Background material for Promoting Problem Solving Through Parabolas

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1. Introduction.

We introduce the parabola and some basic simple results related to it. The figures have all been created using the very versatile (and free) software package *GeoGebra*. If you wish to experiment with GeoGebra visit www.geogebra.org, download and install the software

2. Backgound knowledge.

There are a number of basic facts and techniques that you will need to be familiar with in order to understand the material.

2.1 Quadratic equations.

The quadratic equation in x

$$ax^2 + bx + c = 0 \quad , \quad a \neq 0 \tag{1}$$

has

(i) real roots if and only if $b^2 - 4ac \geq 0$;

(ii) no real roots if and only if $b^2 - 4ac < 0$;

(iii) equal roots if and only if $b^2 - 4ac = 0$;

(iv) equal and opposite roots if and only if $b = 0$.

(In case (ii) there are two *complex* roots.) The formula for the roots of the quadratic equation (1) are given by

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a} \quad . \tag{2}$$

If α and β are the roots of the quadratic equation (1) then

$$\alpha + \beta = -b/a \quad \text{and} \quad \alpha\beta = c/a \quad .$$

In this module it will be very helpful to know the sum and product of the roots of a quadratic equation (1) *without* finding the roots explicitly using the formula in (2).

2.2 Pythagoras' Theorem.

In any right-angled triangle, the area of the square whose side is the hypotenuse (the side opposite the right angle) is equal to the sum of the areas of the squares whose sides are the two legs (the two sides that meet at a right angle). More specifically, if the lengths of the sides of a right angled triangle are denoted a, b, c, where c represents the length of the hypotenuse, and a and b represent the lengths of the other two sides, then

$$a^2 + b^2 = c^2 \quad .$$

The converse of the theorem can be stated as:

If in a triangle the area of the square on one of the sides equals the sum of the areas of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right. More specifically, if the lengths of the sides of a triangle are denoted a, b, c, and

$$a^2 + b^2 = c^2$$

then between the sides of lengths a and b is a right angle.

2.3 Some algebraic identities.

$$(a+b)^2 \equiv a^2 + b^2 + 2ab$$
$$(a-b)^2 \equiv a^2 + b^2 - 2ab$$
$$(a+b)^2 - (a-b)^2 \equiv 4ab$$

$$a^2 - b^2 \equiv (a-b)(a+b)$$

$$(a+b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 \equiv a^3 - 3a^2b + 3ab^2 - b^3$$

$$a^3 - b^3 \equiv (a-b)(a^2 + ab + b^2)$$
$$a^3 + b^3 \equiv (a+b)(a^2 - ab + b^2) \ .$$

3. Coordinates and distance.

We will assume familiarity with the system of rectangular Cartesian coordinates with two perpendicular straight lines, X′OX, Y′OY, in the plane, intersecting in a point O, as shown in **Figure 1**. They are called **axes of coordinates**, or just **axes**, and O is called the **origin**. If we wish to specify a point on X′OX by its distance from O we use the convention of counting distances from O to points on the right of O as positive, and distances from O to points on the left of O as negative. Similarly, points on Y′OY are described by their distances from O, with a positive sign if they are above X′OX, and otherwise a negative one.

Once the points on the axes have been labelled in this way, we may label the other points in the plane. If P is any point, and PM, PN are perpendiculars drawn to the axes from P, then the (signed) distances $\overrightarrow{\mathrm{OM}}$, $\overrightarrow{\mathrm{ON}}$ are uniquely determined: they are denoted by x, y. Conversely, given two numbers x, y the positions of M, N on the axes can be found, and so the point P can be located.

The two numbers x, y will label P as we required: they are called the **coordinates** of P, and we say that P is the point with coordinates (x, y), or, more simply, P is the point (x, y). We also refer to the point $\mathrm{P}(x, y)$. The x-coordinate and y-coordinate of P are called, respectively, the **abscissa** and **ordinate** of P.

Now suppose $\mathrm{P}_1(x_1, y_1)$ and $\mathrm{P}_2(x_2, y_2)$ are any two points in the plane. Let the perpendiculars from $\mathrm{P}_1, \mathrm{P}_2$ meet the axes in $\mathrm{M}_1, \mathrm{M}_2$ and $\mathrm{N}_1, \mathrm{N}_2$ respectively and let R be the foot of the perpendicular from P_1 to $\mathrm{P}_2\mathrm{M}_2$, as shown in **Figure 2**. Then (by considering all possible positions of P_1 and P_2 relative to one another and to the axes) the distances:

