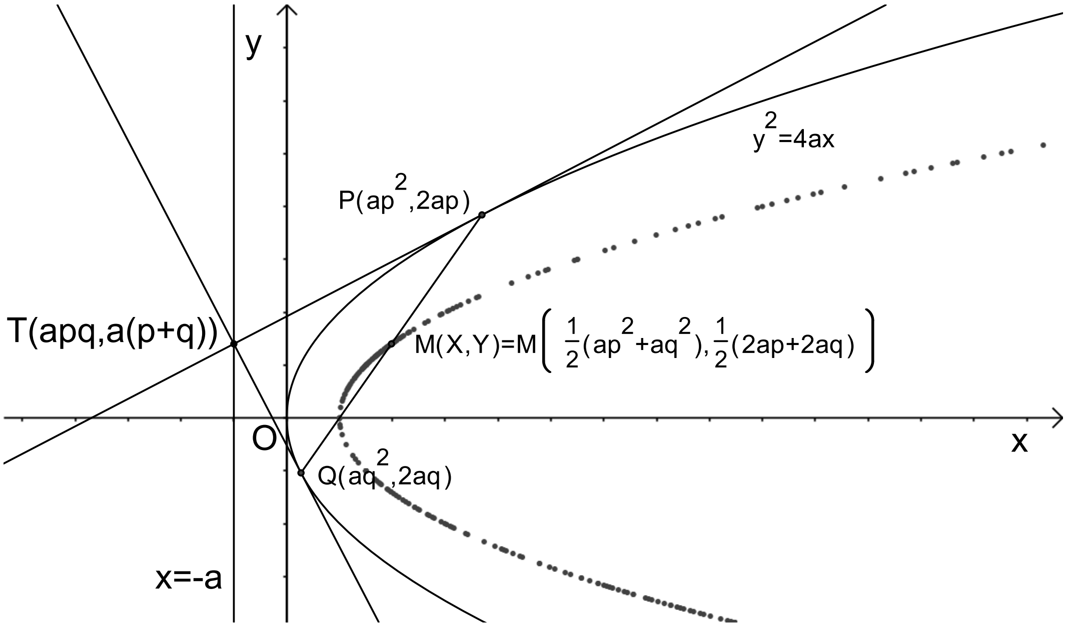
i.e.

$$Y^2 = 2aX - 2a^2 \ .$$

The equation of the locus of the point (X, Y) is therefore

$$y^2 = 2a(x - a) \ .$$

From the form of this equation, which can be written as $y^2 = 4(\frac{1}{2}a)(x - a)$, we see that the locus is a parabola with vertex at $(a, 0)$, focus $(a/2 + a, 0) = (3a/2, 0)$, directrix $x = -a/2 + a = a/2$ and latus rectum $4 \times \frac{1}{2}a = 2a$. (Note, again, that the focus, vertex, directrix and latus rectum of $y^2 = 4bx$ are $(b, 0)$, $(0, 0)$, $x = -b$ and $4b$, respectively.



Problem 18 - locus (Geogebra file)

19. H is a fixed point on the axis of the parabola $y^2 = 4ax$ whose focus is S$(a, 0)$. A variable chord PQ passes through H, the tangents at P and Q meet in T, and R is the foot of the perpendicular from T to PQ. Find the locus of R.

$\boxed{\textbf{Solution}}$ Let the parabola be $y^2 = 4ax$, P and Q the points with parameters p and q, and H the point $(h, 0)$. The equation of PQ is

$$x - \tfrac{1}{2}(p + q)y + apq = 0 \tag{1}$$

and, since this passes through $(h, 0)$,

$$h + apq = 0 \ . \tag{2}$$

(To keep the symmetry we continue to work with both p and q, noting this relation for future use.)

The tangents at P and Q meet in T$(apq, a(p + q))$. The gradient of PQ is $2/(p + q)$, and so the equation of the line through T perpendicular to PQ is therefore

$$\frac{y - a(p + q)}{x - apq} = -\tfrac{1}{2}(p + q)$$

i.e.

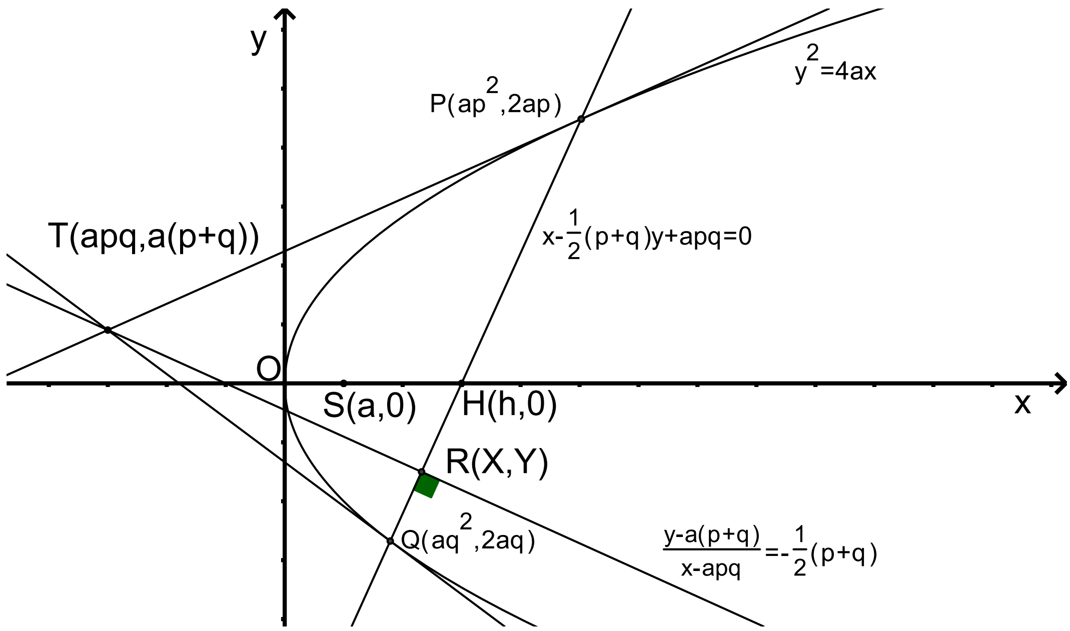
$$(p + q)x + 2y = apq(p + q) + 2a(p + q)$$

i.e.

$$(p + q)x + 2y = (p + q)(apq + 2a) \ .$$

Thus if R is (X, Y), X and Y satisfy this equation, i.e.

$$(p + q)X + 2Y = (p + q)(apq + 2a) \tag{3}$$



Problem 19 (Geogebra file)

and the equation of PQ from (1), i.e.

$$X - \tfrac{1}{2}(p+q)Y + apq = 0 \ . \tag{4}$$

If we now use relation (2) to substitute $apq = -h$ in (3) and (4) then these become

$$(p+q)X + 2Y = (p+q)(2a-h) \ \text{ and } \ X - \tfrac{1}{2}(p+q)Y - h = 0 \ . \tag{5}$$

To find X and Y separately from equations (5) would be rather tedious. However, since our ultimate aim is an equation in X and Y, not involving p or q, it is simpler to eliminate $p+q$ from the equations in (5) by first rewriting as

$$(p+q)(X+h-2a) = -2Y \ \text{ and } \ (p+q)Y = 2X - 2h \ . \tag{6}$$

and then equating $p+q$ obtained from the equations in (6), we have

$$\frac{-2Y}{X+h-2a} = \frac{2X-2h}{Y} \ .$$

This gives

$$X^2 - h^2 - 2aX + 2ah + Y^2 = 0$$

so the equation of the locus of R is

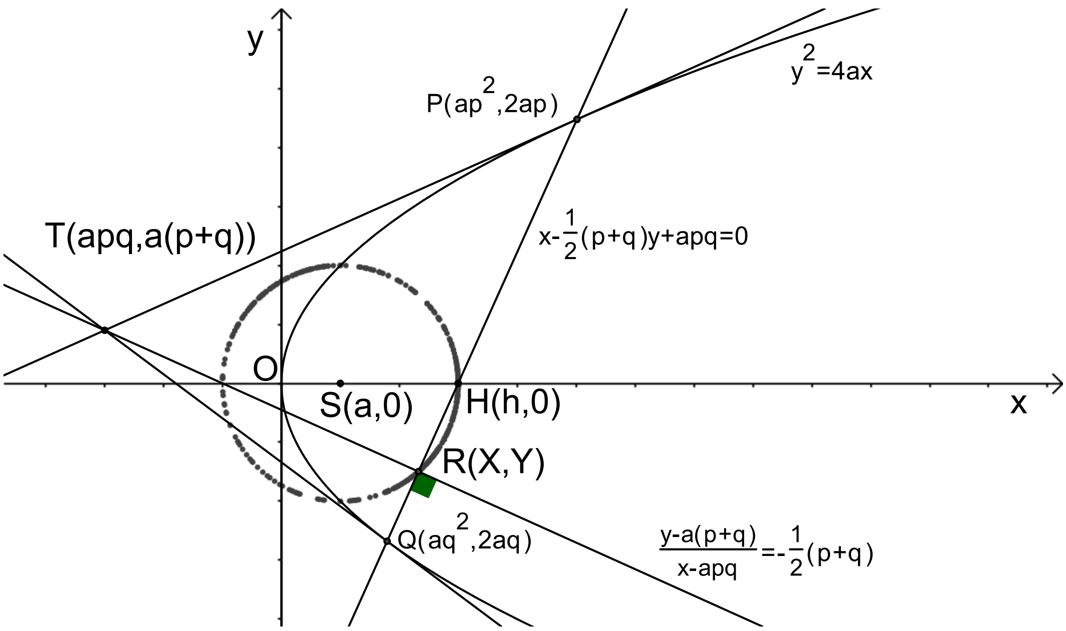
$$x^2 + y^2 - 2ax + 2ah - h^2 = 0$$

i.e.

$$(x-a)^2 + y^2 = (a-h)^2 \ .$$

We identify this as a circle whose centre is $(a, 0)$, which is S, and radius is $|a-h|$, which is SH.

20. If the normal to the parabola at P meets the axis of the parabola at G and GP is produced, beyond P, to Q so that P is the mid-point of GQ, show that the equation of the locus of Q is $y^2 = 16a(x+2a)$.



Problem 19 - locus (Geogebra file)

Solution

Let P$(at^2, 2at)$ be a general point on the parabola $y^2 = 4ax$. The equation of the normal through P is given by

$$y + tx = at^3 + 2at$$

which meets the axis of the parabola (the x-axis) where $y = 0$, i.e.

$$0 + tx = at^3 + 2at$$

i.e.

$$x = at^2 + 2a$$

. assuming $t \neq 0$, i.e. P is *not* at the origin. Hence the coordinates of G are G$(at^2+2a, 0)$.

Denoting by Q(X, Y) the point on QP extended so that P is the mid-point of GQ, then

$$at^2 = \tfrac{1}{2}\left((at^2 + 2a) + X\right)$$

$$2at = \tfrac{1}{2}\left(0 + Y\right)$$

Hence

$$X = at^2 - 2a \qquad (i)$$

and

$$Y = 4at \qquad (ii)$$

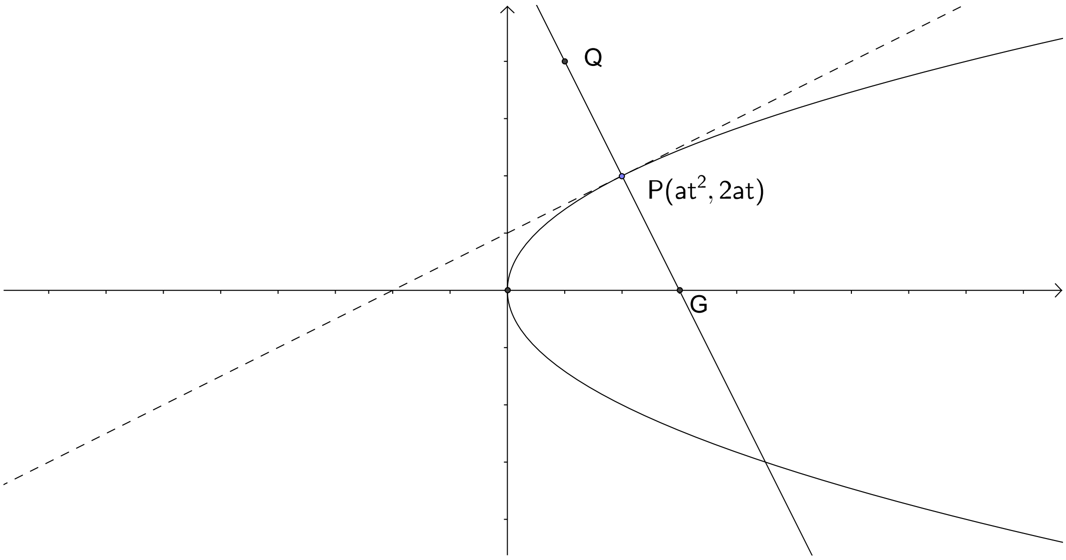
From (ii) $t = \dfrac{Y}{4a}$, and substituting this into (i) and simplifying gives the locus of the point Q as

$$\boxed{Y^2 = 16a(X + 2a)}$$

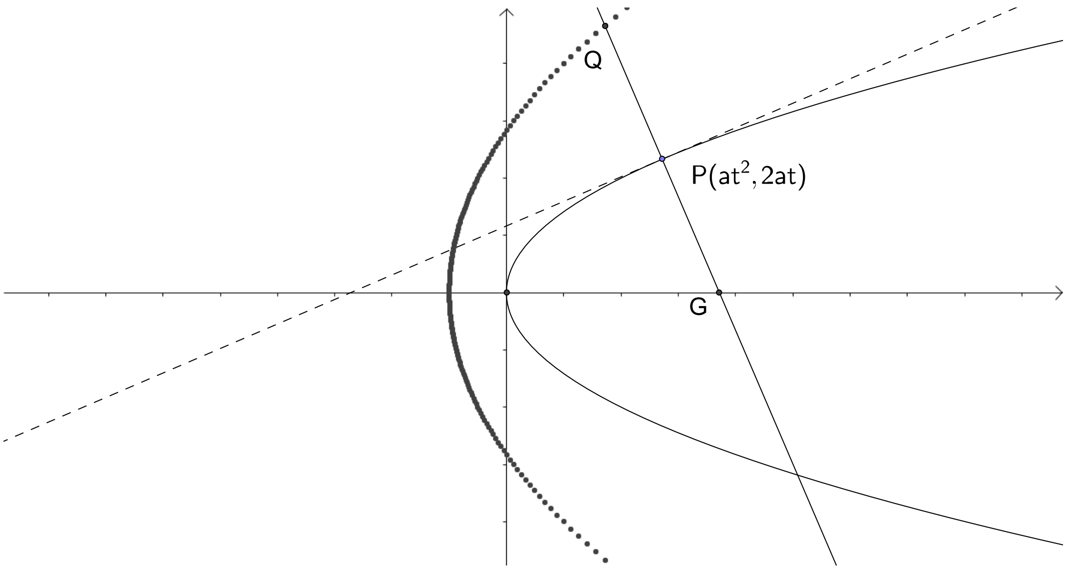
21. Find the coordinates of N, the foot of the perpendicular drawn from the origin to the tangent to the parabola at a general point P. Show that, as P varies, the locus of N is the curve $x(x^2 + y^2) + ay^2 = 0$.

Solution

Let P$(at^2, 2at)$ be a general point on the parabola $y^2 = 4ax$.



Problem 20 (Geogebra file)



Problem 20 - locus (Geogebra file)

The tangent at P has equation

$$x - ty + at^2 = 0 \qquad (i)$$

with slope $1/t$. The gradient of the line through O and perpendicular to the tangent has slope $-t$ and hence has equation

$$\frac{y-0}{x-0} = -t$$

i.e.

$$y + tx = 0 \qquad (ii)$$

The coordinates of the foot of the perpendicular from O to the tangent has coordinates N(X, Y) given by (i) and (ii):

$$X - tY + at^2 = 0 \qquad (iii)$$

$$Y + tX = 0 \qquad (iv)$$

Substituting for $Y = -tX$ from (iv) into (iii), and rearranging, gives

$$X = -\frac{at^2}{1+t^2} \qquad (v)$$

and substituting (v) into (iv) gives

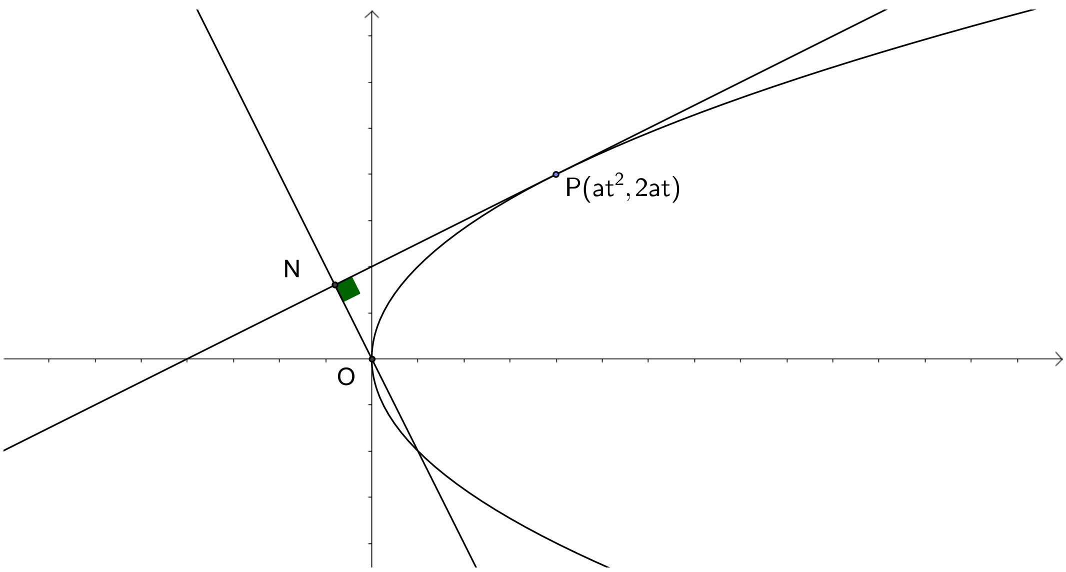
$$Y = \frac{at^3}{1+t^2} \qquad (vi)$$

To determine the locus of N as P varies we eliminate t from (iii) and (iv) by substituting $t = -Y/X$ from (iv) (assuming $X \neq 0$) into (iii) and simplifying to give

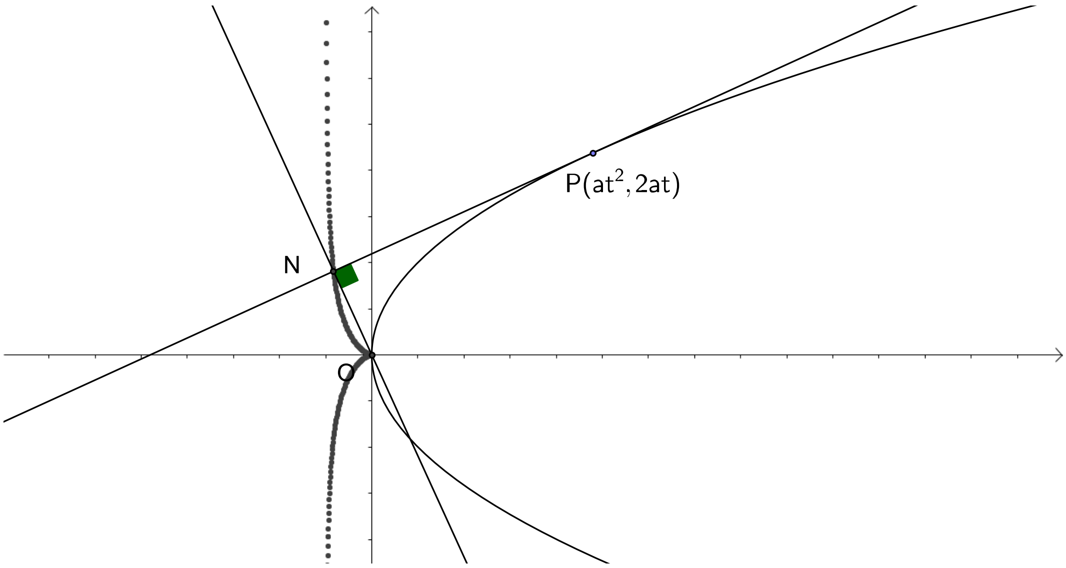
$$\boxed{X(X^2+Y^2) + aY^2 = 0}$$

.

(Note that if $X = 0$ then from (iv) we have $Y = 0$ and the point $(0,0)$ is automatically on the locus of N above.)



Problem 21 (Geogebra file)



Problem 21 - locus (Geogebra file)

22. P is any point on the parabola $y^2 = 4ax$, and O is the origin; Q is the foot of the perpendicular from P to the y-axis, R is the foot of the perpendicular from Q to OP, and QR produced meets the x-axis at K. Prove that K is a fixed point, and find its coordinates. Prove also that the locus of R is a circle, and find its centre.

Solution

Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$. The slope of the line OP is $\dfrac{2at-0}{at^2-0} = \dfrac{2}{t}$, and hence the slope of the line through Q perpendicular to OP, for which R is the foot, is $\dfrac{-1}{2/t} = -\frac{1}{2}t$. Thus the equation of the line through Q perpendicular to OP is

$$\frac{y-2at}{x-0} = -\tfrac{1}{2}t$$

i.e.

$$y = 2at - \tfrac{1}{2}xt \qquad (i)$$

The line in (i) meets the x-axis where $y = 0$, i.e.

$$0 = 2at - \tfrac{1}{2}xt$$

i.e.

$$t\left(2a - \tfrac{1}{2}x\right) = 0$$

which as t varies implies that $x = 4a$, and hence K has coordinates $K(4a, 0)$ and is a fixed point.

Now the equation of the line through OP is

$$\frac{y-0}{x-0} = \frac{2}{t}$$

i.e.

$$y = \frac{2x}{t} \qquad (ii)$$

and hence the coordinates of the point of intersection $R(X, Y)$ satisfy (i) and (ii), i.e.

$$Y = t\left(2a - \tfrac{1}{2}X\right) \qquad (iii)$$

$$Y = \frac{2X}{t} \qquad (iv)$$

Eliminating t between (iii) and (iv) by multiplying them together we have

$$\begin{aligned} Y^2 &= 2X(2a - \tfrac{1}{2}X) \\ &= 4aX - X^2 \end{aligned}$$

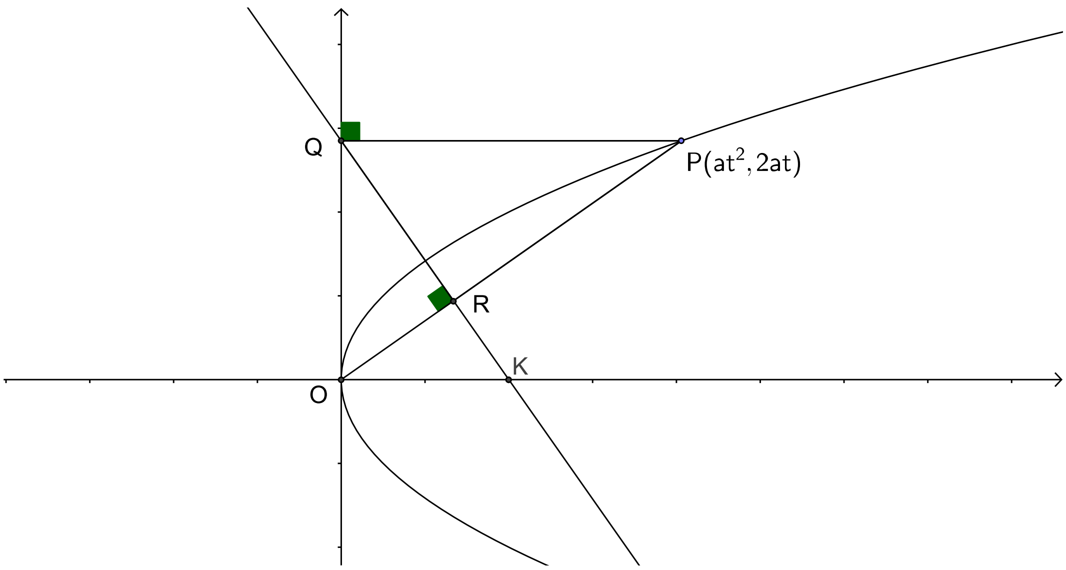
i.e.

$$X^2 - 4aX + Y^2 = 0$$

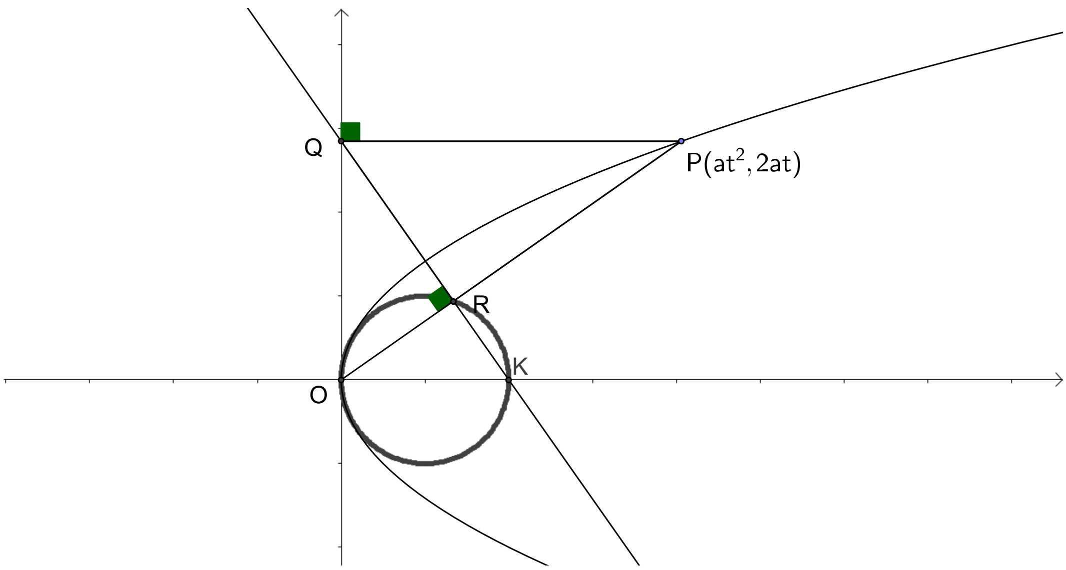
which can be written as

$$\boxed{(X - 2a)^2 + Y^2 = (2a)^2}$$

i.e. R lies on a circle centre $(2a, 0)$ and radius $2a$.



Problem 22 (Geogebra file)



Problem 22 - locus (Geogebra file)

23. Find the coordinates of R, the point of intersection of the tangents to the parabola at P and Q, the points with parameters p and q. Show that, if PQ touches the parabola $y^2 = 2ax$, the point R lies on the parabola $y^2 = 8ax$.

Let P$(ap^2, 2ap)$ and Q$(aq^2, 2aq)$ be two points on the parabola $y^2 = 4ax$. The equation of the tangents at P and Q are, respectively,

$$x - py + ap^2 = 0$$

$$x - qy + aq^2 = 0$$

so that if R(X, Y) is the point of intersection of these then

$$X - pY + ap^2 = 0 \qquad (i)$$

$$X - qY + aq^2 = 0 \qquad (ii)$$

Subtracting $(i) - (ii)$ gives

$$(q - p)Y = a(q^2 - p^2)$$

i.e.

$$\begin{aligned} Y &= a\left(\frac{q^2 - p^2}{q - p}\right) \\ &= a\frac{(q-p)(p+q)}{q-p} \\ &= a(p+q) \qquad (iii) \end{aligned}$$

Substituting for Y from (iii) into (i) gives

$$\begin{aligned} X &= p \times a(p+q) - ap^2 \\ &= apq \end{aligned}$$

Hence the coordinates of the point of intersection of the tangents are R$(apq, a(p+q))$.

Alternatively from (i) and (ii) given the point R(X, Y) one can draw from this two tangents to the parabola which touch it at P$(ap^2, 2ap)$ and Q$(aq^2, 2aq)$ where p, q are the roots of the quadratic equation (in t) given by

$$X - tY + at^2 = 0$$

Now the sum of the roots of this quadratic equation is:

$$p + q = \frac{Y}{a}$$

and the product of the roots is:

$$pq = \frac{X}{a}$$

and hence the coordinates of the point of intersection of the tangents at P and Q are R$(X, Y) = (apq, a(p+q))$.

Consider now the parabola $y^2 = 2ax$, which can be written as $y^2 = 4(\frac{1}{2}a)x$, and consider the tangent at a general point T$((\frac{1}{2}a)t^2, 2(\frac{1}{2}a)t) =$ T$(\frac{1}{2}at^2, at)$ given by

$$x - ty + \tfrac{1}{2}at^2 = 0 \qquad (iv)$$

If this meets the parabola $y^2 = 4ax$ at two points P$(ap^2, 2ap)$ and Q$(aq^2, 2aq)$ (so that the chord PQ is a tangent to the parabola $y^2 = 2ax$) then the coordinates of these two points satisfy (iv), i.e.

$$ap^2 - t(2ap) + \tfrac{1}{2}at^2 = 0 \qquad (v)$$

$$aq^2 - t(2aq) + \tfrac{1}{2}at^2 = 0 \qquad (vi)$$

i.e. p, q are roots of the quadratic equation (in u):

$$au^2 - 2atu + \tfrac{1}{2}at^2 = 0 \qquad (vii)$$

Hence the sum of the roots:

$$p + q = \frac{2at}{a} = 2t \qquad (viii)$$

and the product of the roots:

$$pq = \frac{\frac{1}{2}at^2}{a} = \tfrac{1}{2}t^2 \qquad (ix)$$

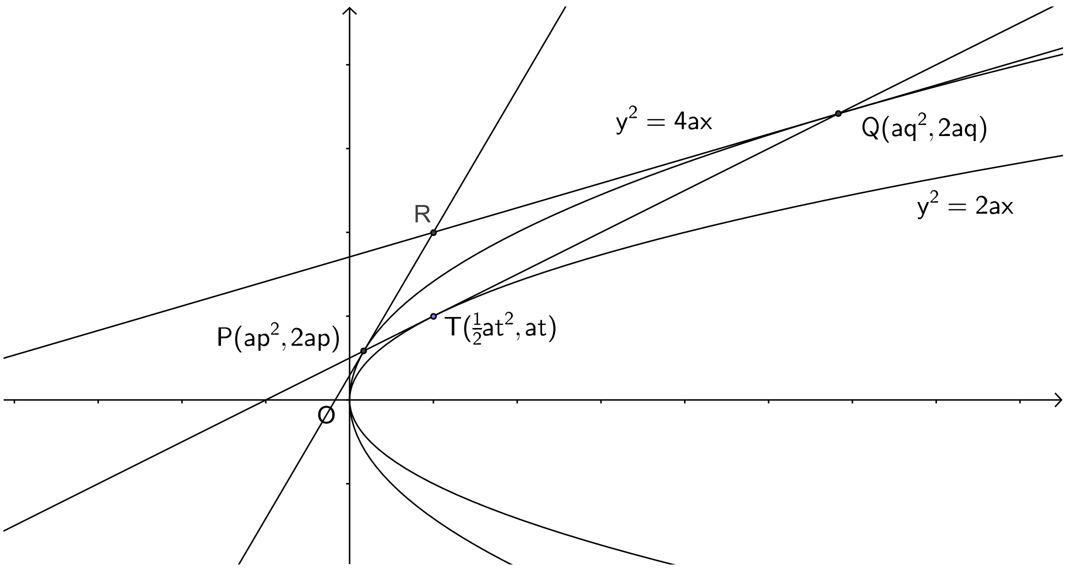
However, the point of intersection of the tangents $\mathrm{R}(X, Y) = \mathrm{R}(apq, a(p+q))$, and hence from $(viii)$ and (ix)

$$X = apq = \tfrac{1}{2}at^2 \qquad (x)$$

$$Y = a(p+q) = 2at \qquad (xi)$$

Eliminating t from (x) and (xi) gives the locus of the point of intersection of tangents as:

$$\boxed{Y^2 = 8aX}$$



Problem 23 (Geogebra file)

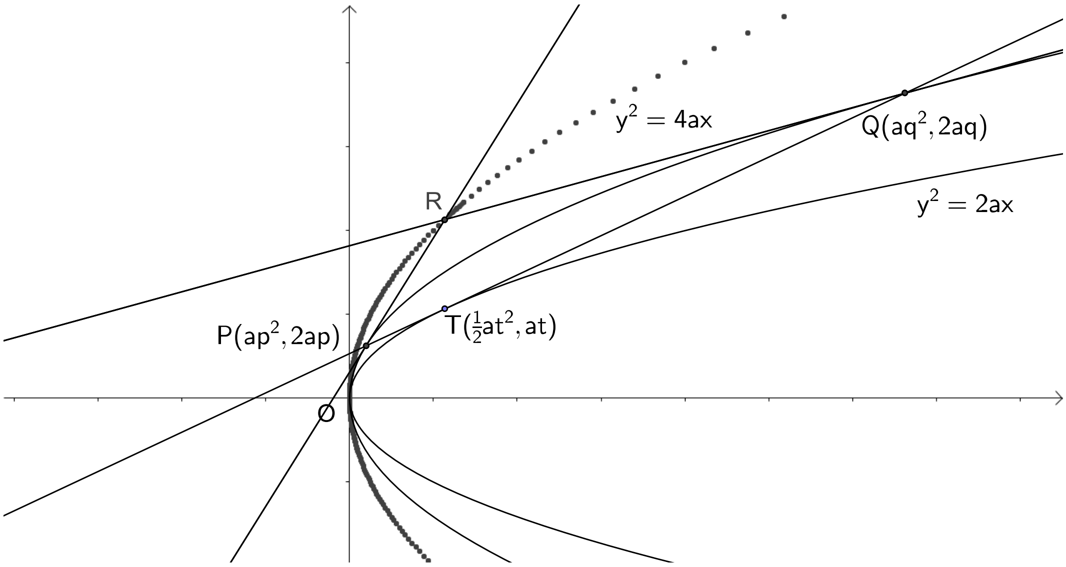
24. A tangent to the parabola $y^2 = 4ax$ at the point P meets the parabola $y^2 = 4bx$ at the points Q and R, and the tangents at P and Q meet at U. Find the locus of U as P varies.
Solution In this Problem there are two parabolas. We take the general point on $y^2 = 4ax$ as $\mathrm{P}(ap^2, 2ap)$, as usual, and on $y^2 = 4bx$ the general point will be $\mathrm{T}(bt^2, 2bt)$, with parameter t.

The equation of the tangent to $y^2 = 4ax$ at the point p is

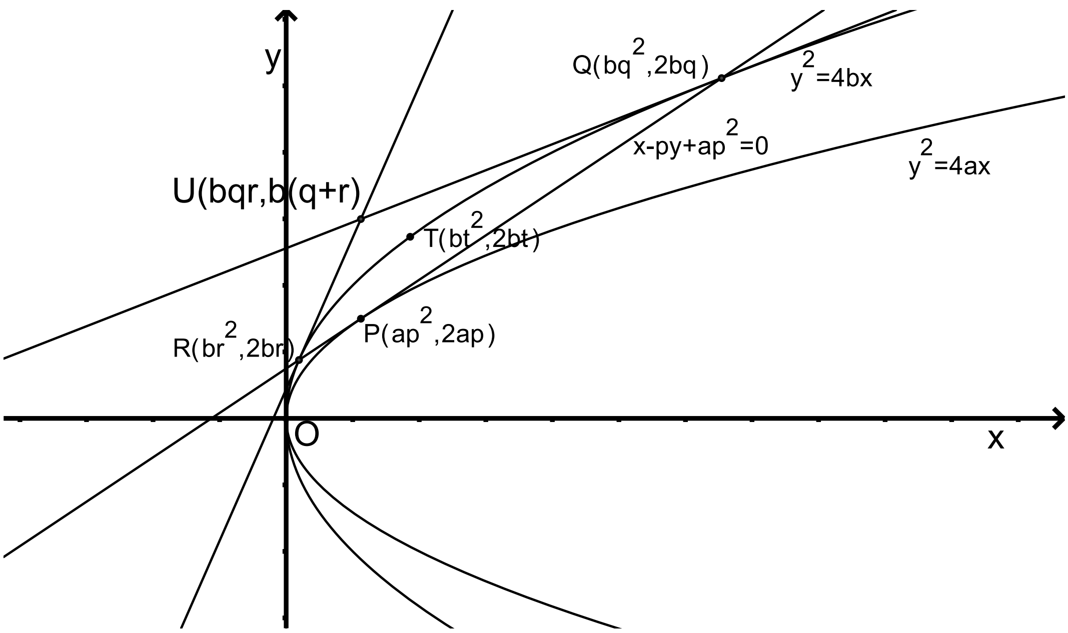
$$x - py + ap^2 = 0 \ .$$

This meets $y^2 = 4bx$ in points whose parameters are the roots of the equation

$$bt^2 - 2bpt + ap^2 = 0$$



Problem 23 - locus (Geogebra file)



Problem 24 (Geogebra file)

where this equation is thought of as a quadratic in t, the value of p for the moment being fixed.

If the roots are q and r, then

$$q + r = 2p \text{ and } qr = ap^2/b \ .$$

The tangents to $y^2 = 4bx$ at q and r meet in the point $(bqr, b(q + r))$.

If U is the point (X, Y) then

$$X = bqr = ap^2 \text{ and } Y = b(q + r) = 2bp \ .$$

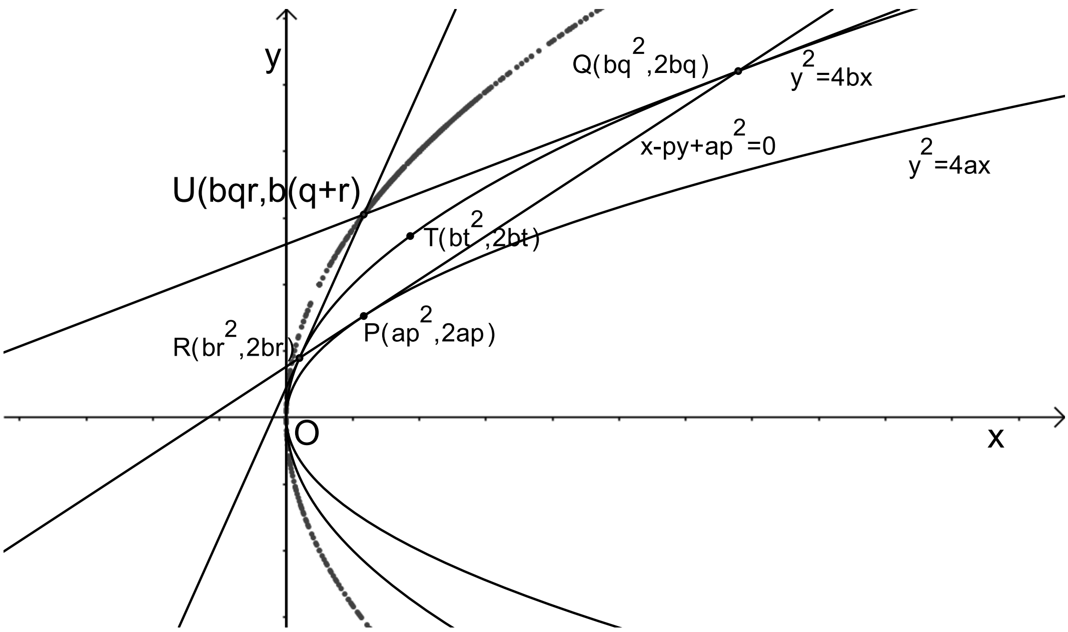
Eliminating p

$$\frac{X}{a} = \left(\frac{Y}{2b}\right)^2$$

and so the equation of the locus of U is

$$ay^2 = 4b^2x \text{ i.e. } y^2 = 4(b^2/a)x$$

another parabola, with the same vertex and axis as the two given ones, but with a different focus at $(b^2/a, 0)$ (unless $a = b$).



Problem 24 - locus (Geogebra file)

25. Find the locus of the mid-points of chords which subtend a right angle at the vertex of the parabola.
$\boxed{\textbf{Solution}}$ As usual the vertex is $(0, 0)$ and the equation of the parabola is $y^2 = 4ax$. Suppose that a typical chord joins the points P and Q with parameters p and q.

The gradient of OP is

$$2ap/ap^2 = 2/p \ , \text{ since } p \neq 0 \ .$$

Similarly, the gradient of OQ is $2/q$. Since OP and OQ are perpendicular

$$(2/p)(2/q) = -1$$

i.e.

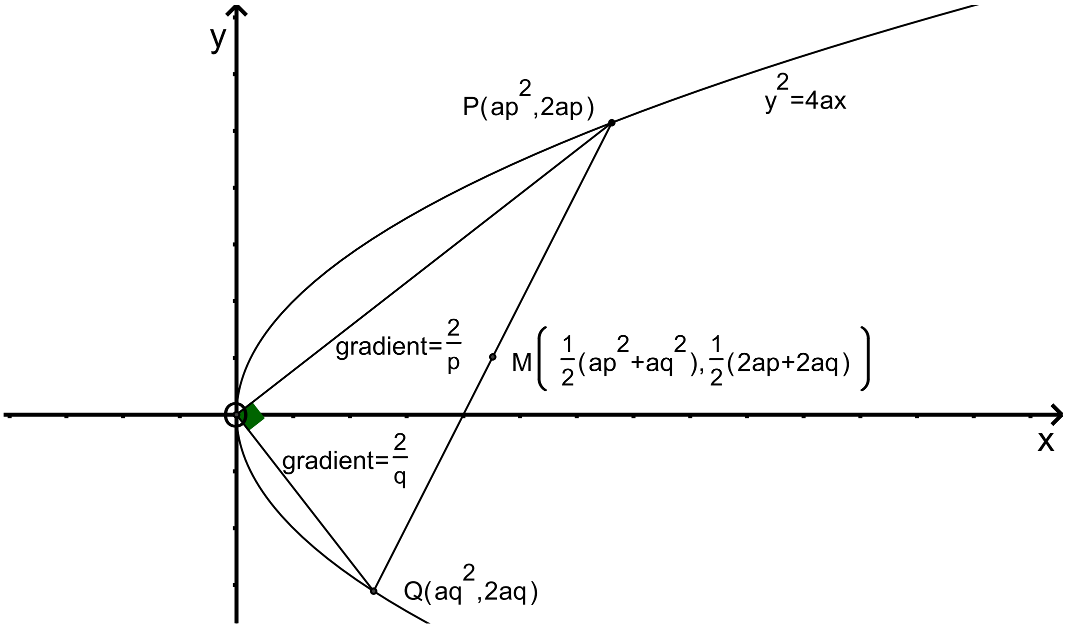
$$pq = -4 \ .$$

Now if M(X, Y) is the mid-point of the chord

$$X = \tfrac{1}{2}(ap^2 + aq^2) \ \text{ and } \ Y = \tfrac{1}{2}(2ap + 2aq)$$

i.e.

$$p^2 + q^2 = \frac{2X}{a} \ \text{ and } \ p + q = \frac{Y}{a} \ .$$



Problem 25 (Geogebra file)

To find the equation of the locus of (X, Y), we eliminate p and q, using the identity $(p+q)^2 \equiv p^2 + q^2 + 2pq$ and the fact that $pq = -4$, i.e.

$$\left(\frac{Y}{a}\right)^2 = \frac{2X}{a} - 8$$

i.e.