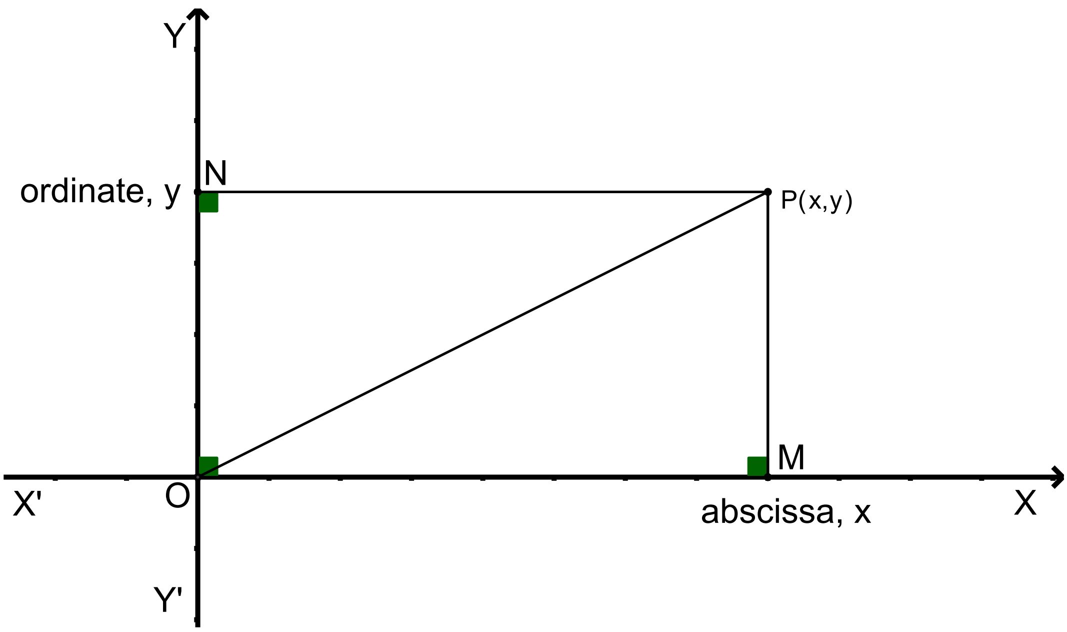

Figure 1: Cartesian Coordinates (Geogebra file)

$$|\overrightarrow{\mathrm{P_1R}}| = |\overrightarrow{\mathrm{M_1M_2}}| = |x_2 - x_1| \text{ and } |\overrightarrow{\mathrm{RP_2}}| = |\overrightarrow{\mathrm{N_1N_2}}| = |y_2 - y_1|$$

and by Pythagoras' theorem

$$|\overrightarrow{\mathrm{P_1P_2}}|^2 = |\overrightarrow{\mathrm{P_1R}}|^2 + |\overrightarrow{\mathrm{RP_2}}|^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

i.e.

$$(\mathrm{P_1P_2})^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad .$$

In particular if P is (x, y) and O is the origin $(0, 0)$, then

$$\mathrm{OP}^2 = (x - 0)^2 + (y - 0)^2$$

and so

$$\mathrm{OP}^2 = x^2 + y^2 \quad .$$

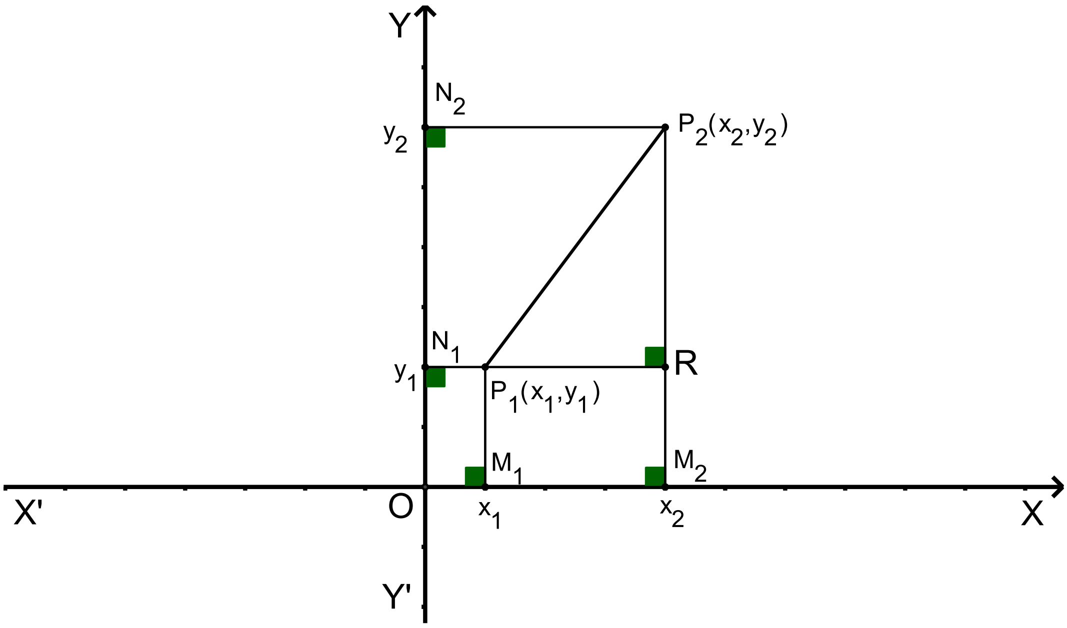

Figure 2: Distance from two points (Geogebra file)

4. Equation of the locus of a point.

A point P(x, y) may, with suitable choice of x and y, lie anywhere in the plane. If we impose some rule on x and y, we limit the region of the plane where P may be. For example, we have already seen that if x and y are both positive, P lies in the first quadrant. Again, if $x = 0$, P lies on Y$'$OY; or, if $y = 0$, P lies on X$'$OX. For example, as we shall see shortly, if we have the rule: P is a fixed distance of 2 from the point $(3, 1)$ and radius 2, then it will be found that x and y satisfy the equation $x^2 + y^2 - 6x - 2y + 6 = 0$. Such an equation is called the **equation of the locus of P**; in this case it is the equation of the circle, centre $(3, 1)$ and radius 2.

There are now two associated problems:

(a) We may be told some geometrical fact which limits the positions of P, and asked to find what relation is then satisfied by x and y, the coordinates of P. For example, P may be given as being a fixed distance from a fixed point, i.e. a circle, or P may be equidistant from two fixed points, i.e. a straight line. This problem of finding an equation does not usually present many difficulties.