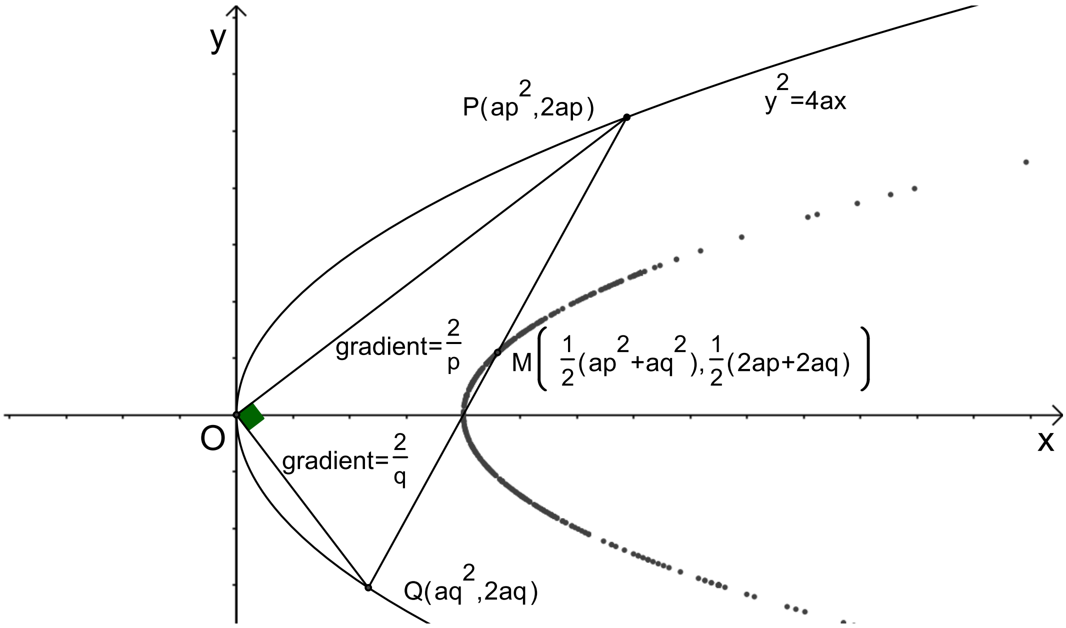
$$Y^2 = 2aX - 8a^2 \ .$$

The equation of the locus of the point (X, Y) is therefore

$$y^2 = 2a(x - 4a) \ .$$

From the form of this equation, which can be written as $y^2 = 4(\frac{1}{2}a(x - 4a)$, we see that the locus is a parabola with vertex at $(4a, 0)$, focus $(a/2 + 4a, 0) = (9a/2, 0)$, directrix $x = -a/2 + 4a = 7a/2$ and latus rectum $4 \times \frac{1}{2}a = 2a$. (Note that the focus, vertex, directrix and latus rectum of $y^2 = 4bx$ are $(b, 0)$, $(0, 0)$, $x = -b$ and $4b$, respectively.

26. The chord PQ of a parabola passes through the focus, and the normals at P and Q meet in R. If P and Q have parameters p and q show that $pq + 1 = 0$. Find the coordinates of R, and hence find its locus.



Problem 25 - locus (Geogebra file)

Solution

Equation of chord: $x - \frac{1}{2}(p+q)y + apq = 0$ (end points P$(ap^2, 2ap)$, Q$(aq^2, 2aq)$, as usual.)
Focal chord passes through S$(a, 0)$, i.e.

$$a - \frac{1}{2}(p+q) \times 0 + apq = 0$$

i.e.

$$a(1 + pq) = 0$$

i.e.

$$pq = -1$$

The point of intersection of the normals at P and Q has coordinates R(X, Y) given by

$$X = a(p^2 + pq + q^2 + 2) = a((p+q)^2 - pq + 2)$$

$$Y = -apq(p+q)$$

and with $pq = -1$ these become

$$X = a((p+q)^2 + 3) \qquad (i)$$

$$Y = a(p+q) \qquad (ii)$$

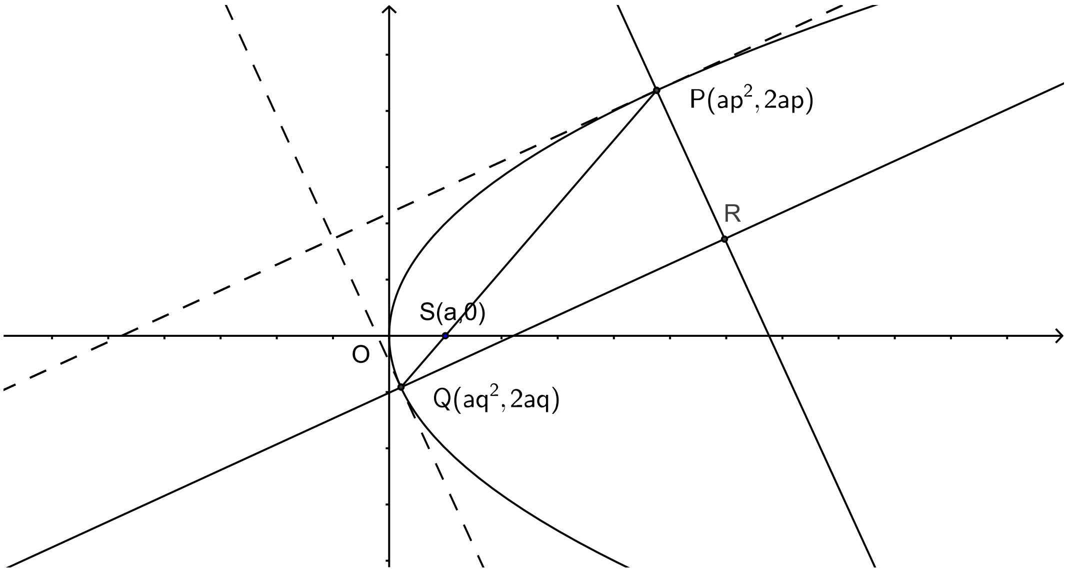
Eliminating $p + q$ from (i) and (ii) gives the locus of the point of intersection of normals at the ends a focal chord as:

$$\boxed{Y^2 = a(X - 3a)}$$

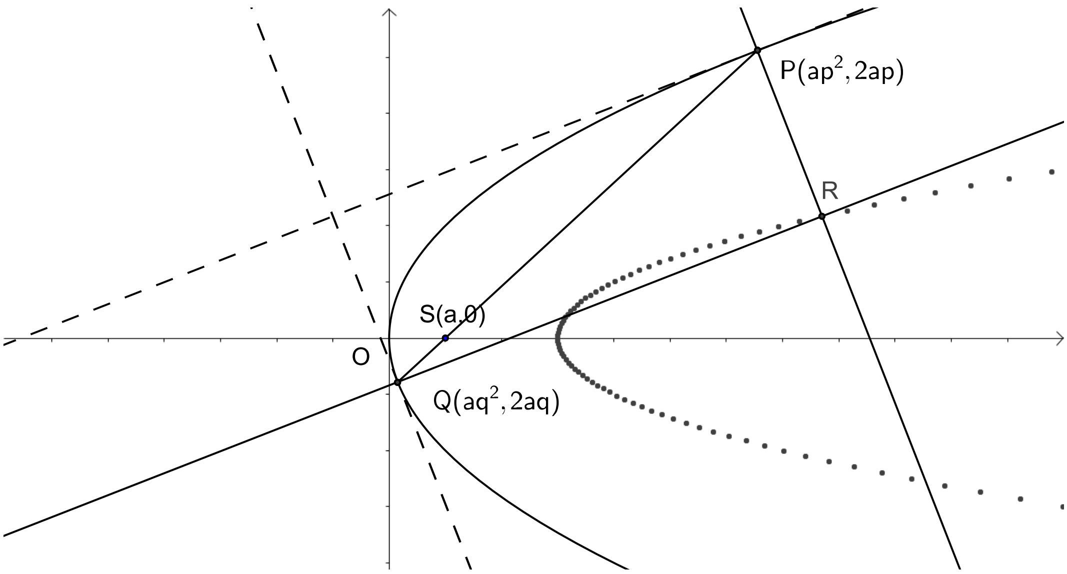
27. If two perpendicular tangents meet at T and the corresponding normals meet at N, show that, for all such pairs of tangents, TN is parallel to the axis of the parabola, and find the locus of N.

Solution

Let P$(ap^2, 2ap)$ and Q$(aq^2, 2aq)$ be two points on the parabola $y^2 = 4ax$. The point of intersection of the tangents at P and Q has coordinates T$(apq, a(p+q))$ and the point of



Problem 26 (Geogebra file)



Problem 26 - locus (Geogebra file)

intersection of the normals at P and Q has coordinates $\mathrm{N}(a(p^2+pq+q^2+2), -apq(p+q)) = \mathrm{N}(a((p+q)^2 - pq + 2), -apq(p+q))$. If the tangents at P and Q are perpendicular then their slopes satisfy:

$$\frac{1}{p} \times \frac{1}{q} = -1$$

i.e. $pq = -1$. Thus the points T and N have coordinates $T(-a, a(p+q))$ and $\mathrm{N}(a((p+q)^2+3), a(p+q))$, and since the y coordinate of both T and N is $a(p+q)$ then TN is horizontal and parallel to the axis of the parabola (the x-axis).

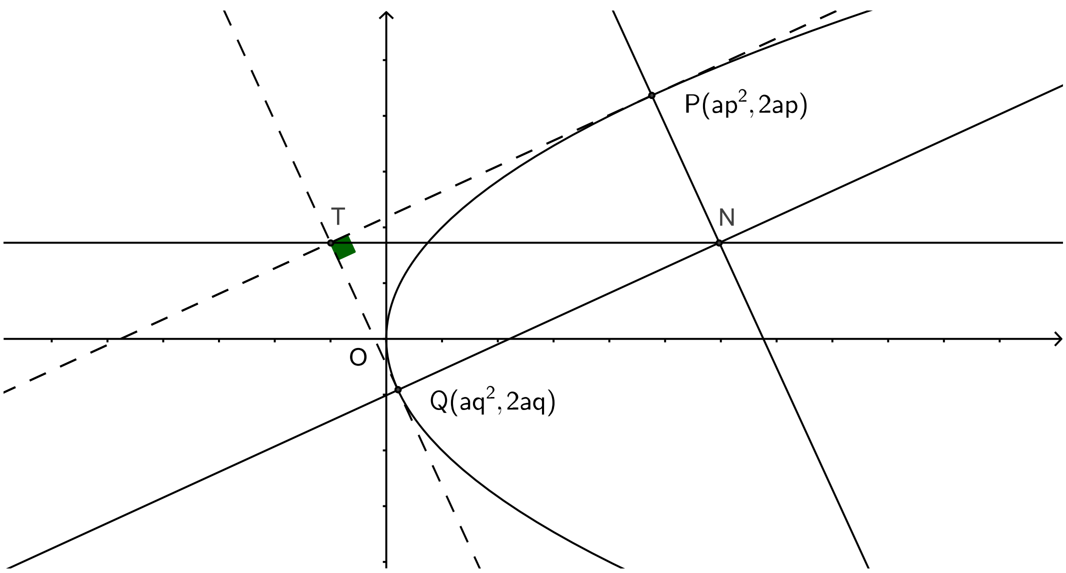
Denoting the coordinates of N by $\mathrm{N}(X, Y)$ we have

$$X = a((p+q)^2 + 3) \qquad (i)$$

$$Y = a(p+q) \qquad (ii)$$

and eliminating $p+q$ from (i) and (ii) gives the locus of the point of intersection of the normals (which are also perpendicular) as

$$\boxed{Y^2 = a(X - 3a)}$$



Problem 27 (Geogebra file)

28. Find the locus of the point of intersection of the normals at two points on a parabola when the chord joining them subtends a right angle at the vertex.

Solution

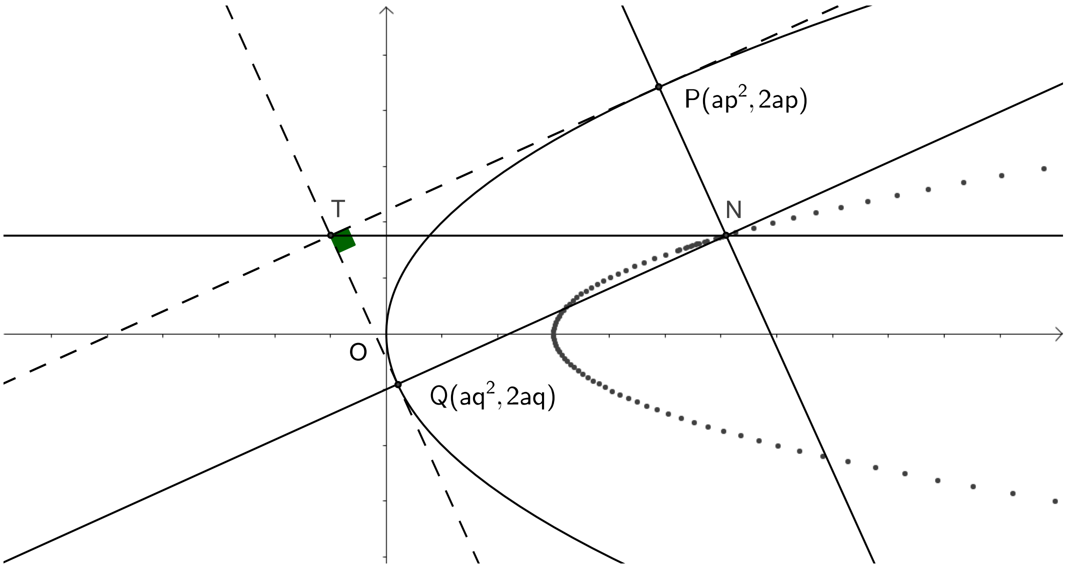
Let $\mathrm{P}(ap^2, 2ap)$, $\mathrm{Q}(aq^2, 2aq)$ be two points on the parabola $y^2 = 4ax$. If O denotes the vertex $(0,0)$ then the slope of the line OP is

$$\frac{2ap - 0}{ap^2 - 0} = \frac{2}{p}$$

and similarly the slope of the line OQ is $\dfrac{2}{q}$.

Now since the chord PQ subtends a right angle at the vertex then these lines are perpendicular, i.e.

$$\frac{2}{p} \times \frac{2}{q} = -1$$



Problem 27 - locus (Geogebra file)

and hence $pq = -4$.

Denoting by $\mathrm{N}(X, Y)$ the point of intersection of the normals at P and Q, then we have, as usual,

$$X = a((p+q)^2 - pq + 2) \qquad (i)$$

$$Y = -apq(p+q) \qquad (ii)$$

However, since $pq = -4$, (i) and (ii) become

$$X = a((p+q)^2 + 6) \qquad (iii)$$

$$Y = 4a(p+q) \qquad (iv)$$

Subsituting for $p+q$ from (iv) into (iii) gives

$$X = a\left(\left(\frac{Y}{4a}\right)^2 + 6\right)$$

and hence the locus of the points of intersection of normals is

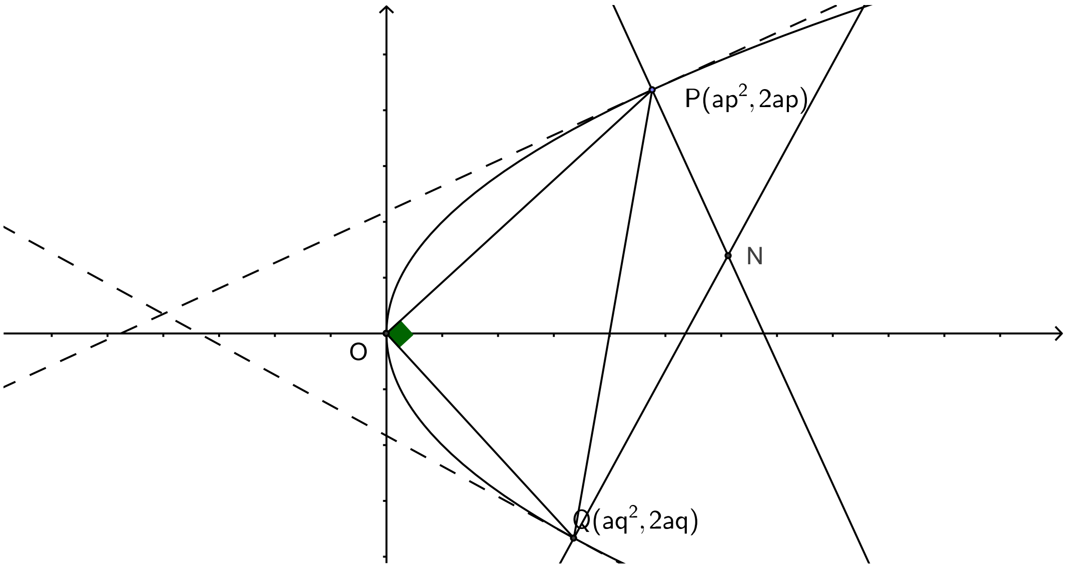
$$\boxed{Y^2 = 16a(X - 6a)}$$

29. P and Q are two points on a parabola whose focus is S. The tangents at P and Q meet at T, and the normals at these points meet at N. R is the mid-point of TN. Prove that the angle TSR is a right angle. If the chord PQ passes through the focus of the parabola, prove that the locus of R is a parabola with its axis coinciding with that of the first parabola, and vertex S.

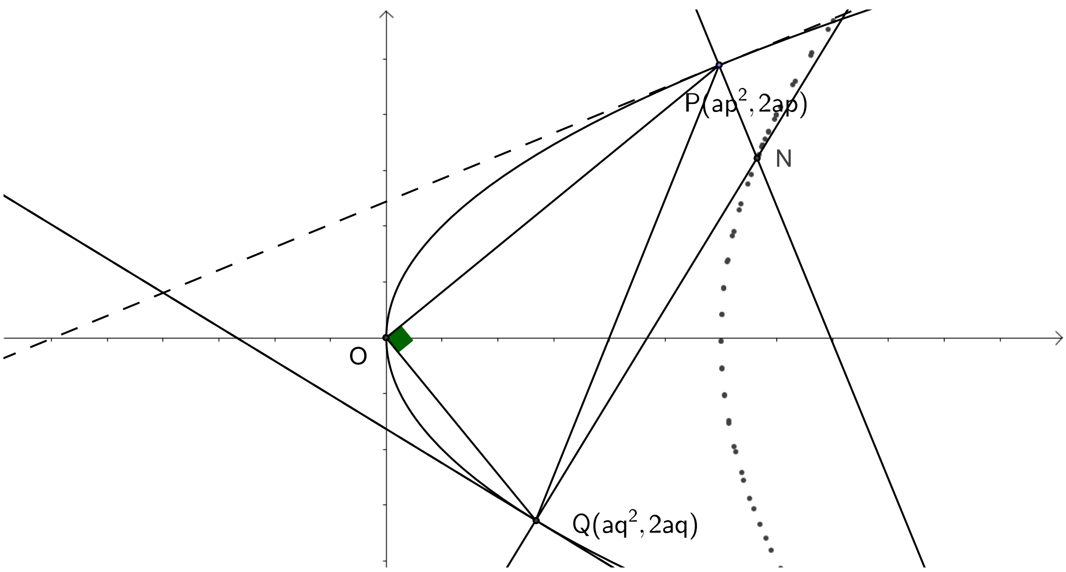
Solution

Let $\mathrm{P}(ap^2, 2ap)$, $\mathrm{Q}(aq^2, 2aq)$ be two points on the parabola $y^2 = 4ax$. The coordinates of the point of intersection T of the tangents at P and Q are $\mathrm{T}(apq, a(p+q))$, and of the point of intersection of the normals $\mathrm{N}((a(p+q)^2 - pq + 2), -apq(p+q))$. Hence the coordinates of the mid-point, (X, Y) are given by

$$\begin{aligned} X &= \tfrac{1}{2}\left(apq + a((p+q)^2 - pq + 2)\right) \\ &= \tfrac{1}{2}a\left((p+q)^2 + 2)\right) \qquad (i) \end{aligned}$$



Problem 28 (Geogebra file)



Problem 28 - locus (Geogebra file)

$$\begin{aligned} Y &= \tfrac{1}{2}\left(a(p+q) - apq(p+q)\right) \\ &= \tfrac{1}{2}a(1-pq)(p+q) \qquad (ii) \end{aligned}$$

The slope of the line SR is therefore

$$\frac{\frac{1}{2}a(1-pq)(p+q) - 0}{\frac{1}{2}a\left((p+q)^2+2)\right) - a} = \frac{1-pq}{p+q} \qquad (iii)$$

and the slope of TS is

$$\frac{0 - a(p+q)}{a - apq} = \frac{p+q}{pq-1} \qquad (iv)$$

and the product of the slopes in (iii) and (iv) is

$$\frac{1-pq}{p+q} \times \frac{p+q}{pq-1} = -1$$

and hence the angle TSR is a right angle.

If the chord PQ passes through the focus S$(a, 0)$ then from the equation of the chord

$$x - \tfrac{1}{2}a(p+q)y + apq = 0$$

we have in particular that

$$a - \tfrac{1}{2}a(p+q) \times 0 + apq = 0$$

i.e.

$$a(1+pq) = 0$$

and hence $pq = -1$.

In this case, from (i) and (ii) the coordinates of the mid-point of TN become

$$X = \tfrac{1}{2}a\left((p+q)^2+2)\right) \qquad (v)$$
$$Y = a(p+q) \qquad (vi)$$

and substituting for $p+q$ from (vi) into (v) gives

$$X = \tfrac{1}{2}a\left(\left(\frac{Y}{a}\right)^2 + 2)\right)$$

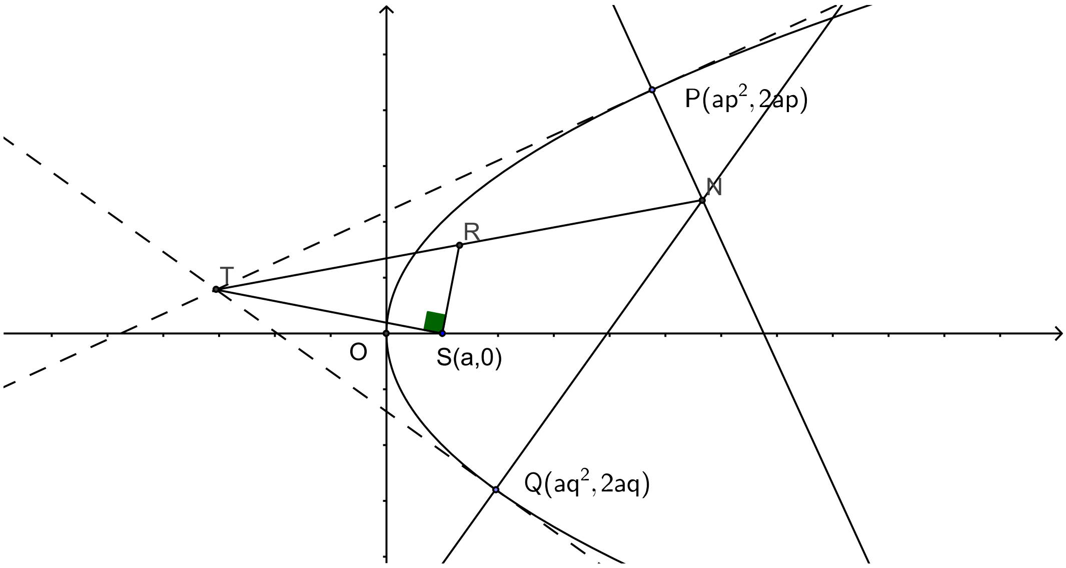
and hence the locus of R in this case is

$$\boxed{Y^2 = 2a(X-a)}$$

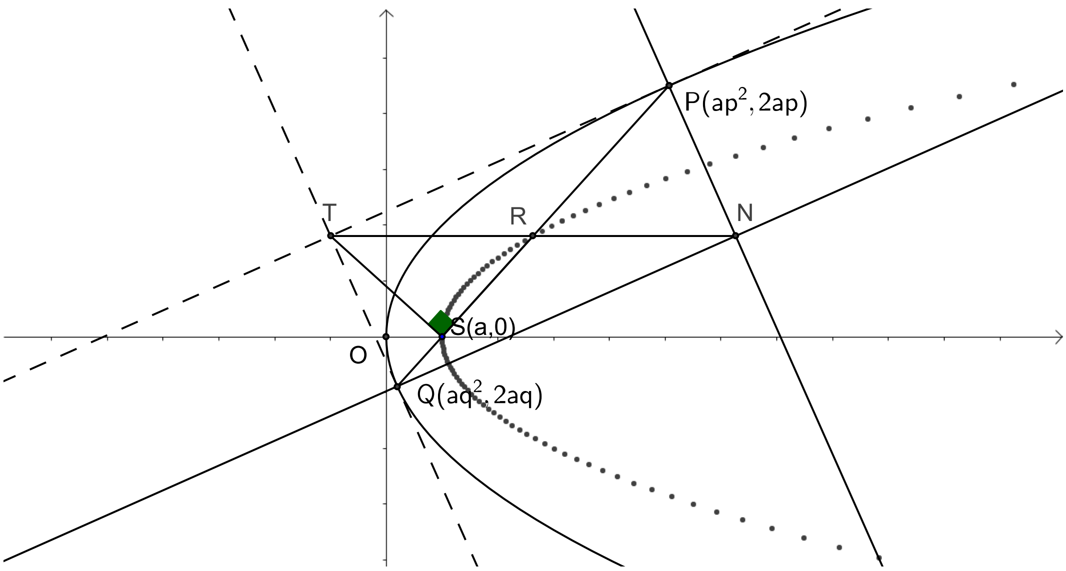
This is a parabola with axis $y = 0$, coinciding with the axis of the original parabola $y^2 = 4ax$, and a vertex at $x = 0 \;\; y = 0$, i.e. at the point $(a, 0)$, which is the focus of the original parabola.

30. If OP, OQ are a variable pair of perpendicular chords through the vertex O of the parabola $y^2 = 4ax$, prove that the chord PQ cuts the axis in a fixed point. Find the equation of the locus of the point of intersection of the normals at P and Q to the parabola.

Solution



Problem 29 (Geogebra file)



Problem 29 - locus (Geogebra file)

Let $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$ be two points on the parabola $y^2 = 4ax$. The slope of the chord OP is

$$\frac{2ap - 0}{ap^2 - 0} = \frac{2}{p}$$

and similarly the slope of the chord OQ is $\dfrac{2}{q}$. If these are perpendicular then

$$\frac{2}{p} \times \frac{2}{q} = -1$$

and hence $pq = -4$.

Now the chord PQ has equation

$$x - \tfrac{1}{2}a(p+q)y + apq = 0$$

and since $pq = -4$ this can be written as

$$x - \tfrac{1}{2}a(p+q)y - 4a = 0$$

This crosses the axis, $y = 0$, where

$$x - \tfrac{1}{2}a(p+q) \times 0 - 4a = 0$$

i.e. $x = 4a$, and hence at a fixed point $(4a, 0)$

The point of intersection of the normals, $N(X, Y)$, has coordinates

$$X = a((p+q)^2 - pq + 2) \qquad (i)$$

$$Y = -apq(p+q) \qquad (ii)$$

However, since $pq = -4$, (i) and (ii) become

$$X = a((p+q)^2 + 6) \qquad (iii)$$

$$Y = 4a(p+q) \qquad (iv)$$

Subsituting for $p + q$ from (iv) into (iii) gives

$$X = a\left(\left(\frac{Y}{4a}\right)^2 + 6\right)$$

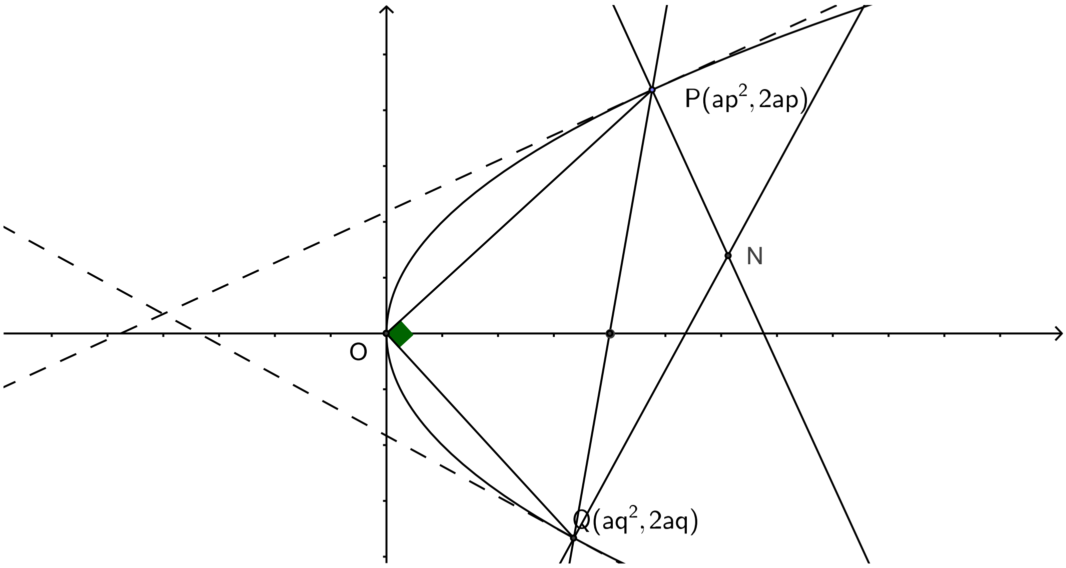
and hence the locus of the points of intersection of normals is

$$\boxed{Y^2 = 16a(X - 6a)}$$

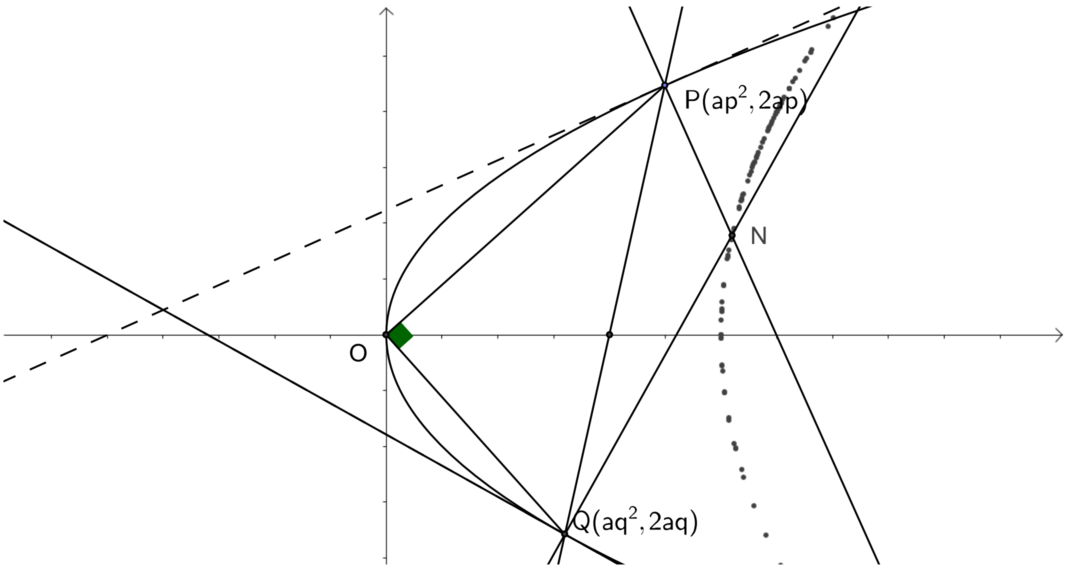
31. Find in each case the equation of the locus of the point of intersection of tangents to a parabola $y^2 = 4ax$ at P and Q, when (a) PQ is perpendicular to the tangent at P; (b) the tangents or normals at P and Q are perpendicular; (c) PQ passes through a fixed point on the directrix; (d) PQ is a focal chord.

Solution

In all cases let $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$ be two points on the parabola $y^2 = 4ax$, so that coordinates of the point of intersection of the tangents at P and Q are $T(X, Y)$ where



Problem 30 (Geogebra file)



Problem 30 - locus (Geogebra file)

$$X = apq \qquad (i)$$

$$Y = a(p+q) \qquad (ii)$$

(a) The slope of PQ is

$$\frac{2aq - 2ap}{aq^2 - ap^2} = \frac{2(q-p)}{(q-p)(p+q)} = \frac{2}{p+q}$$

and the slope of the tangent at P is $\frac{1}{p}$, and if these are perpendicular

$$\frac{1}{p} \times \frac{2}{p+q} = -1$$

i.e.

$$p(p+q) = -2 \qquad (iii)$$

or

$$p^2 + pq = -2 \qquad (iv)$$

(Alternatively, since the perpendicular to the tangent at P is the normal, whose slope is $-p$, then $\frac{2}{p+q} = -p$, giving the same expression in (iii) or (iv).) Subsituting for pq from (iv) into (i) gives

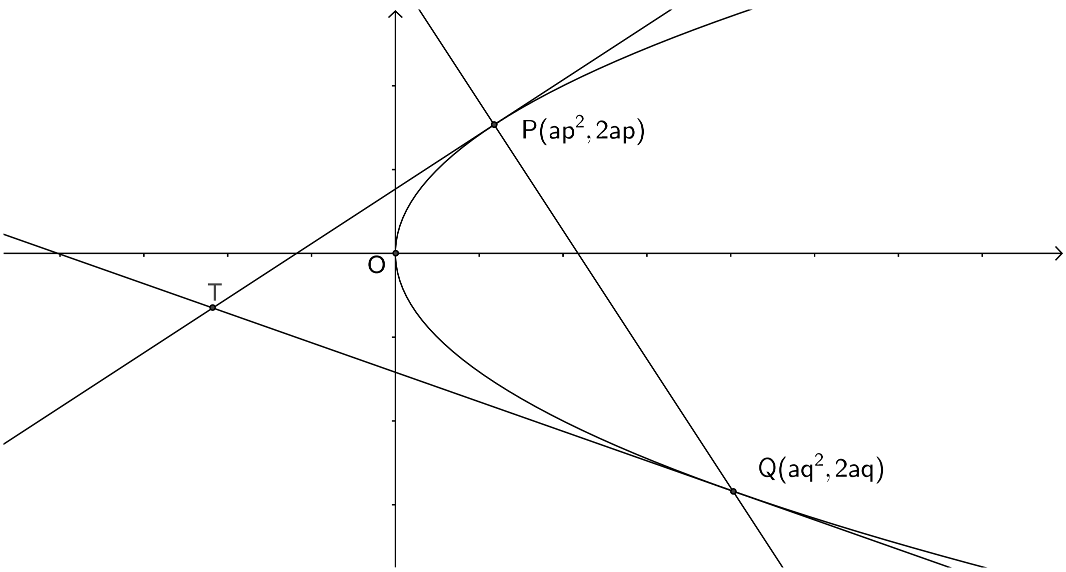
$$X = a(-2 - p^2) \qquad (v)$$

and for $p+q$ from (iii) into (ii) gives

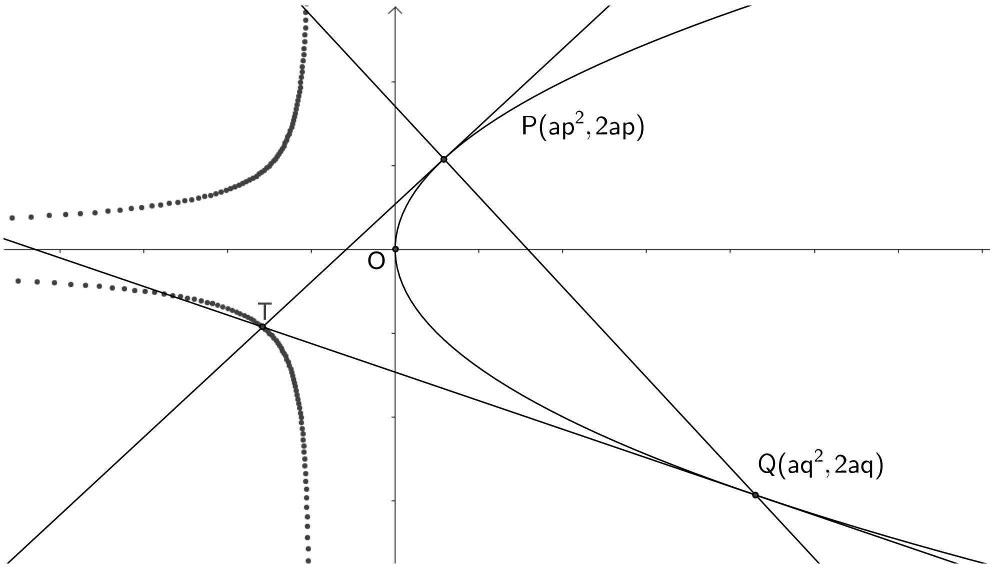
$$Y = -\frac{2a}{p} \qquad (vi)$$

and eliminating p from (v) and (vi) gives the locus

$$\boxed{XY^2 + 2aY^2 + 4a^3 = 0}$$



Problem 31a (Geogebra file)



Problem 31a - locus (Geogebra file)

(b) If the tangents or normals at P and Q are perpendicular the product of their slopes is

$$\frac{1}{p} \times \frac{1}{q} = -1$$

or

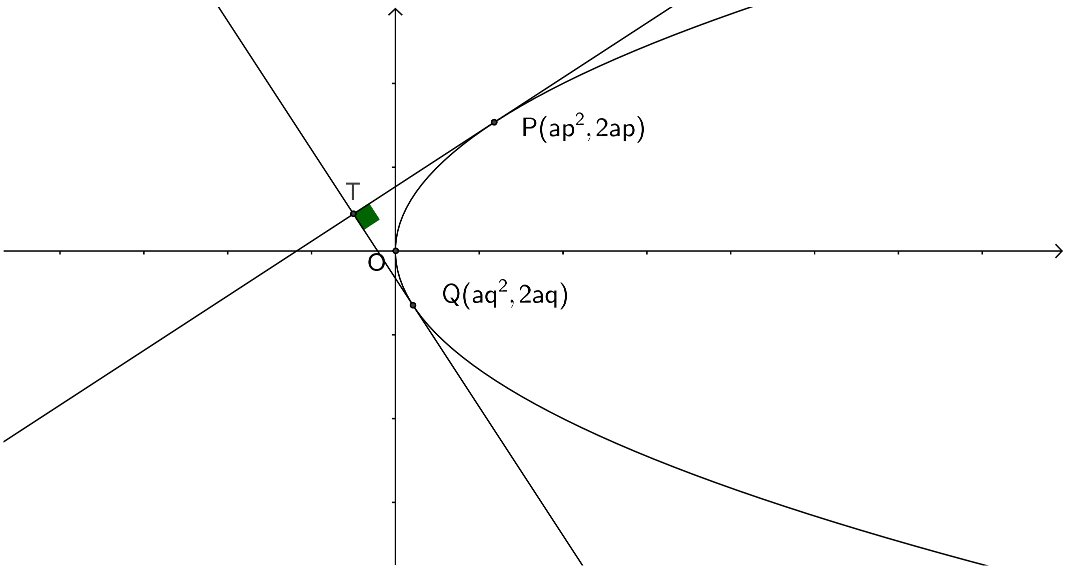
$$(-p) \times (-q) = -1$$

i.e. $pq = -1$.

Substituting $pq = -1$ into (i) gives

$$\boxed{X = -a}$$

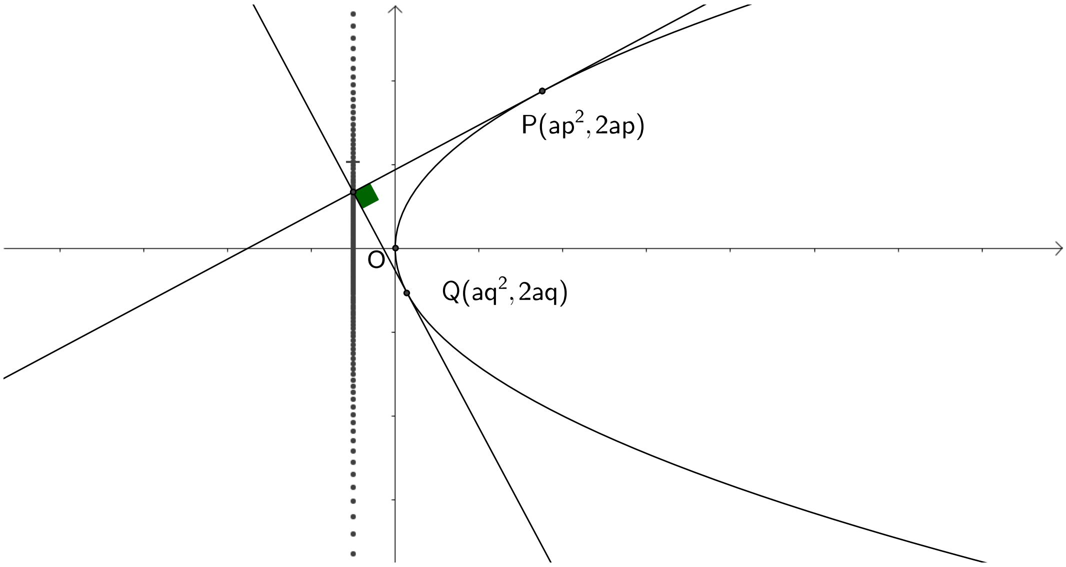
as the locus, which is the directrix.