The converse is more complicated:

(b) An equation connecting x and y is given, and the problem is to find out on what geometrical locus P lies. This is a question of identification and familarity with standard loci such as conics.

It is with one or other of these problems that coordinate geometry often deals.

5. The (straight) line.

The simplest locus of a point in a plane is a **straight line**, and we usually call it simply **a line**. In this section we obtain the equation of a line in various forms.

5.1 Gradient of a line. If any two points on a line are selected, then the line segment joining them makes a constant angle with a fixed direction, and the angle is independent of the two particular points selected on the line. This is a precise way of saying that any line has a constant slope. It is customary to measure the angle α which a line makes with the x-axis, and then $\tan\alpha$ is called the **gradient** of the line, as shown in **Figure 3**.

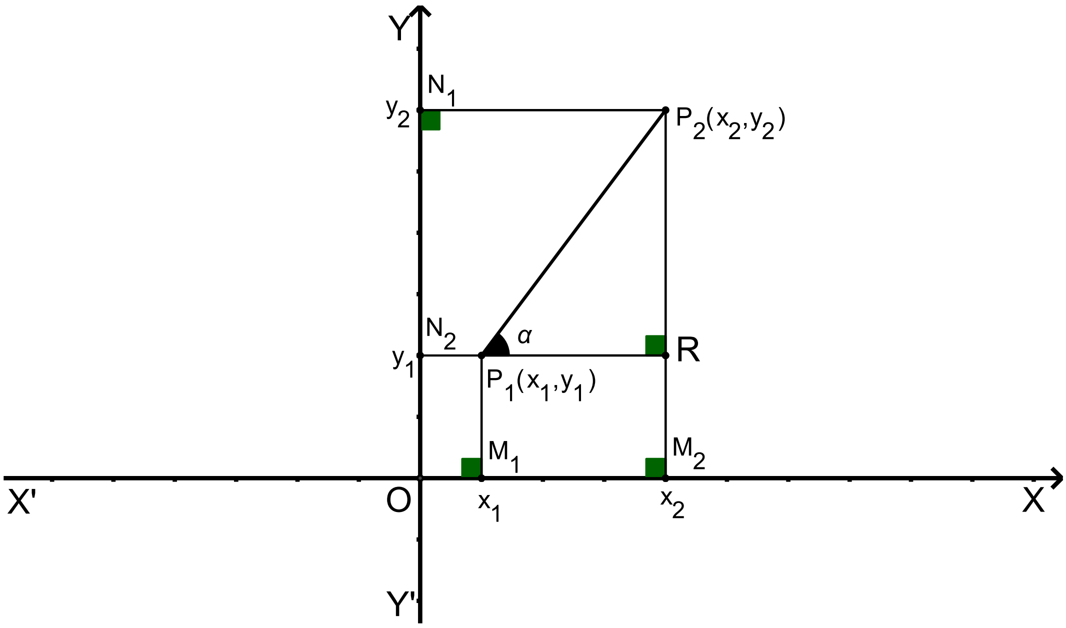

Figure 3: Gradient of a line (Geogebra file)

To find the gradient of the line joining $\mathrm{P}_1(x_1, y_1)$ and $\mathrm{P}_2(x_2, y_2)$, draw lines $\mathrm{P}_1\mathrm{M}_1, \mathrm{P}_2\mathrm{M}_2$ perpendicular to OX, and let the perpendicular from P_1 to $\mathrm{P}_2\mathrm{M}_2$ meet it in R. If $\mathrm{P}_1\mathrm{P}_2$ makes an angle α with OX, angle $\mathrm{P}_2\mathrm{P}_1\mathrm{R} = \alpha$, and

$$\tan\alpha = \frac{y_2 - y_1}{x_2 - x_1}$$

This then is the gradient of $\mathrm{P}_1\mathrm{P}_2$ which will hold for any positions of $\mathrm{P}_1, \mathrm{P}_2$ in the plane.

We notice that lines which make an acute angle with the positive direction of OX have a positive gradient, while lines which make an obtuse angle with OX have a negative gradient.

5.2 Parallel and perpendicular lines. A set of parallel lines all make the same angle α with the x-axis, and so all the lines have the same gradient. Conversely, if two lines have the same gradient, they are parallel. The condition for lines with gradients m and m' to be parallel is therefore

$$m = m' \ .$$

In particular, all lines parallel to OX have zero gradient, for $\alpha = 0$ and so $\tan\alpha = 0$. For lines parallel to OY, $\alpha = \frac{\pi}{2}$, and so $\tan\alpha$ is infinite.

Now consider two lines which are perpendicular, and suppose their gradients are m amd m'. If the first line makes an angle α with OX, and the second line makes an angle α' with OX, then

$\alpha' = \alpha + \frac{\pi}{2}$. So

$$\begin{aligned} \tan\alpha' &= \tan(\alpha + \tfrac{\pi}{2}) \\ &= -\cot\alpha \\ &= -1/\tan\alpha \ . \end{aligned}$$

Therefore since $\tan\alpha = m$ and $\tan\alpha' = m'$, the relation between the gradients is

$$m' = -1/m \quad \text{or} \quad mm' = -1 \ .$$

Conversely, if this relation is satisfied, the lines with gradients m amd m' are perpendicular.

5.3 Equation of a line. If a line passes through A(h, k) and has gradient m it is uniquely determined, and so the equation satisfied by the coordinates of any point P(x, y) on it can be found. The gradient of AP is $(y-k)/(x-h)$ and this is m, therefore (see **Figure 4**)

$$\frac{y-k}{x-h} = m$$

i.e.

$$y - k = m(x-h) \ .$$

This is the equation satisfied by the coordinates of any point P on the line; it is called the **equation of the line**.