Problem 31b (Geogebra file)

(c) If the chord PQ passes through a fixed point, F, on the directrix $x = -a$, with coordinates F$(-a, k)$, then from the equation of the chord PQ:

$$x - \tfrac{1}{2}(p+q)y + apq = 0$$



Problem 31b - locus (Geogebra file)

we have

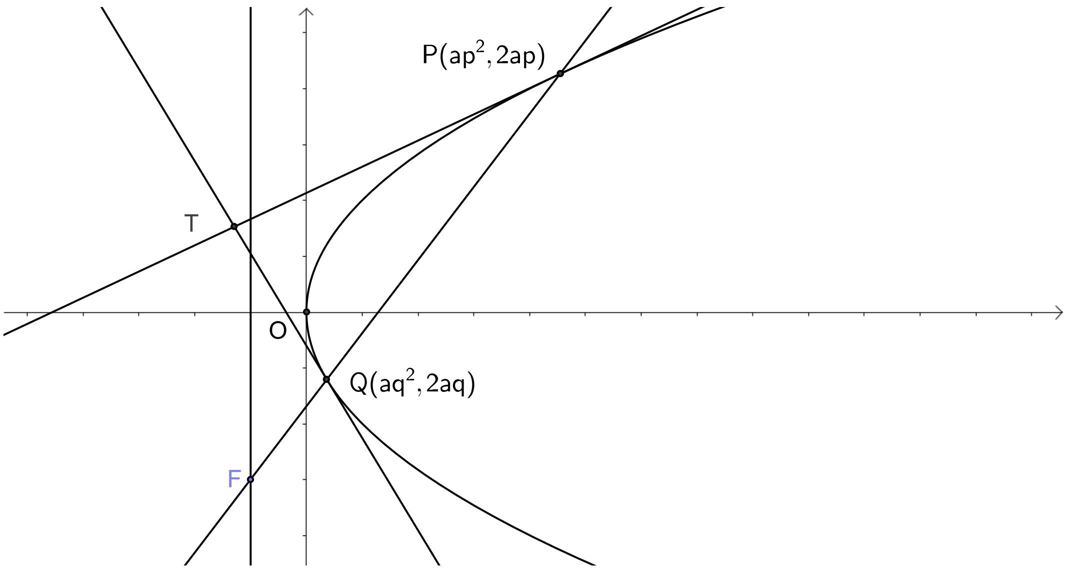
$$-a - \tfrac{1}{2}(p+q)k + apq = 0 \qquad (vii)$$

However, given that the point of intersection of the tangents has coordinates $\mathrm{T}(X, Y) = (apq, a(p+q))$, we have from (vii)

$$-a - \tfrac{1}{2}\left(\frac{Y}{a}\right)k + X = 0$$

giving the locus as the straight line

$$\boxed{2aX - kY = 2a^2}$$

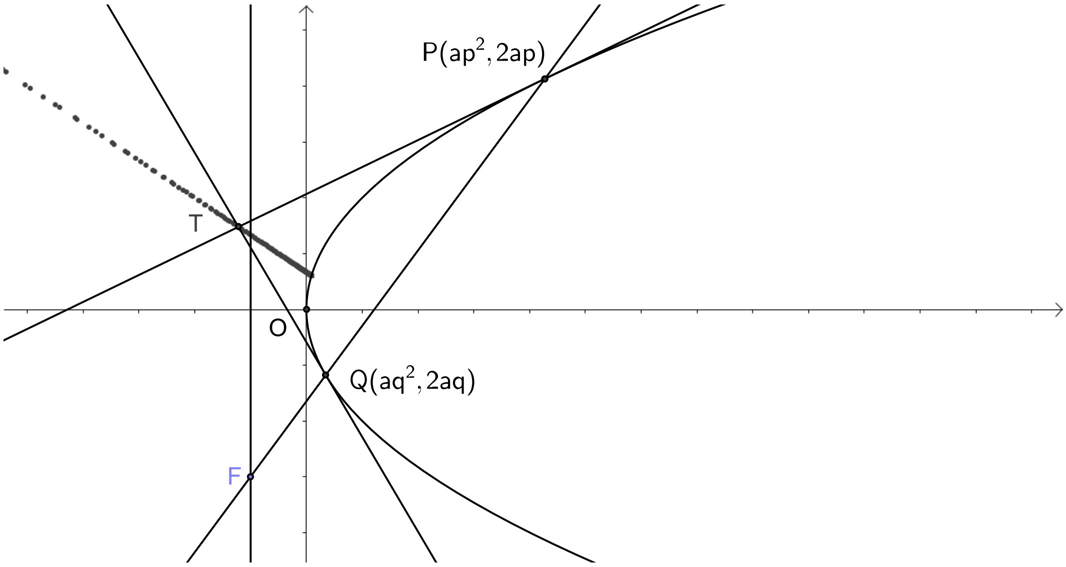


Problem 31c (Geogebra file)

(d) If PQ is a focal chord, which passes through the focus $\mathrm{S}(a, 0)$, then from the equation of the chord

$$x - \tfrac{1}{2}(p+q)y + apq = 0$$

we have



Problem 31c - locus (Geogebra file)

$$a - \tfrac{1}{2}(p+q)\times 0 + apq = 0$$

i.e.

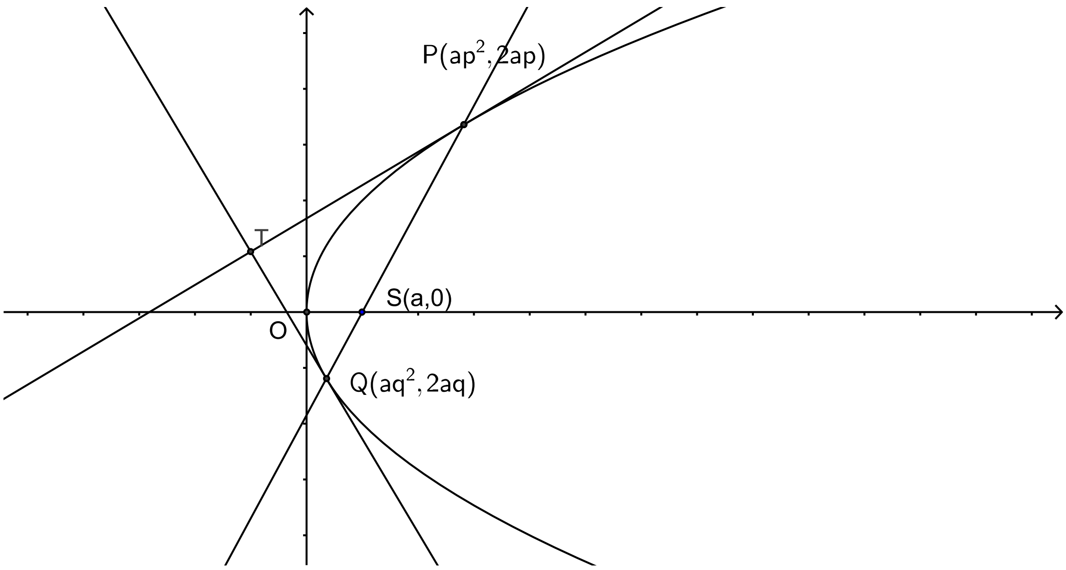
$$a(1+pq) = 0$$

i.e. $pq = -1$.

Substituting $pq = -1$ into (i) gives

$$\boxed{X = -a}$$

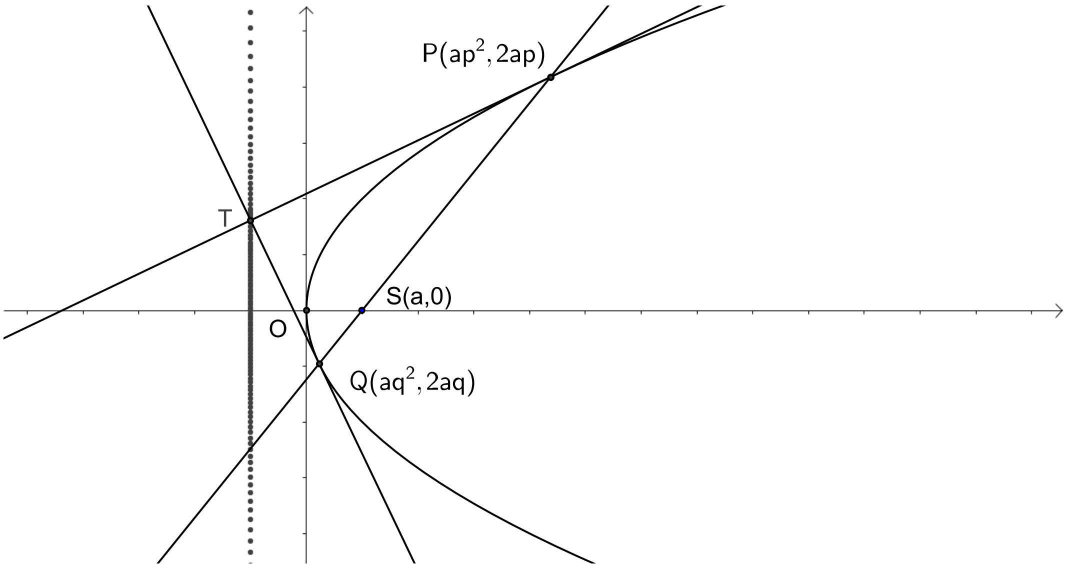
as the locus, which is the directrix.



Problem 31d (Geogebra file)

32. Find in each case the equation of the locus of the point of intersection of normals to a parabola $y^2 = 4ax$ at P and Q, when (a) PQ is perpendicular to the tangent at P; (b) the tangents or normals at P and Q are perpendicular; (c) PQ passes through a fixed point on the directrix; (d) PQ is a focal chord.

Solution



Problem 31d - locus (Geogebra file)

In all cases let P$(ap^2, 2ap)$, Q$(aq^2, 2aq)$ be two points on the parabola $y^2 = 4ax$, so that coordinates of the point of intersection of the normals at P and Q is N(X, Y) where

$$X = a((p+q)^2 - pq + 2) \qquad (i)$$
$$Y = -apq(p+q) \qquad (ii)$$

or

$$X = a(p^2 + q^2 + pq + 2) \qquad (iii)$$
$$Y = -apq(p+q) \qquad (iv)$$

(a) The slope of PQ is

$$\frac{2aq - 2ap}{aq^2 - ap^2} = \frac{2(q-p)}{(q-p)(p+q)} = \frac{2}{p+q}$$

and the slope of the tangent at P is $\frac{1}{p}$, and if these are perpendicular

$$\frac{1}{p} \times \frac{2}{p+q} = -1$$

i.e.

$$p(p+q) = -2 \qquad (v)$$

or

$$p^2 + pq = -2 \qquad (vi)$$

(Alternatively, since the perpendicular to the tangent at P is the normal, whose slope is $-p$, then $\frac{2}{p+q} = -p$, giving the same expression in (v) or (vi).) Subsituting for $p^2 + pq$ from (vi) into (iii) gives

$$X = aq^2 \qquad (vii)$$

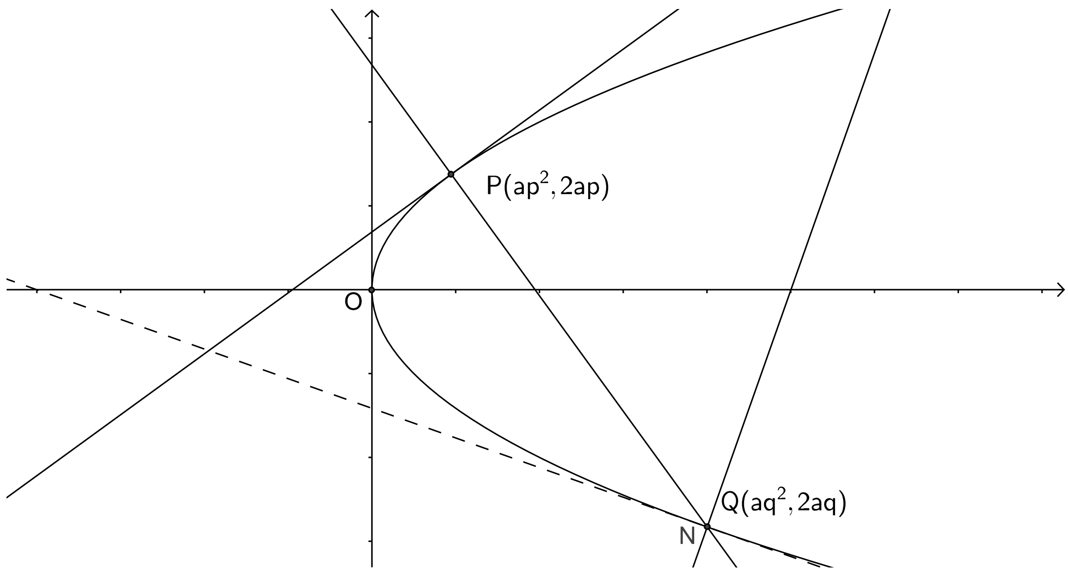
and for $p + q$ from (v) into (ii) gives

$$Y = 2aq \qquad (viii)$$

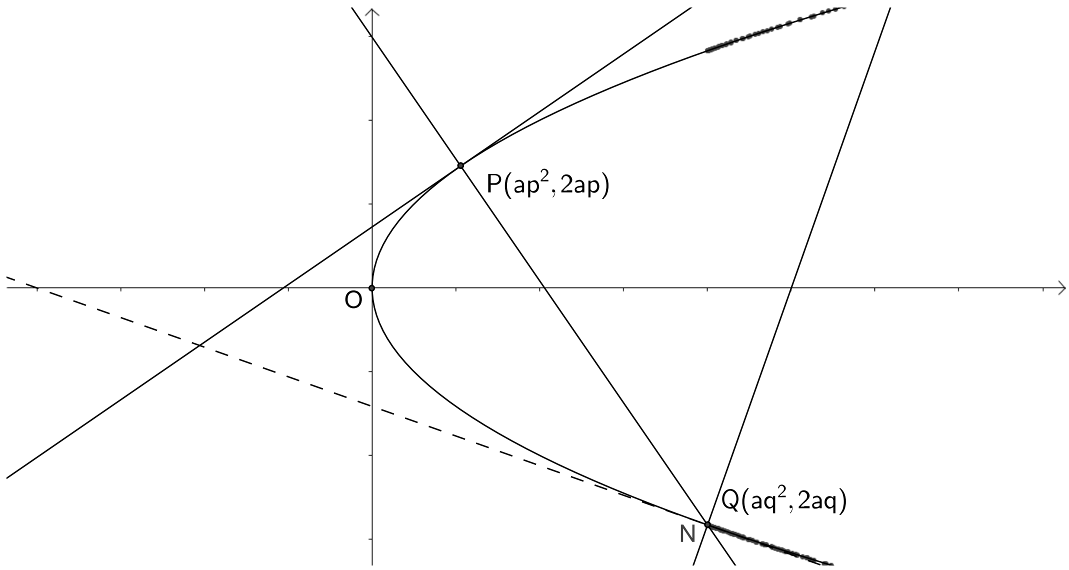
and eliminating p from (vii) and $(viii)$ gives the locus

$$\boxed{Y^2 = 4aX}$$

which is precisely the same locus as the original parabola, and from (vii) and $(viii)$ the point of intersection of the normals is the point Q. This is obvious since the chord is perpendicular to the tangent at P, and hence the chord coincides with the normal at P. Since this passes through Q, then clearly the point of intersection of the normals at P and Q is Q!



Problem 32a (Geogebra file)



Problem 32a - locus (Geogebra file)

(b) If the tangents or normals at P and Q are perpendicular the product of their slopes is

$$\frac{1}{p} \times \frac{1}{q} = -1$$

or

$$(-p) \times (-q) = -1$$

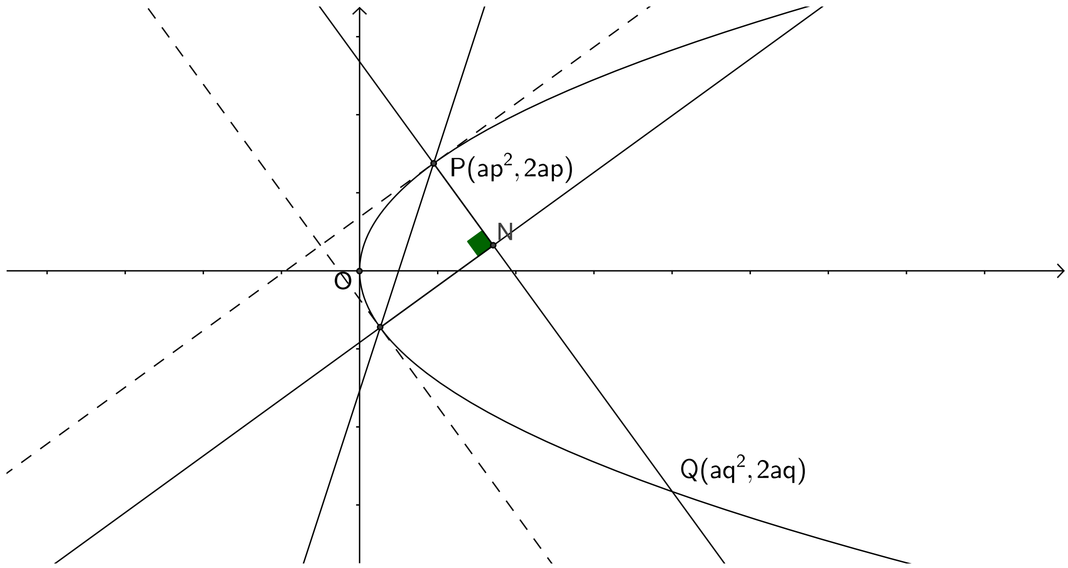
i.e. $pq = -1$.

Substituting $pq = -1$ into (i) and (ii) gives

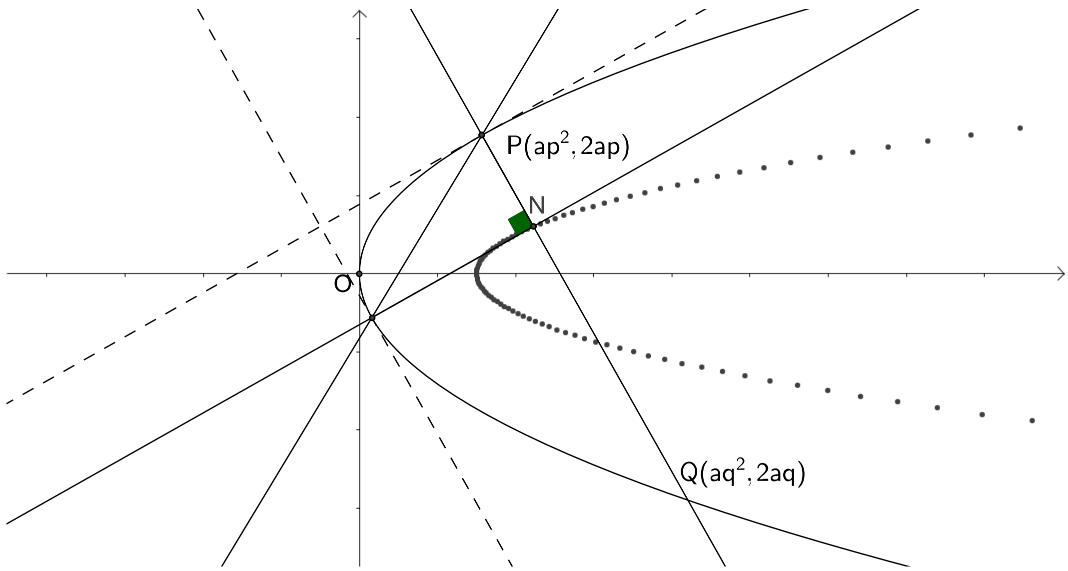
$$X = a((p+q)^2 + 3) \qquad (viii)$$
$$Y = a(p+q) \qquad (ix)$$

and eliminating $p+q$ in $(viii)$ and (ix) gives the locus

$$\boxed{Y^2 = a(X - 3a)}$$



Problem 32b (Geogebra file)



Problem 32b - locus (Geogebra file)

(c) If the chord PQ passes through a fixed point, F, on the directrix $x = -a$, with coordinates F$(-a, k)$, then from the equation of the chord PQ:

$$x - \tfrac{1}{2}(p+q)y + apq = 0$$

we have

$$-a - \tfrac{1}{2}(p+q)k + apq = 0 \qquad (x)$$

i.e.

$$pq = 1 + \frac{k(p+q)}{2a} \qquad (xi)$$

Substituting pq from (xi) into (i) and (ii) gives

$$X = a\left((p+q)^2 + 1 - \frac{k(p+q)}{2a}\right)$$
$$Y = a\left(-\frac{k(p+q)^2}{2a} - (p+q)\right)$$

which can be written as

$$X = a\left(u^2 + 1 - \frac{ku}{2a}\right) \qquad (xii)$$
$$Y = a\left(-\frac{ku^2}{2a} - u\right) \qquad (xiii)$$

where $u = p + q$, giving a parametric form for the locus. However, forming $2k \times (xii) + 4a \times (xiii)$, and rearranging, gives

$$(4a^2 + k^2)u = 2ka - 2kX - 4aY \qquad (xiv)$$

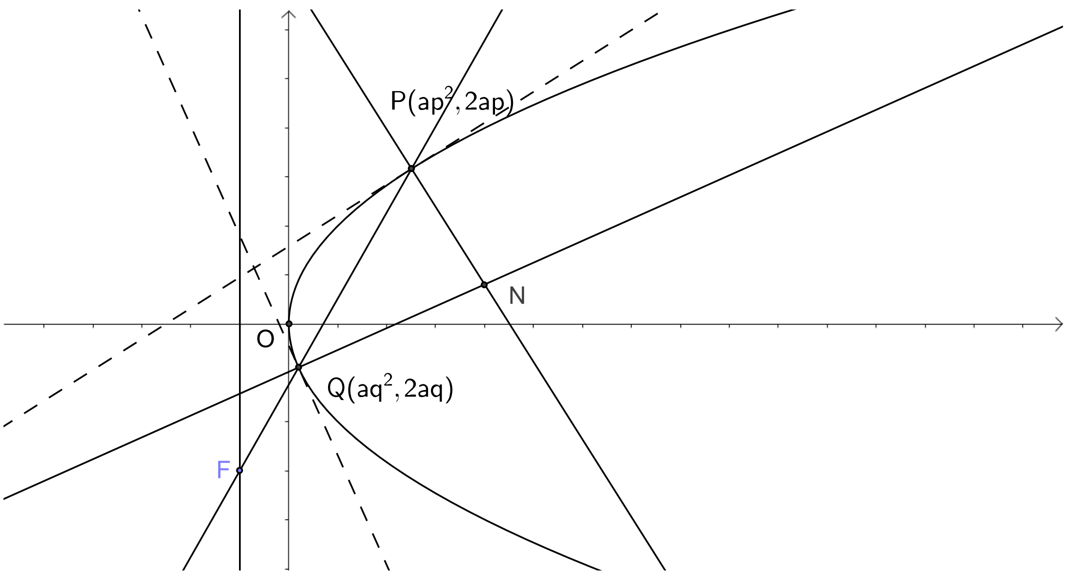
and forming $4a \times (xii) - 2k \times (xiii)$, and rearranging, gives

$$(4a^2 + k^2)u^2 = 4aX - 2kY - 4a^2 \qquad (xv)$$

Finally, substituting for u from (xiv) into (xv) gives the locus

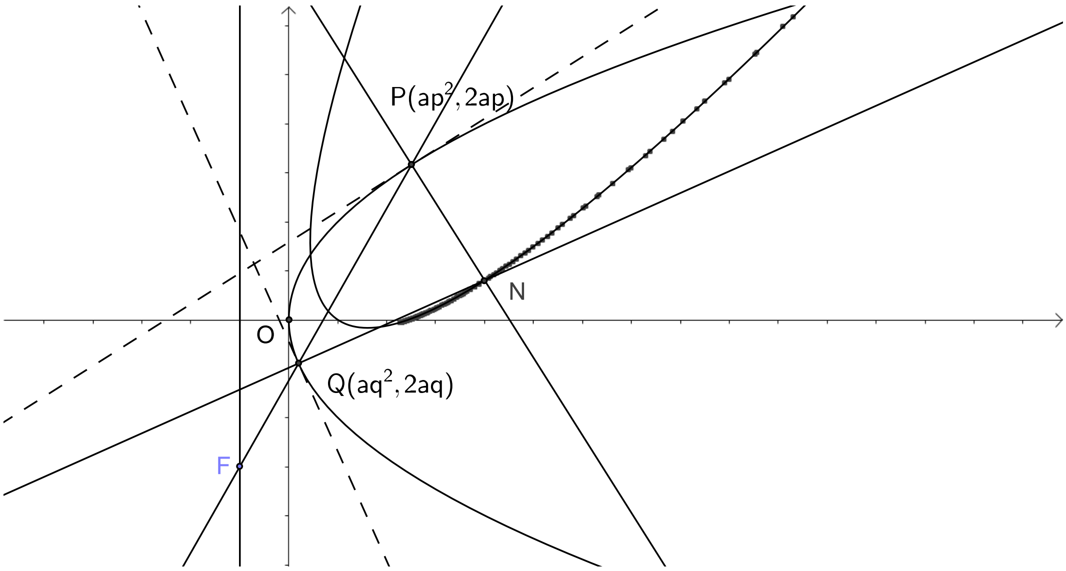
$$\boxed{(4a^2 + k^2)(4aX - 2kY - 4a^2) = (2ka - 2kX - 4aY)^2}$$

(Note that this is a parabola whose axis is $2ka - 2kx - 4ay = 0$, and vertex at the point of intersection of the axis, $2ka - 2kx - 4ay = 0$, and the tangent at the vertex, $4ax - 2ky - 4a^2 = 0$, which is readily shown to be the point $(a, 0)$, i.e. the focus of the original parabola.)



Problem 32c (Geogebra file)

(d) If PQ is a focal chord, which passes through the focus S$(a, 0)$, then from the equation of the chord



Problem 32c - locus (Geogebra file)

$$x - \tfrac{1}{2}(p+q)y + apq = 0$$

we have

$$a - \tfrac{1}{2}(p+q) \times 0 + apq = 0$$

i.e.

$$a(1 + pq) = 0$$

i.e. $pq = -1$.

Substituting $pq = -1$ into (i) and (ii) gives

$$X = a((p+q)^2 + 3)$$

$$Y = a(p+q)$$

and eliminating $p + q$ gives the locus

$$\boxed{Y^2 = a(X - 3a)}$$

33. If the normals at two points on a parabola meet on the same parabola, show that the chord joining the two points meets the axis in a fixed point.

Solution

A normal to the parabola $y^2 = 4ax$ at the point with coordinates $(at^2, 2at)$ has equation

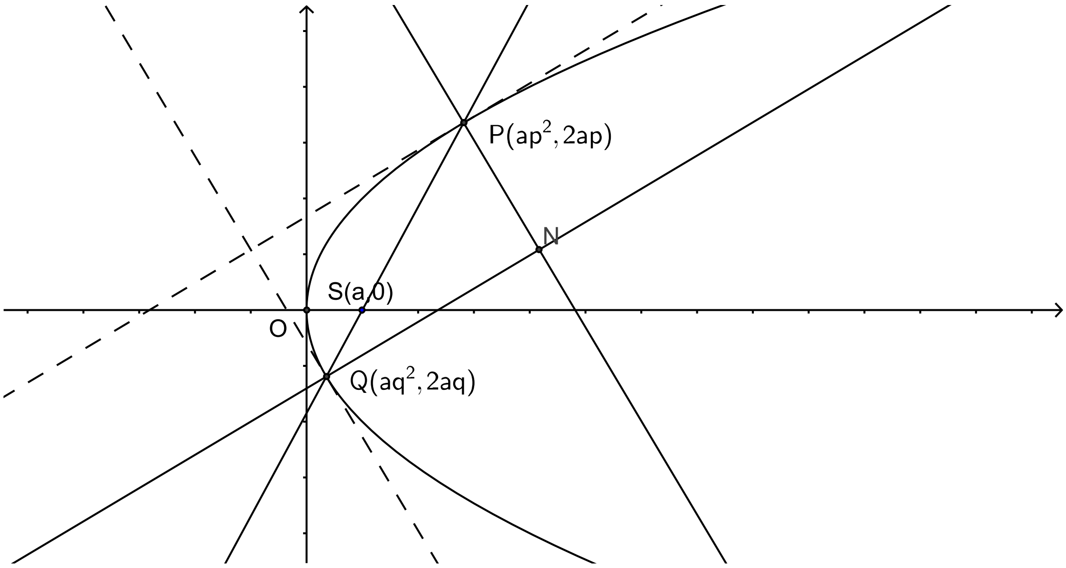
$$y + tx = at^3 + 2at$$

so that in general for any point (X, Y) there are three points P$(ap^2, 2ap)$, Q$(aq^2, 2aq)$, R$(ar^2, 2ar)$ whose normals pass through this point, where p, q, r are the roots of the cubic equation

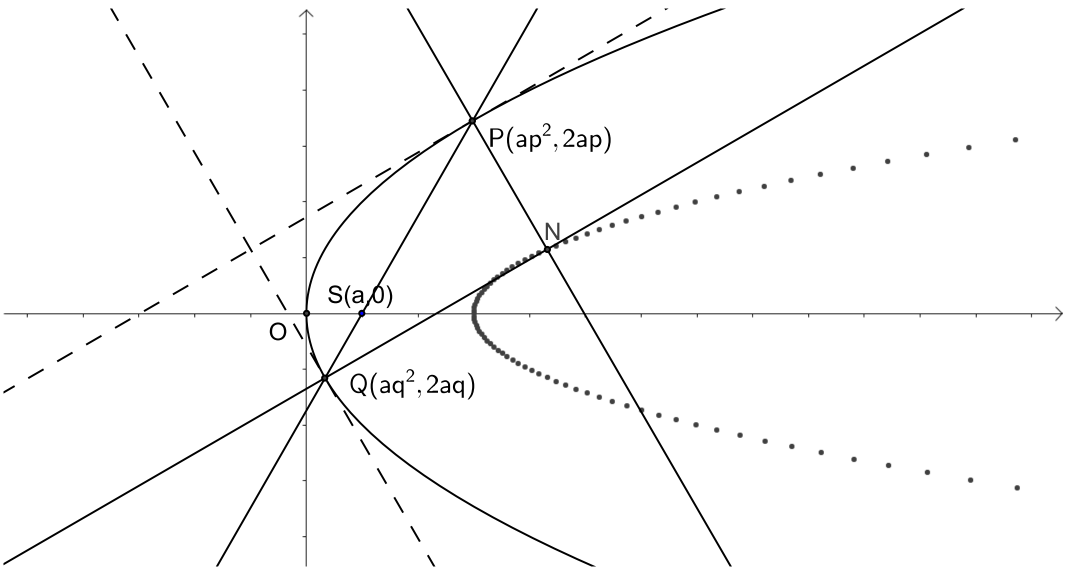
$$at^3 + (2a - X)t - Y = 0 \qquad (i)$$

From the standard results for the roots of a cubic equation we have from (i)

$$pqr = \frac{Y}{a} \qquad (ii)$$



Problem 32d (Geogebra file)



Problem 32d - locus (Geogebra file)

However, if the point of intersection of the two normals at P, Q lies on the parabola, then this point coincides with R, and hence $(X, Y) = (ar^2, 2ar)$, so that from (ii)

$$pq = 2$$

and the chord joining P and Q with equation

$$x - \tfrac{1}{2}(p+q)y + apq = 0$$

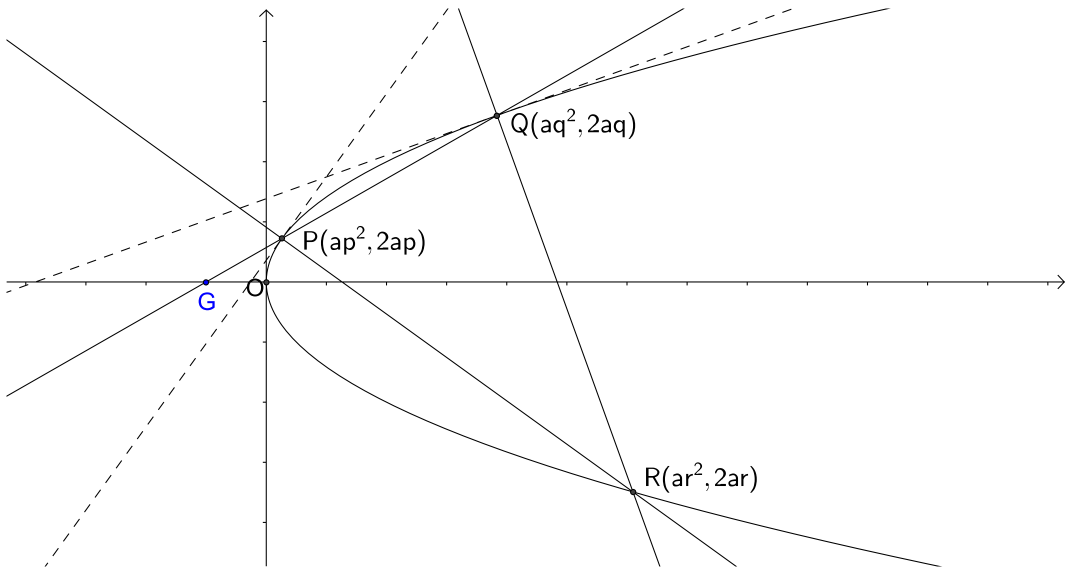
i.e.

$$x - \tfrac{1}{2}(p+q)y + 2a = 0$$

meets the axis of the parabola (the x-axis) where

$$x - \tfrac{1}{2}(p+q) \times 0 + apq = 0$$

i.e. $x = -2a$, a fixed point, $\mathrm{G}(-2a, 0)$.



Problem 33 (Geogebra file)

34. Consider the parabola $y^2 = 4ax$, where $a > 0$, and a variable focal chord PQ. The tangents at P and Q meet at T, and the normals at P and Q meet at N. Determine the locus of the mid-point of TN.

Solution

Let $\mathrm{P}(ap^2, 2ap)$, $\mathrm{Q}(aq^2, 2aq)$ be two points on the parabola $y^2 = 4ax$. The coordinates of the point of intersection T of the tangents at P and Q are $\mathrm{T}(apq, a(p+q))$, and of the point of intersection of the normals $\mathrm{N}((a(p+q)^2 - pq + 2), -apq(p+q))$. Hence the coordinates of the mid-point, (X, Y) are given by

$$\begin{aligned} X &= \tfrac{1}{2}\left(apq + a((p+q)^2 - pq + 2)\right) \\ &= \tfrac{1}{2}a\left((p+q)^2 + 2\right) \qquad (i) \end{aligned}$$

$$\begin{aligned} Y &= \tfrac{1}{2}\left(a(p+q) - apq(p+q)\right) \\ &= \tfrac{1}{2}a(1-pq)(p+q) \qquad (ii) \end{aligned}$$

If the chord PQ passes through the focus $\mathrm{S}(a, 0)$ then from the equation of the chord

$$x - \tfrac{1}{2}a(p+q)y + apq = 0$$

we have in particular that

$$a - \tfrac{1}{2}a(p+q) \times 0 + apq = 0$$

i.e.

$$a(1+pq) = 0$$

and hence $pq = -1$.

In this case, from (i) and (ii) the coordinates of the mid-point of TN become

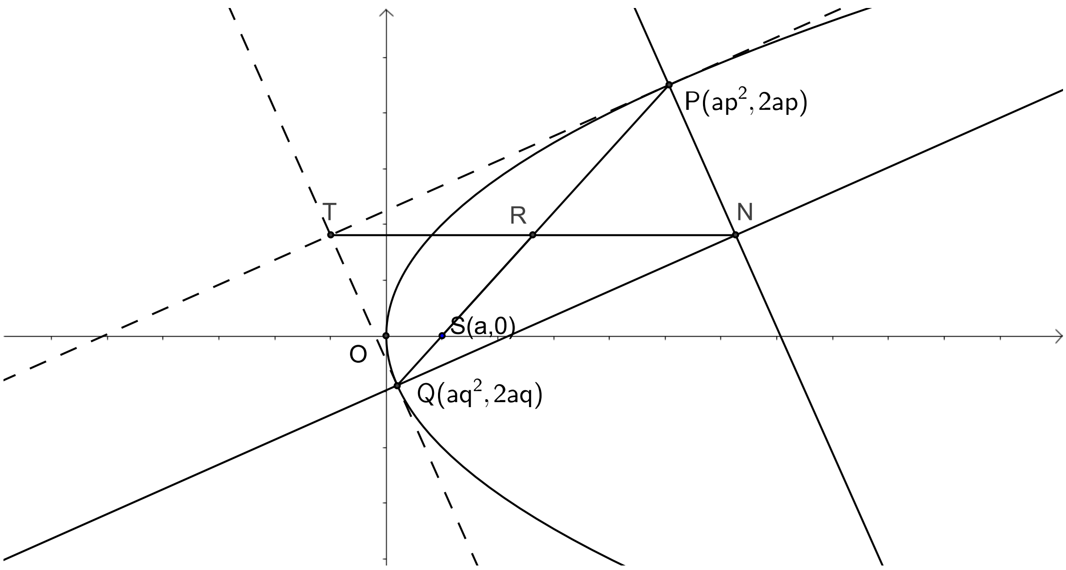
$$X = \tfrac{1}{2}a\left((p+q)^2 + 2)\right) \qquad (iii)$$
$$Y = a(p+q) \qquad (iv)$$

and substituting for $p+q$ from (iv) into (iii) gives

$$X = \tfrac{1}{2}a\left(\left(\frac{Y}{a}\right)^2 + 2)\right)$$

and hence the locus of R is

$$\boxed{Y^2 = 2a(X-a)}$$



Problem 34 (Geogebra file)

35. P and Q are two points on the parabola $y^2 = 4ax$ and PQ passes through the foot of the directrix on the x-axis, i.e. the point $(-a, 0)$. Determine the locus of the mid-point of PQ.

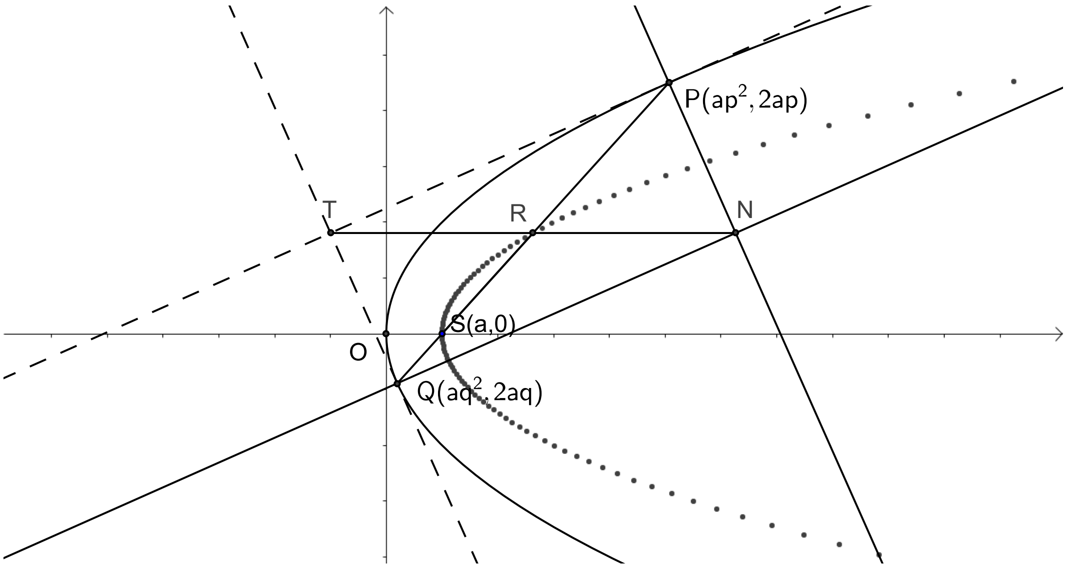
Solution

If P$(ap^2, 2aq)$ and Q$(aq^2, 2aq)$ are two points on the parabola $y^2 = 4ax$ where the chord, with equation

$$x - \tfrac{1}{2}(p+q)y + apq = 0$$

passes through the foot of the directrix on the x-axis, i.e. the point G$(-a, 0)$, then

$$-a - \tfrac{1}{2}(p+q) \times 0 + apq = 0$$



Problem 34 - locus (Geogebra file)

i.e. $pq = 1$.

The mid-point M(X, Y) of the chord PQ has coordinates given by

$$X = \tfrac{1}{2}(ap^2 + aq^2) \quad Y = \tfrac{1}{2}(2ap + 2aq)$$

i.e.

$$X = \tfrac{1}{2}a(p^2 + q^2) \quad Y = a(p + q)$$

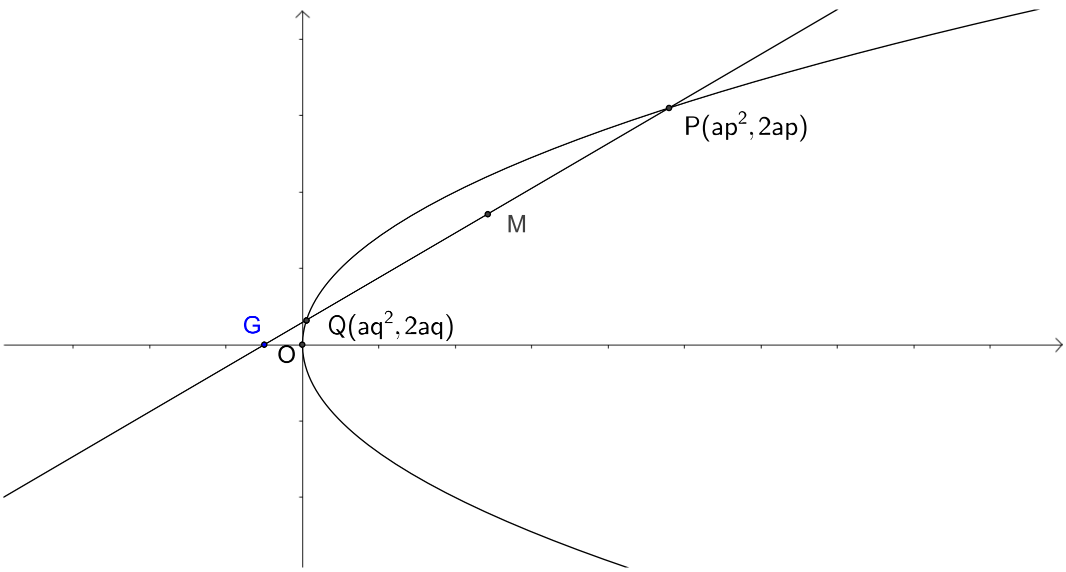
where $pq = 1$.

Using $(p+q)^2 \equiv p^2 + q^2 + 2pq$, we have the locus of the mid-points given by

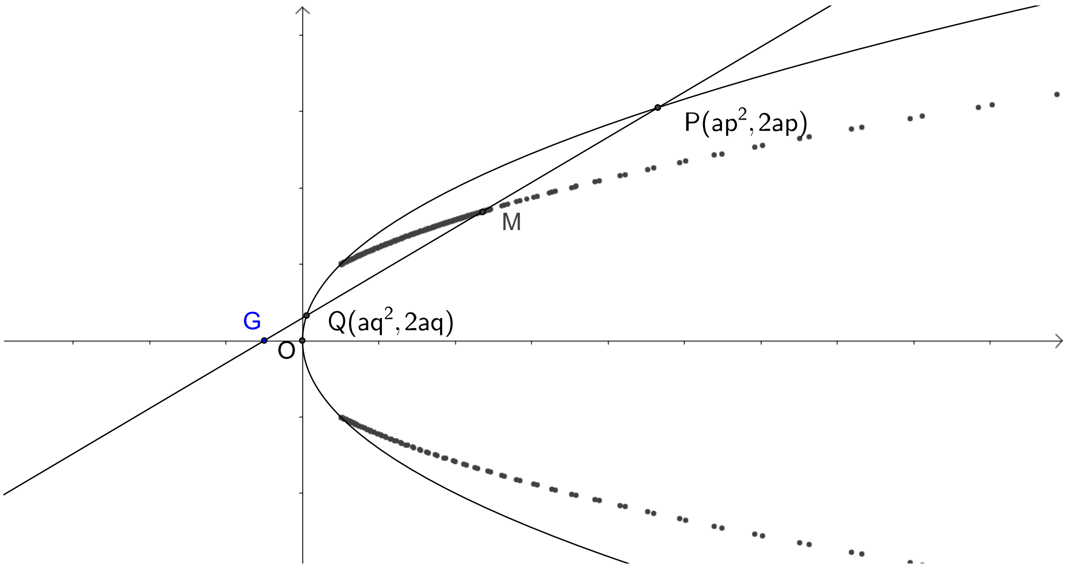
$$\left(\frac{Y}{a}\right)^2 = \frac{X}{a} + 2 \times 1$$

i.e.

$$\boxed{Y^2 = 2a(X + a)}$$



Problem 35 (Geogebra file)



Problem 35 - locus (Geogebra file)

36. P and Q are two points on the parabola $y^2 = 4ax$ and PQ passes through the foot of the directrix on the x-axis, i.e. the point $(-a, 0)$. Determine the locus of the point of intersection of the tangents at P and Q.