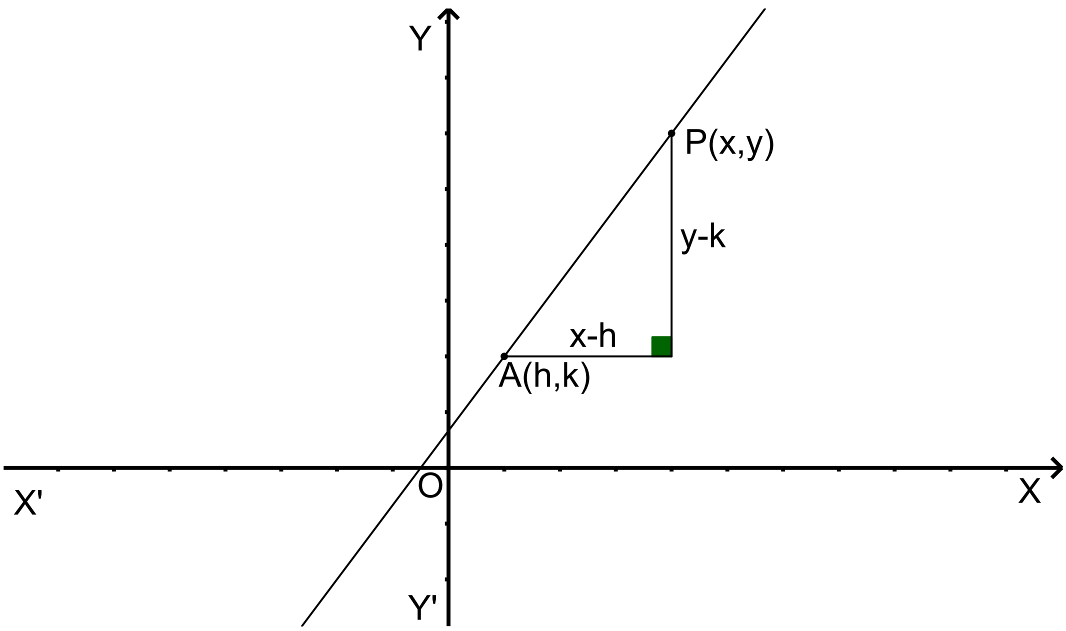

Figure 4: Equation of a line - the gradient form (Geogebra file)

The form of this equation is important. It can be written

$$mx - y + k - mh = 0$$

and we see that it contains a term in x, i.e. mx, a term in y, i.e. $(-y)$, and a constant term (which does not involve x or y), i.e. $k - mh$. There are no other terms, like x^2 or xy. An equation of this type is called a **linear** equation.

Conversely, we consider the general linear equation

$$ax + by + c = 0$$

and prove that, as its name suggests, it always represents a line. Since the equation gives a relation satisfied by some, but not all, values of x and y, not both of a and b can be zero. If $a = 0$, and $b \neq 0$, the equation is

$$y = -c/b$$

and this is satisfied by all the points with a fixed ordinate, $-c/b$, on a line parallel to OX. Similarly, if $b = 0$ and $a \neq 0$, the equation is

$$x = -c/a$$

and represents a line parallel to OY.

In the general case, when $a \neq 0$ and $b \neq 0$, suppose (x_1, y_1) and (x_2, y_2) are two points whose coordinates satisfy the equation, so that

$$ax_1 + by_1 + c = 0$$

and

$$ax_2 + by_2 + c = 0 \ .$$

Subtracting, $a(x_2 - x_1) + b(y_2 - y_1) = 0$, and so

$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{a}{b} \ .$$

It follows that the gradient of the line joining any two points on the locus has the constant value $-a/b$, and so the locus is that of a line.

We refer to 'the line $ax + by + c = 0$' instead of 'the line whose equation is $ax + by + c = 0$'.

5.4 Special forms of the equation of the line. There are a number of useful alternative forms in which the equation of a line may be written. The form chosen in any particular case will depend either on what information is given about the line, or on what is to be found out about it. We now list these various forms, the first two of which have already been found above.

(a) *The standard form*, $ax + by + c = 0$.

The equation of any line can be written in this form.

(b) *The gradient form*, $y - k = m(x - h)$.

This form is useful because it can be written down at once if the gradient m and one point (h, k) on the line are known. Conversely, from this form the gradient of the line and the coordinates of a particular point on it can be seen at a glance. The equation of a line parallel to the y-axis, however, cannot be put into this form.

(c) *The gradient-intercept form*

This standard form is obtained when the point given on the line is $(0, c)$. That is, the *intercept* c which the line makes on the y-axis is given, and also the gradient m. Then as above the equation takes the form (see **Figure 5**)

$$y - c = m(x - 0)$$

or

$$y = mx + c \ .$$

When the equation is written in this form (which is possible for all lines except those parallel to the y-axis), the gradient m and the intercept c on the y-axis can be found by inspection.

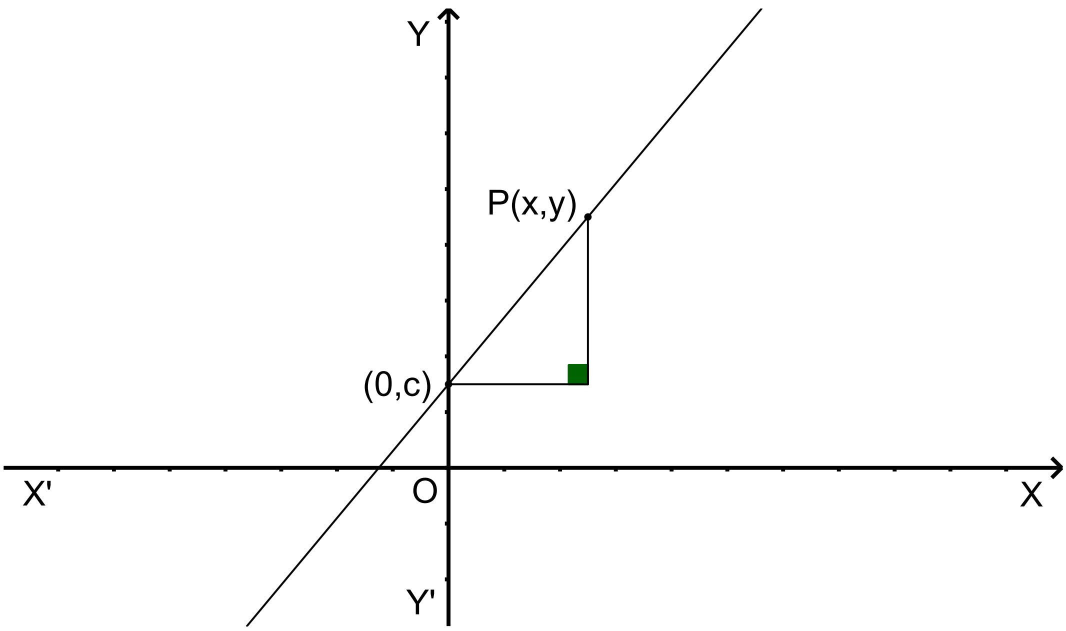

Figure 5: Equation of a line - the gradient-intercept form (Geogebra file)

(d) *The line joining two points.*

If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lie on the line, its gradient is

$$\frac{y_2 - y_1}{x_2 - x_1} \quad .$$

Substituting this value for m and the given point as (x_1, y_1) in the gradient form, we obtain the equation of the line as

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad ,$$

or equivalently

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \quad ,$$

where the left hand side is the gradient of P_1P and the right hand side is the gradient of P_1P_2, where $P(x, y)$ and lies on the line joining P_1 and P_2 (see **Figure 6**).

(e) *The intercept form*

The equation of the line through the points $A(a, 0)$ and $B(0, b)$ is (applying the result just obtained)

$$y - 0 = \frac{b - 0}{0 - a}(x - a) \quad .$$

This reduces to

$$\frac{x}{a} + \frac{y}{b} = 1 \quad .$$

This is called the *intercept form*, because it can be written down at once if the lengths of the intercepts a on the x-axis and b on the y-axis are known (see **Figure 7**). Conversely, from the equation in this form the values of a and b can be seen at once. The equations of all lines other than those through the origin can be written in this form.

5.5 Coordinates of the mid-point of a line segment. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be two points, where $x_1 < x_2$ and $y_1 < y_2$. With $R(x, y)$ the mid-point of P_1P_2 and α the angle P_1P_2

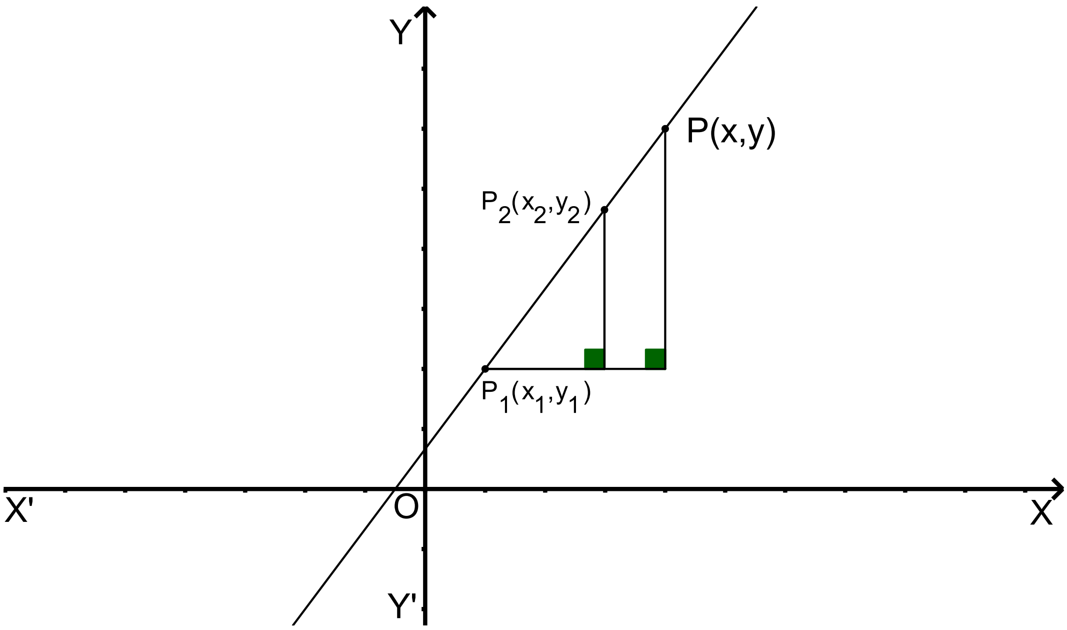

Figure 6: Equation of a line - the line joining two points (Geogebra file)