Solution

If P$(ap^2, 2aq)$ and Q$(aq^2, 2aq)$ are two points on the parabola $y^2 = 4ax$ where the chord, with equation

$$x - \tfrac{1}{2}(p+q)y + apq = 0$$

passes through the foot of the directrix on the x-axis, i.e. the point G$(-a, 0)$, then

$$-a - \tfrac{1}{2}(p+q) \times 0 + apq = 0$$

i.e. $pq = 1$.

The coordinates of the point of intersection of the tangents at P and Q are T(X, Y), where

$$X = apq \qquad (i)$$

$$Y = a(p+q)$$

where $pq = 1$.

Hence the locus is given by (i) as

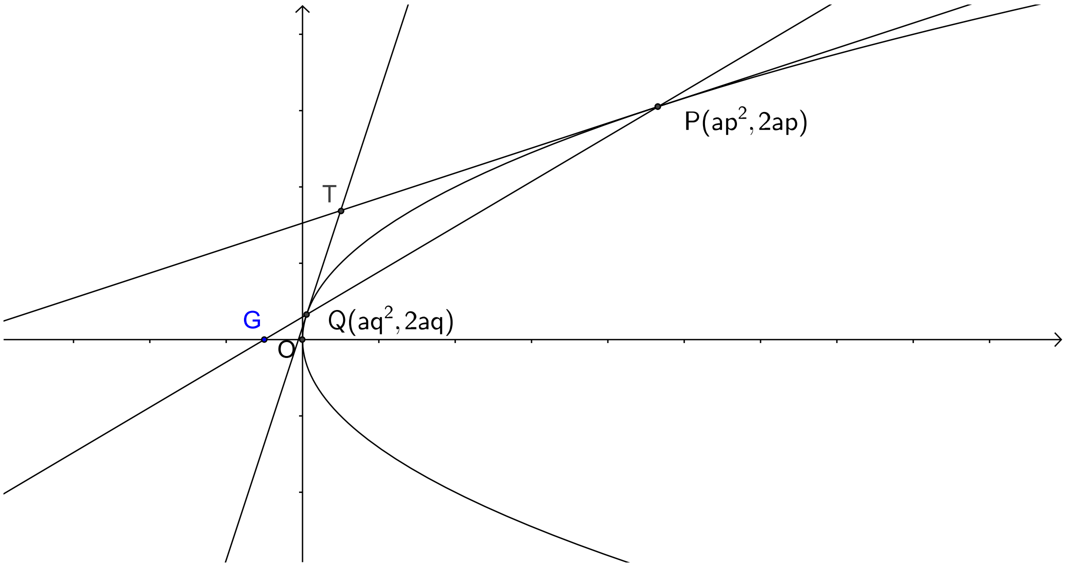
$$\boxed{X = a}$$

37. P and Q are two points on the parabola $y^2 = 4ax$ and PQ passes through the foot of the directrix on the x-axis, i.e. the point $(-a, 0)$. Determine the locus of the point of intersection of the normals at P and Q.

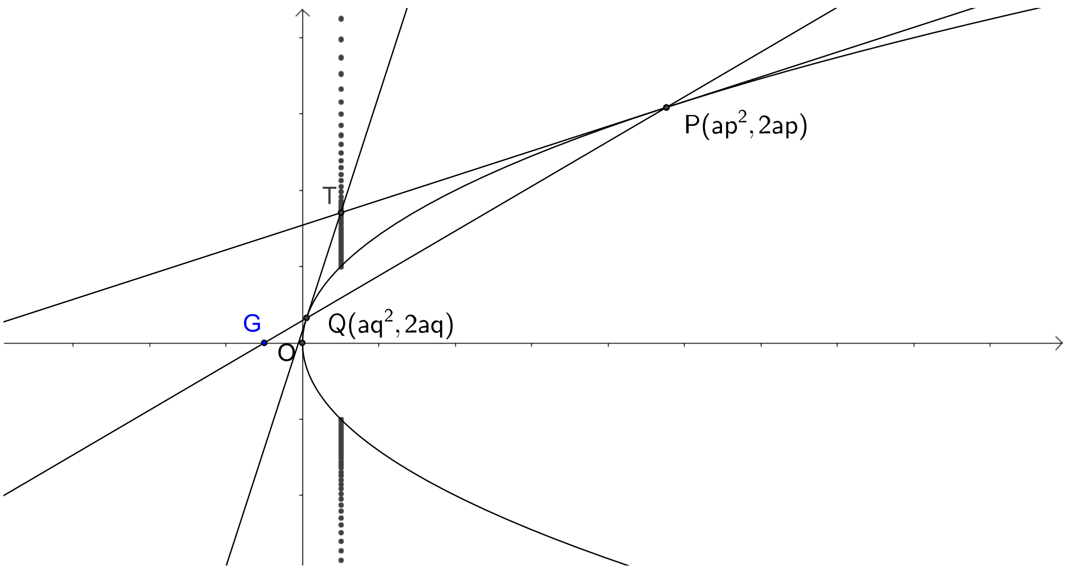
Solution

If P$(ap^2, 2aq)$ and Q$(aq^2, 2aq)$ are two points on the parabola $y^2 = 4ax$ where the chord, with equation

$$x - \tfrac{1}{2}(p+q)y + apq = 0$$



Problem 36 (Geogebra file)



Problem 36 - locus (Geogebra file)

passes through the foot of the directrix on the x-axis, i.e. the point G$(-a, 0)$, then

$$-a - \tfrac{1}{2}(p+q) \times 0 + apq = 0$$

i.e. $pq = 1$.

The coordinates of the point of intersection of the normals at P and Q are N(X, Y), where

$$X = a((p+q)^2 - pq + 2)$$

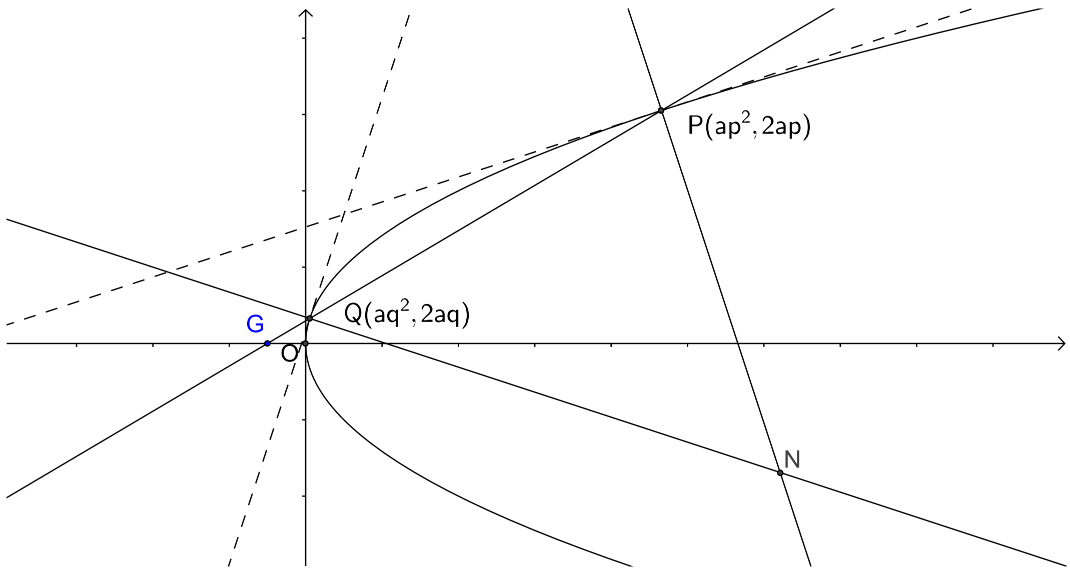
$$Y = -apq(p+q)$$

where $pq = 1$, and hence can be rewritten as

$$X = a((p+q)^2 + 1) \qquad (i)$$
$$Y = -a(p+q) \qquad (ii)$$

By substituting for $p+q$ from (ii) into (i), the locus is given by

$$\boxed{Y^2 = a(X - a)}$$



Problem 37 (Geogebra file)

38. A tangent to the parabola $y^2 = 4ax$ at the point P meets the parabola $y^2 = 4bx$ at the points Q and R, where $2b > a > 0$. Determine the equation of the locus of the mid-point of QR.

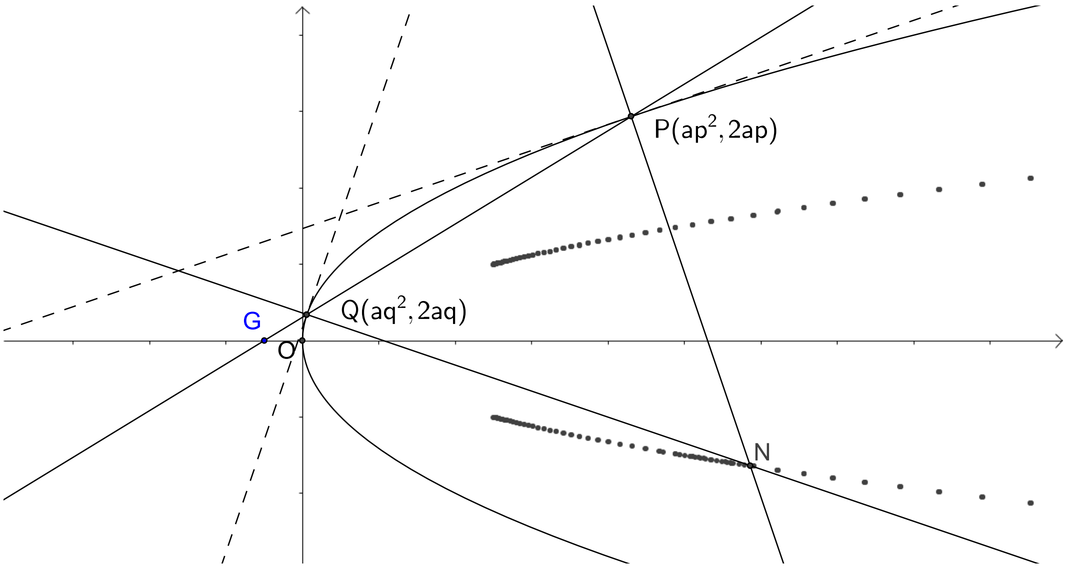
Solution

Let P$(ap^2, 2ap)$ be a point on the parabola $y^2 = 4ax$. The tangent at P has equation

$$x - py + ap^2 = 0 \qquad (i)$$

If this meets the parabola $y^2 = 4bx$ at the points Q and R with coordinates Q$(bq^2, 2bq)$ and R$(br^2, 2br)$, then from (i)

$$bq^2 - 2bqp + ap^2 = 0 \qquad (ii)$$



Problem 37 - locus (Geogebra file)

$$br^2 - 2brp + ap^2 = 0 \qquad (iii)$$

From (ii) and (iii) we have that q, r are the roots of the quadratic equation (in t):

$$bt^2 - 2btp + ap^2 = 0 \qquad (iv)$$

Now the coordinates of the mid-point M(X, Y) of QR are given by

$$\begin{aligned} X &= \tfrac{1}{2}(bq^2 + br^2) \\ &= \tfrac{1}{2}b(q^2 + r^2) \qquad (v) \end{aligned}$$

$$\begin{aligned} Y &= \tfrac{1}{2}(2bq + 2br) \\ &= b(q + r) \qquad (vi) \end{aligned}$$

However, from (iv), the sum and product of the roots q, r are given by

$$q + r = \frac{2bp}{b} = 2p \qquad (vii)$$

$$qr = \frac{ap^2}{b} \qquad (viii)$$

Rewriting (v) we have

$$\begin{aligned} X &= \tfrac{1}{2}b\left((q + r)^2 - 2qr\right) \\ &= \tfrac{1}{2}b\left((2p)^2 - \frac{2ap^2}{b}\right) \\ &= (2b - a)p^2 \qquad (ix) \end{aligned}$$

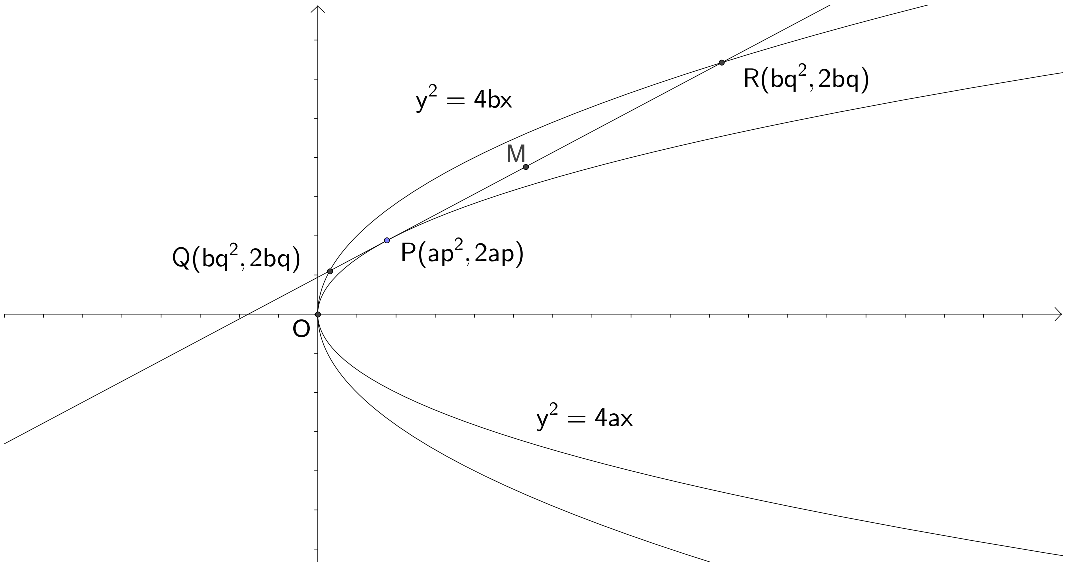
using (vii) and $(viii)$, and rewriting (vi) as

$$Y = 2bp \qquad (x)$$

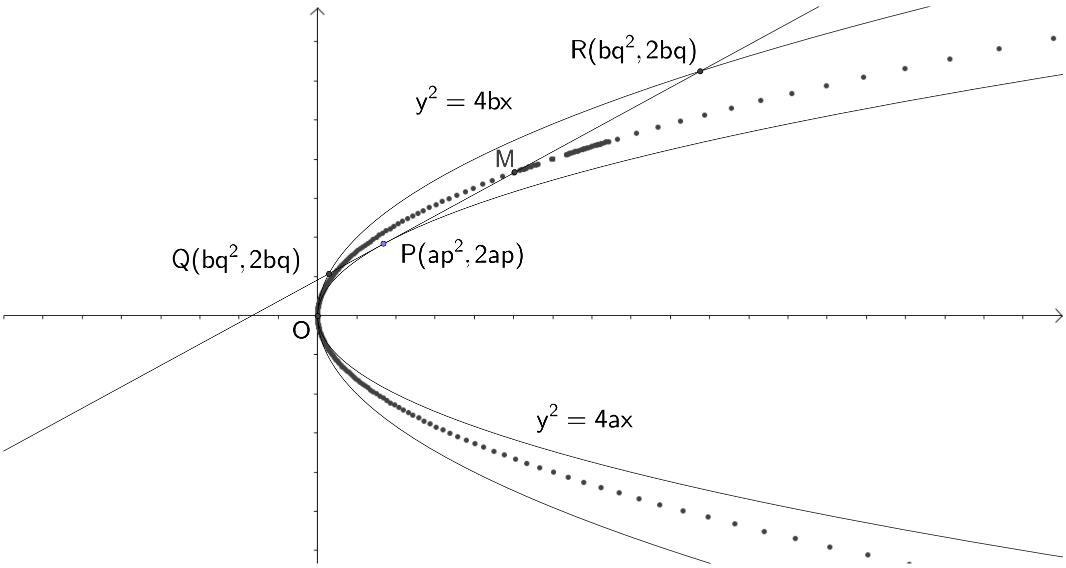
so that eliminating p from (ix) and (x) gives the locus of the mid-point as

$$\boxed{(2b - a)y^2 = 4b^2x}$$

.



Problem 38 (Geogebra file)



Problem 38 - locus (Geogebra file)

39. The normal at a point P of a parabola, focus S, meets the axis in G. Prove that there are two positions of P such that the triangle SPG is equilateral, and that the sides then have length $4a$.

Solution Take the equation of the parabola in the standard form $y^2 = 4ax$, and P as the point $(at^2, 2at)$, then the equation of the normal at P is

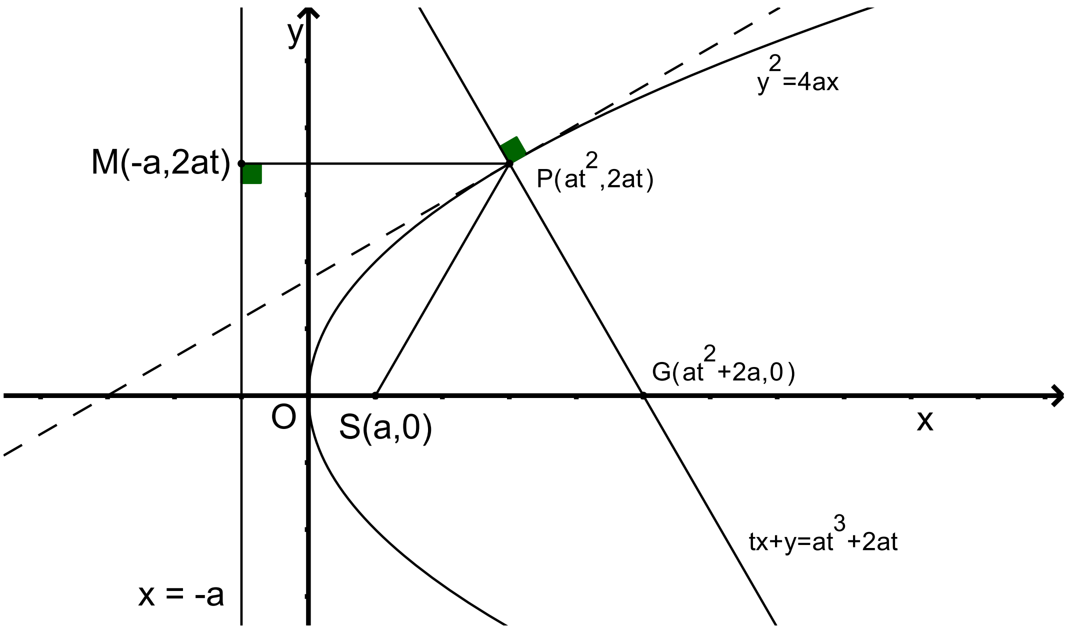
$$tx + y = at^3 + 2at$$

and this meets the axis of the parabola, $y = 0$, in the point

$$\mathrm{G}(at^2 + 2a, 0) \quad .$$

The focus S is the point $(a, 0)$, so that

$$\mathrm{SG}^2 = (at^2 + a)^2 \quad .$$



Problem 39 (Geogebra file)

Now from the definition of the parabola:

$$\begin{aligned} \mathrm{SP}^2 &= \mathrm{PM}^2 \\ &= (at^2 + a)^2 \end{aligned}$$

Therefore SP = SG for every value of t, that is for every position of P. If the triangle SPG is to be equilateral, $\mathrm{SP}^2 = \mathrm{PG}^2$ too, and since

$$\mathrm{PG}^2 = (2a)^2 + (2at)^2$$

we have

$$(2a)^2 + (2at)^2 = (at^2 + a)^2.$$

This reduces to

$$t^4 - 2t^2 - 3 = 0$$

i.e.

$$(t^2 - 3)(t^2 + 1) = 0 \quad .$$

Since t is real. t^2+1 is not zero, so the roots of this equation are

$$t = \pm\sqrt{3} \ .$$

There are thus two positions of P such that the triangle SPG is equilateral, and in each the length of the side is $4a$ since

$$\mathrm{SG} = at^2 + a = 4a \ .$$

40. A variable chord PQ of the parabola $y^2 = 4ax$ is parallel to the line $y = x$. If P and Q are the points $(ap^2, 2ap)$ and $(aq^2, 2aq)$, show that $p + q = 2$, and prove that the locus of the point of intersection of the normals to the parabola at P and Q is a straight line.