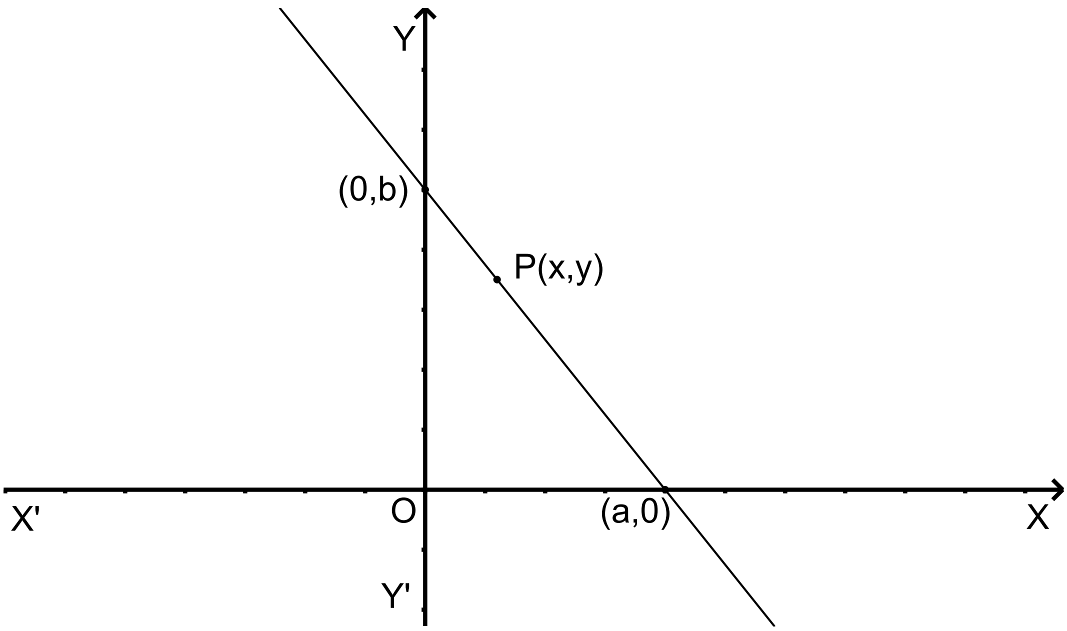

Figure 7: Equation of a line - the intercept form (Geogebra file)

makes with the x-axis, then the coordinates of R are found:

$$\begin{aligned} x &= x_1 + (\tfrac{1}{2}|\mathrm{P}_1\mathrm{P}_2|)\cos\alpha \\ &= x_1 + \tfrac{1}{2}(|\mathrm{P}_1\mathrm{P}_2|\cos\alpha) \\ &= x_1 + \tfrac{1}{2}(x_2 - x_1) \\ &= \tfrac{1}{2}(x_1 + x_2) \ . \end{aligned}$$

Similarly

$$\begin{aligned} y &= y_1 + (\tfrac{1}{2}|\mathrm{P}_1\mathrm{P}_2|)\sin\alpha \\ &= y_1 + \tfrac{1}{2}(|\mathrm{P}_1\mathrm{P}_2|\sin\alpha) \\ &= y_1 + \tfrac{1}{2}(y_2 - y_1) \\ &= \tfrac{1}{2}(y_1 + y_2) \ . \end{aligned}$$

Considering all other possible relative positions of $\mathrm{P}_1(x_1, y_1)$ and $\mathrm{P}_2(x_2, y_2)$ leads to the same result, and hence the coordinates of the mid-point of the line segment joining $\mathrm{P}_1(x_1, y_1)$ and $\mathrm{P}_2(x_2, y_2)$ are $\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$ (see **Figure 8**).

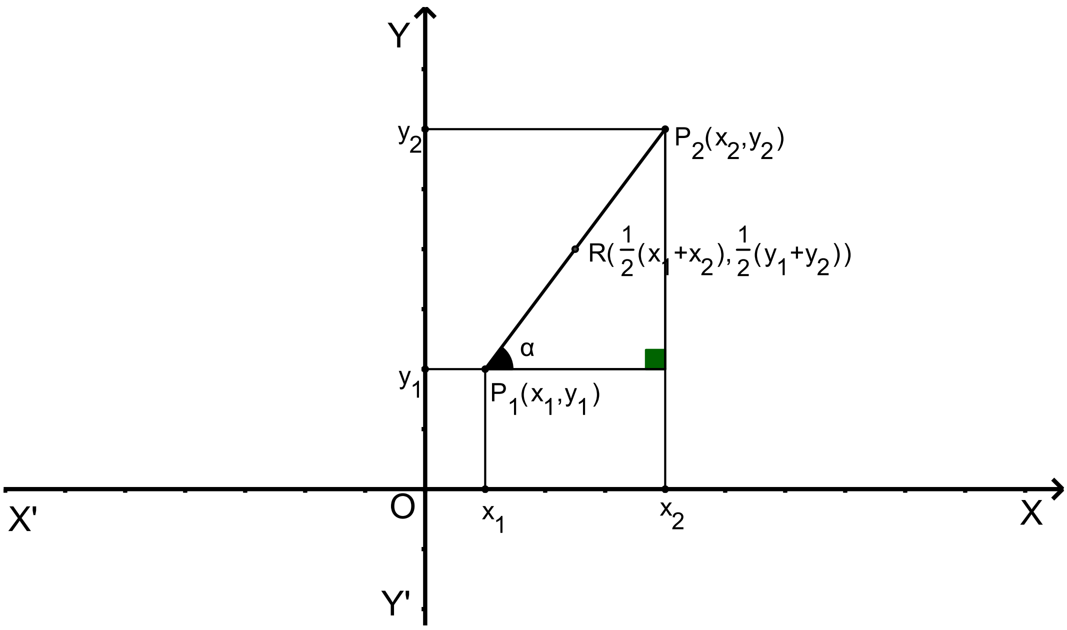

Figure 8: Mid-point (Geogebra file)

7. The parabola.

In the remaining section we shall be studying the parabola which can be defined as the locus of a point which moves so that its distance from a fixed point is equal to its distance from a fixed line.