Solution The gradient of PQ is

$$\frac{2ap - 2aq}{ap^2 - aq^2} = \frac{2}{p+q} \ , \quad \text{since } \ p - q \neq 0 \ .$$

It is given that PQ is parallel to $y = x$, i.e. it has gradient 1. Therefore

$$2/(p+q) = 1 \ , \ \text{i.e.} \ \ p + q = 2$$

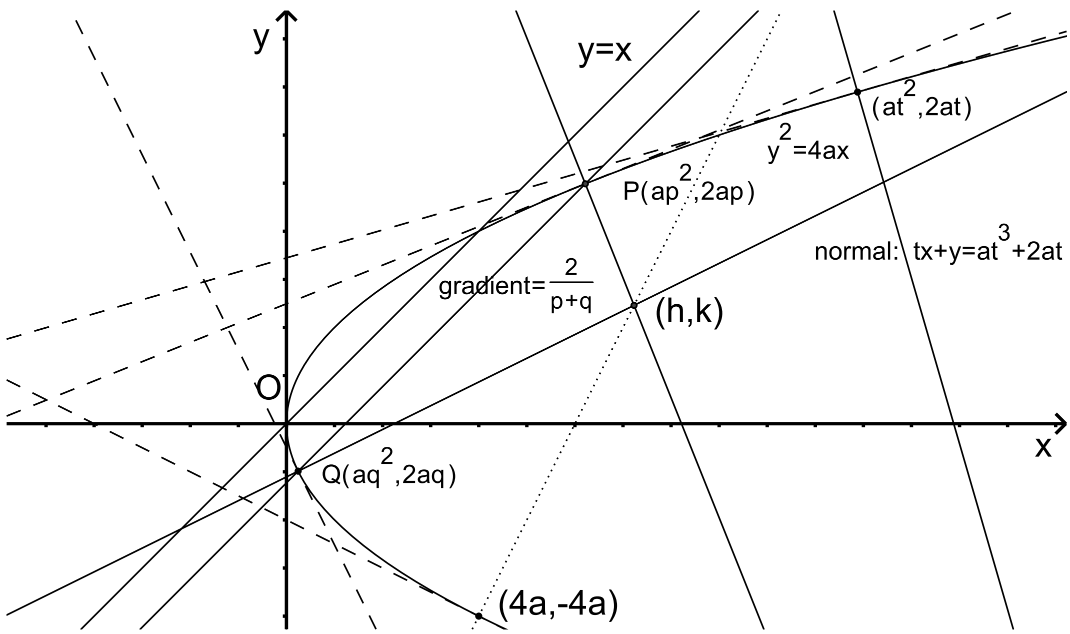
as required.

The equation of the normal to the parabola at the general point $(at^2, 2at)$ is

$$tx + y = at^3 + 2at \ . \qquad (1)$$

This passes through the point (h, k) if

$$at^3 + t(2a - h) - k = 0 \ . \qquad (2)$$



Problem 40 (Geogebra file)

If the roots of (2) are p, q and r then

$$p + q + r = 0 \tag{3}$$
$$qr + rp + pq = 2 - h/a \tag{4}$$
$$pqr = k/a \quad . \tag{5}$$

Now $p + q = 2$, so that from (3), $r = -2$, and from (4)

$$\begin{aligned} 2 - h/a &= pq + rp + rq \\ &= pq + r(p+q) \\ &= pq + (-2) \times (2) \\ &= pq - 4 \end{aligned}$$

so that

$$pq = 6 - h/a \quad . \tag{6}$$

Substituting $r = -2$ in (5) and using (6) gives

$$\begin{aligned} k/a &= pqr \\ &= (6 - h/a) \times (-2) \\ &= -12 + 2h/a \end{aligned}$$

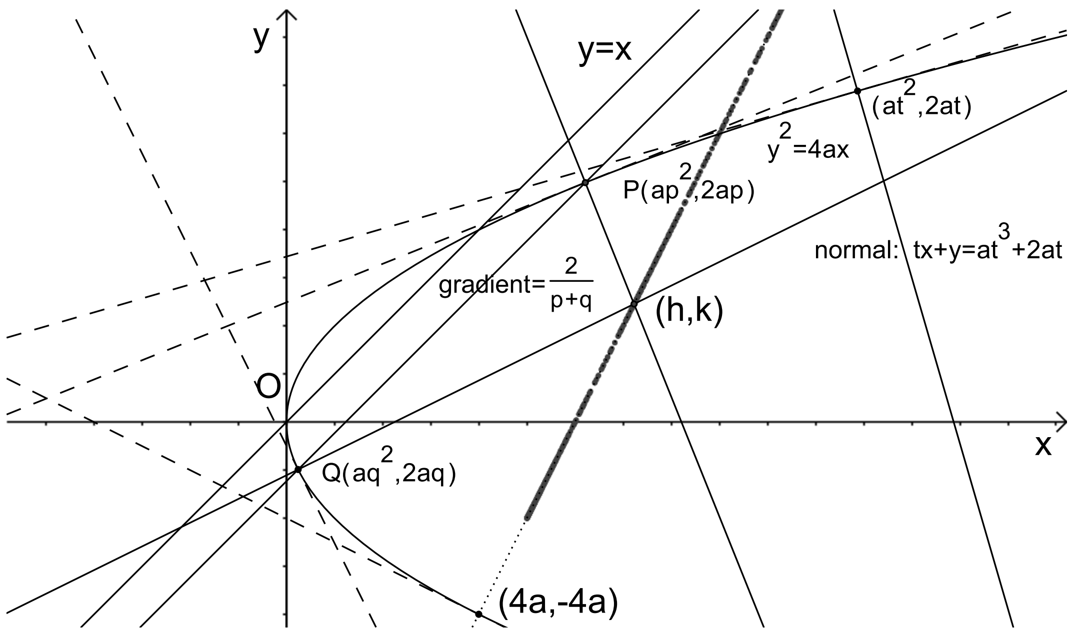
i.e.

$$k = 2h - 12a \quad .$$

It follows that the equation of the locus of (h, k) is

$$y = 2x - 12a \quad ,$$

representing a straight line, as required. Note also that from (1) the equation of the third normal which passes through (h, k) (with $t = r = -2$) is $(-2)x + y = a(-2)^3 + 2a \times (-2)$, i.e. $y = 2x - 12a$, the locus in (7), intersecting the parabola at $(a(-2)^2, 2a \times (-2)) = (4a, -4a)$.



Problem 40 - locus (Geogebra file)

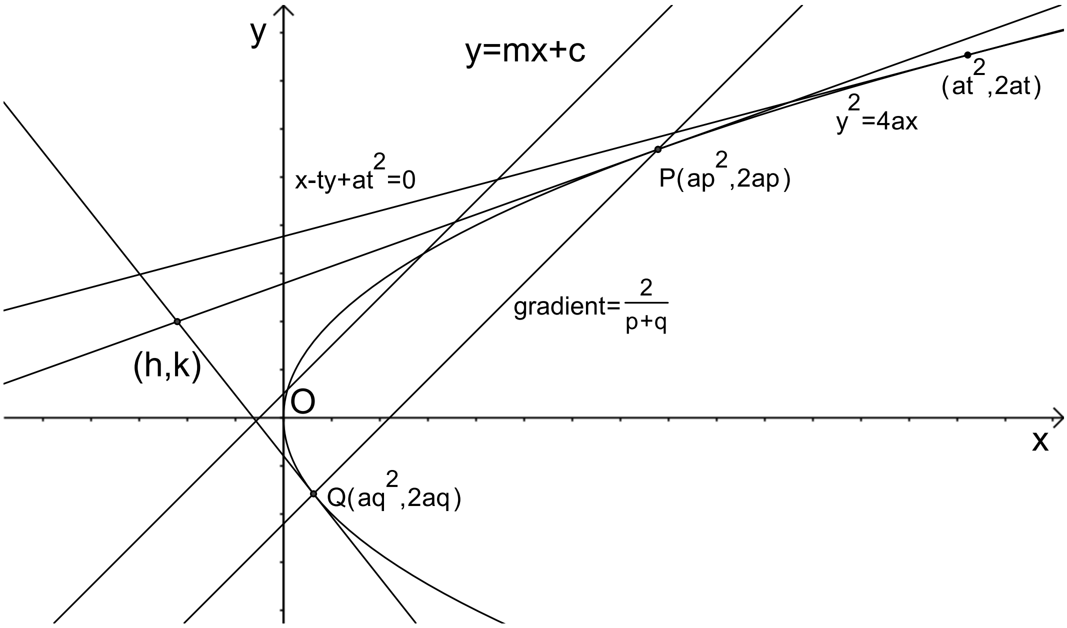
41. A variable chord PQ of the parabola $y^2 = 4ax$ is parallel to a fixed line. If P and Q are the points $(ap^2, 2ap)$ and $(aq^2, 2aq)$, prove that the locus of the point of intersection of the tangents to the parabola at P and Q is a straight line.

Solution The gradient of PQ is

$$\frac{2ap - 2aq}{ap^2 - aq^2} = \frac{2}{p+q} \ , \quad \text{since } \ p - q \neq 0 \ .$$

It is given that PQ is of fixed slope, say m, and hence

$$2/(p+q) = m \ , \ \text{i.e. } \ p + q = \frac{2}{m}$$



Problem 41 (Geogebra file)

The equation of the tangent to the parabola at the general point $(at^2, 2at)$ is

$$x - ty + at^2 = 0 \ .$$

This passes through the point (h, k) if

$$at^2 - tk + h = 0 \ .$$

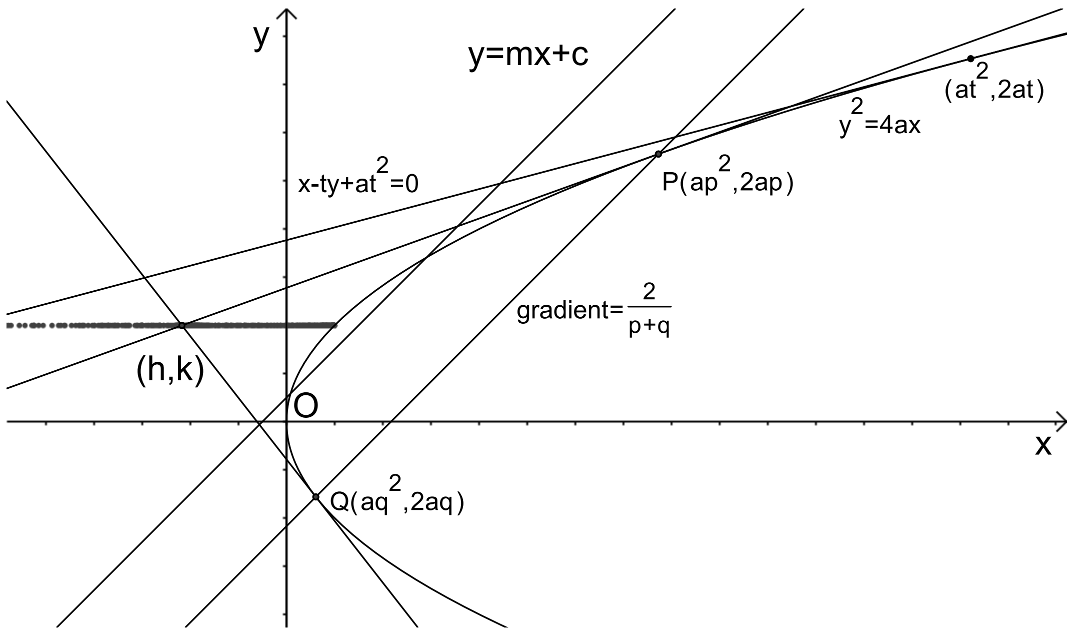
If the roots of this equation are p and q

$$p + q = \frac{k}{a} \ \text{ (and } \ pq = \frac{h}{a} \ \text{).} \tag{1}$$

Hence

$$k = a(p+q) = \frac{2a}{m} \ .$$

It follows that the equation of the locus of (h, k) is a line parallel to the x-axis.



Problem 41 - locus (Geogebra file)

42. A variable straight line with constant gradient m meets the parabola

$$y^2 = 4ax$$

at Q, R. Find the locus of P, the mid-point of QR.
$\boxed{\textbf{Solution}}$ Let the equation of the line be

$$y = mx + c \ . \tag{1}$$

To find the coordinates of Q, R, we would solve the equation of the line and the equation of the parabola

$$y^2 = 4ax$$

simultaneously:

$$(m^2x^2 + 2mxc + c^2) = 4ax$$

i.e.

$$m^2x^2 + (2mc - 4a)x + c^2 = 0 \ . \tag{2}$$

The x-coordinates of Q, R, say x_1, x_2, are the roots of this equation.
However P(X, Y) is the mid-point, and hence

$$X = \frac{1}{2}(x_1 + x_2)$$

and using the expression for the sum of the roots from (2)

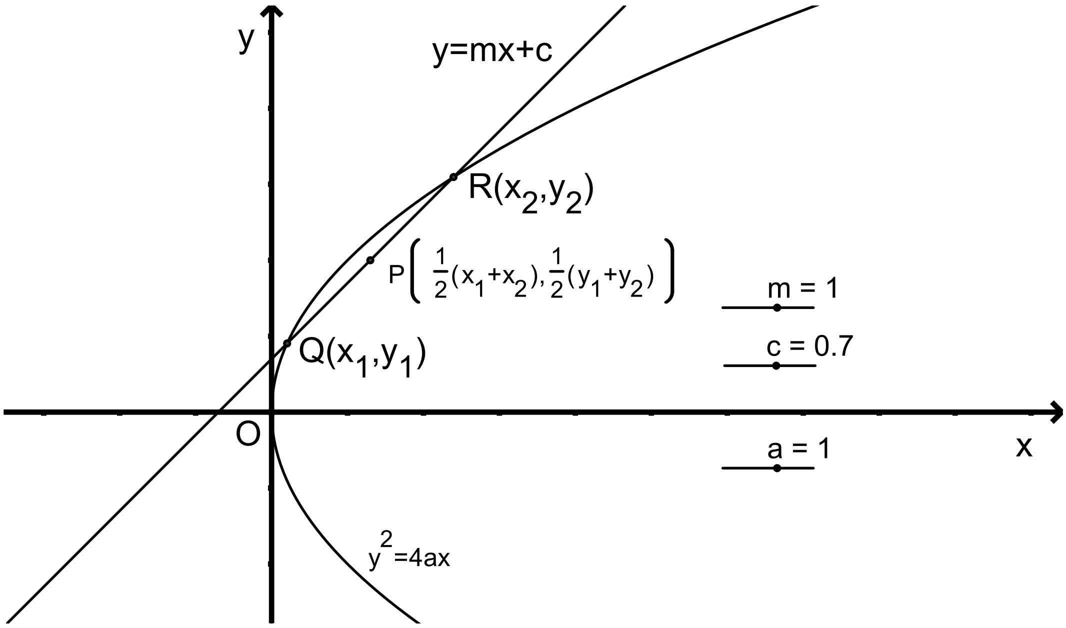
$$X = \frac{1}{2} \times \frac{-(2mc - 4a)}{m^2} = \frac{2a - mc}{m^2} \tag{3}$$

Now the coordinates of P satisfy equation (1), so

$$Y = mX + c \ . \tag{4}$$

Since m is fixed and c is variable we find the locus of P by eliminating c between equations (3) and (4). Substituting

$$c = Y - mX$$



Problem 42 and Problem 43 (Geogebra file)

in equation (3) and rearranging we have

$$m^2X = 2a - m(Y - mX)$$

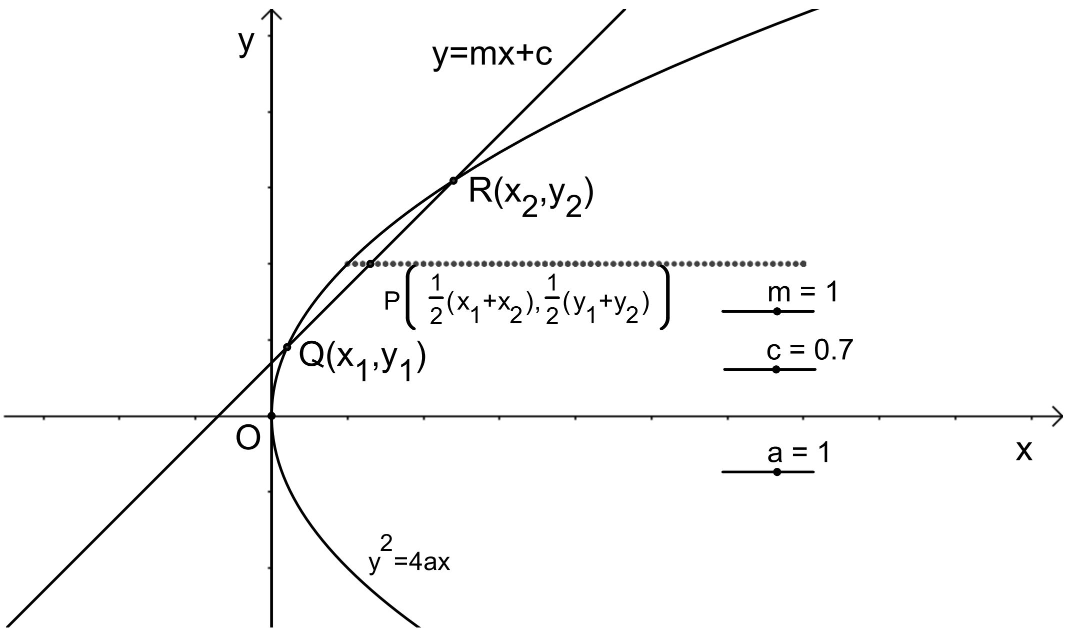
i.e.

$$Y = \frac{2a}{m} \quad .$$

Therefore the locus of P is

$$y = \frac{2a}{m}$$

which is a **diameter** of the parabola $(0, 0)$.



Problem 42 - locus (Geogebra file)

43. A variable straight line with constant intercept c meets the parabola

$$y^2 = 4ax$$

at Q, R. Find the locus of P, the mid-point of QR.

$\boxed{\textbf{Solution}}$ Let the equation of the line be

$$y = mx + c \ . \tag{1}$$

To find the coordinates of Q, R, we would solve the equation of the line and the equation of the parabola

$$y^2 = 4ax$$

simultaneously:

$$(m^2x^2 + 2mxc + c^2) = 4ax$$

i.e.

$$m^2x^2 + (2mc - 4a)x + c^2 = 0 \ . \tag{2}$$

The x-coordinates of Q, R, say x_1, x_2, are the roots of this equation. However P(X, Y) is the mid-point, and hence

$$X = \frac{1}{2}(x_1 + x_2)$$

and using the expression for the sum of the roots from (2)

$$X = \frac{1}{2} \times \frac{-(2mc - 4a)}{m^2} = \frac{2a - mc}{m^2} \tag{3}$$

Now the coordinates of P satisfy equation (1), so

$$Y = mX + c \ . \tag{4}$$

Since c is fixed and m is variable we find the locus of P by eliminating m between equations (3) and (4). Multiplying (3) by X and rearranging we have

$$m^2X^2 - 2aX + cmX = 0 \ . \tag{5}$$

If we now substitute

$$mX = Y - c$$

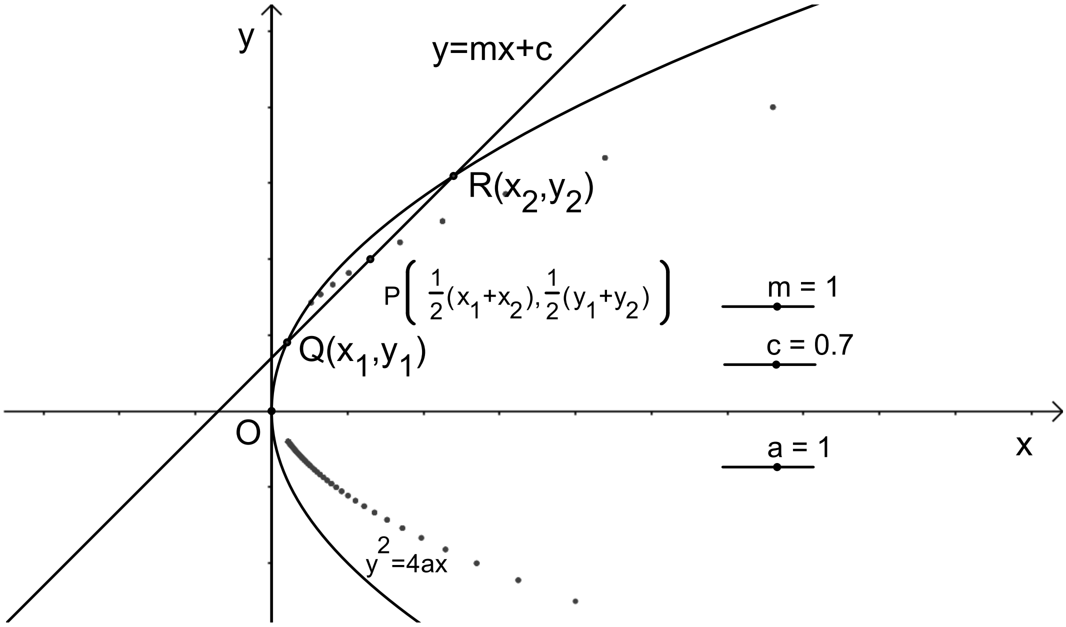
in equation (5) and rearranging and simplifying, we have

$$\left(Y - \frac{c}{2}\right)^2 = 2a\left(X + \frac{c^2}{8a}\right) \ .$$

Therefore the locus of P is

$$\left(y - \frac{c}{2}\right)^2 = 2a\left(x + \frac{c^2}{8a}\right) \ . \tag{6}$$

By comparing (6) with the parabola $y^2 = 4bx$ with focus $(b, 0)$, directrix $x = -b$, vertex $(0, 0)$ and latus rectum $4b$, we see that (6) is a parabola with focus at $(a/2 - c^2/8a, c/2) = ((a + c)/2, 0)$, directrix $x = -a/2 - c^2/8a$, vertex $(-c^2/8a, c/2)$ and latus rectum $2a$.



Problem 43 - locus (Geogebra file)