The fixed point S is called the **focus** and the fixed line is called the **directrix**.
This curve may already have been met in other branches of mathematics. It is the path of a particle projected under gravity in a vacuum - for example the path of a cricket ball (assuming negligible air resistance). It is also the form of the curve which, when rotated about its axis, forms a surface which reflects rays of light or radio waves from a point source into a beam of parallel rays - for example a torch, headlamp or a satellite dish.

Given the focus S of the parabola and the directrix, we can take what axes we find most convenient. First note that the figure formed by the focus and directrix has an axis of symmetry through S perpendicular to the directrix. This we take as the x-axis. If we now plot a few points, as shown in **Figure 20**, using the definition of the locus given in the last section,

$$\frac{\text{SP}}{\text{PM}} = 1$$

an indication of the shape of the curve may be obtained. **Figure 21** shows the complete curve.

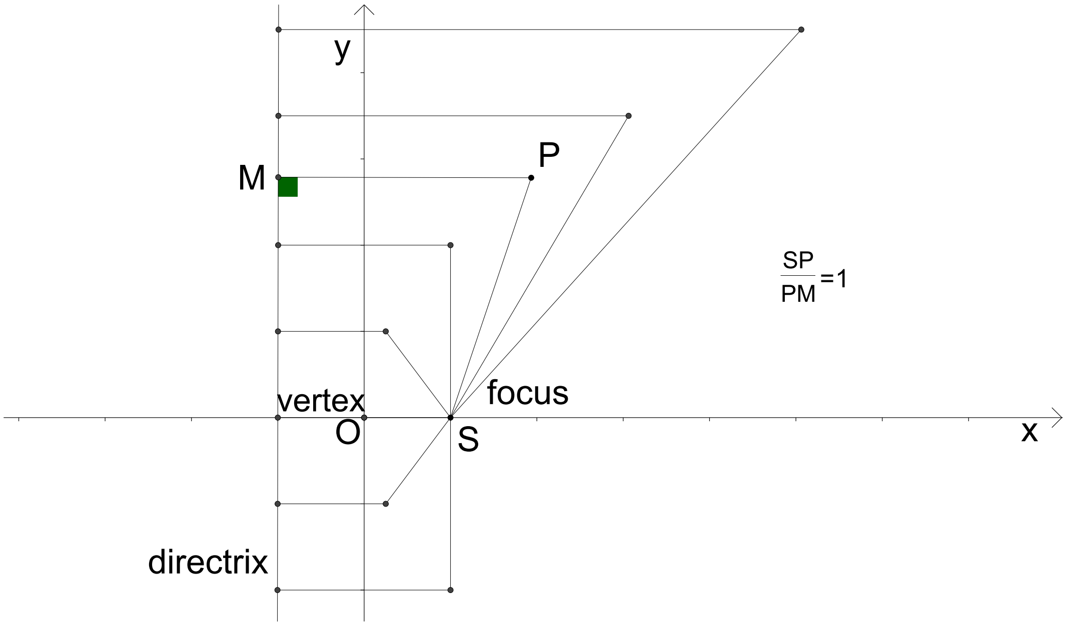

Figure 20: Focus-directrix property of a parabola (Geogebra file)

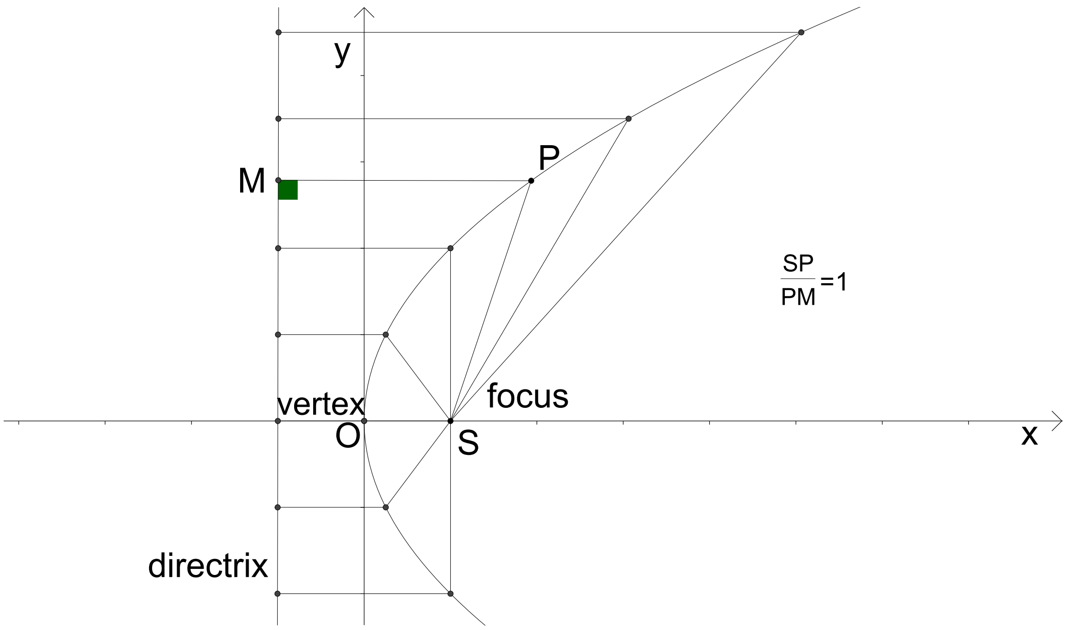

Figure 21: Complete parabola (Geogebra file)

(The plotting may be done very simply using squared paper and a pair of compasses.) It now seems reasonable to take the y-axis through the point on the axis of symmetry mid-way between the focus and directrix. This point is called the **vertex** of the parabola.

Let the distance from the vertex to the focus be $a > 0$, then the focus S is $(a, 0)$ and the directrix is the line $x = -a$. If P(x, y) is any point on the parabola, and M is the foot of the perpendicular from P to the directrix,

$$\text{SP}^2 = (x - a)^2 + y^2$$

and

$$\mathrm{PM} = x + a \quad .$$

But from the definition,

$$\frac{\mathrm{SP}}{\mathrm{PM}} = 1,$$

so

$$\mathrm{SP}^2 = \mathrm{PM}^2 \quad .$$

Substituting in gives

$$(x-a)^2 + y^2 = (x+a)^2$$

i.e.

$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$

and hence

$$y^2 = 4ax \quad ,$$

which is the standard equation of a parabola, as shown in **Figure 22**. This is the equation satisfied by the coordinates of all points on the parabola, and conversely, any point whose coordinates satisfy this equation lies on the curve. We say that $y^2 = 4ax$ is the equation of the parabola (referred to the axes we have chosen).

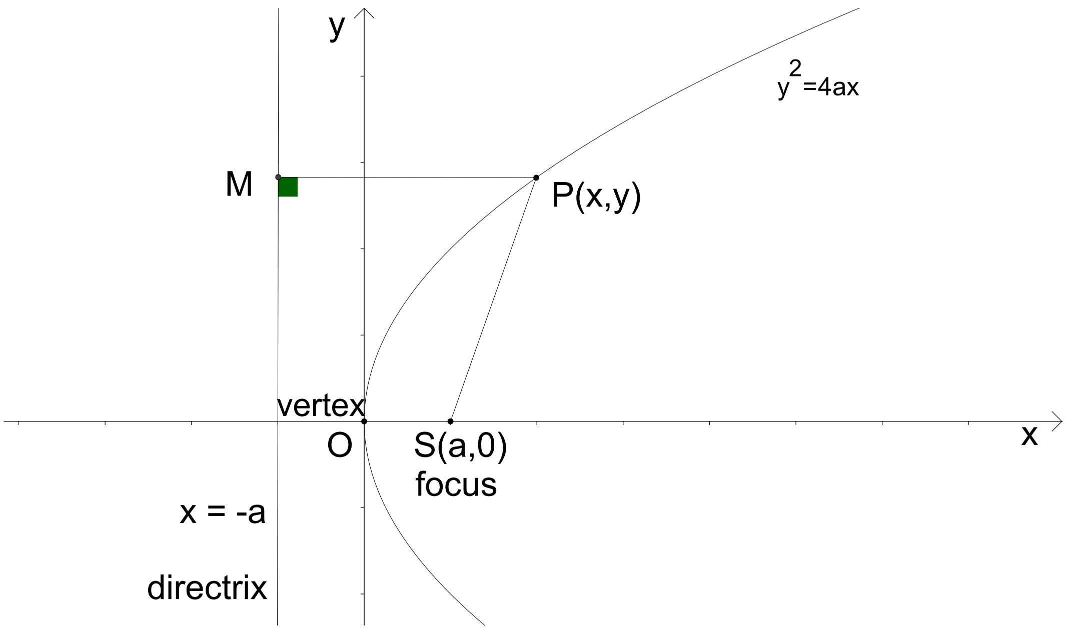

Figure 22: Standard parabola (Geogebra file)

8.1 The shape of the parabola. From the equation, $y^2 = 4ax$, in the standard form, we can deduce a number of facts about the curve.

(i) If in the equation we put $-y$ for y, the equation is unaltered. That is, if $\mathrm{P}(x, y)$ lies on the parabola, so does $\mathrm{P}'(x, -y)$. The curve is thus symmetrical about the x-axis. This line, about which the parabola is symmetrical, is called the axis of the parabola.

(ii) Since a is positive, and $y^2 = 4ax$, x must always be positive. This means that every point on the curve lies to the right of the y-axis.

(iii) When $x = 0$, $y = 0$, so that the curve passes through O. This point, where the parabola meets its axis, is called the vertex of the parabola.

(iv) When x is large, so is y.

(v) A line $x = h$ parallel to the directrix meets the curve where $y = \pm 2\sqrt{ah}$. When $h > 0$ these will be two distinct points. These coincide in the origin when $h = 0$.

(vi) A line $y = k$ parallel to the axis meets the curve in one point $(k^2/4a, k)$.

From elementary calculus we can complete the picture by finding the gradient of the curve at any point. From $y^2 = 4ax$, by differentiation with respect to x (see MA1MM1):

$$2y\frac{dy}{dx} = 4a \quad ,$$

so

$$\frac{dy}{dx} = \frac{2a}{y} \quad .$$

Thus for positive values of y, the gradient decreases as y increases.

With the help of these results, the curve can now be sketched, as shown in **Figure 23**. We notice that all parabolas have the same shape. Varying a merely involves a change of scale, since $y^2 = 4bx$ can be written $(ya/b)^2 = 4a(xa/b)$, i.e. $Y^2 = 4aX$, where $X = (a/b)x$ and $Y = (a/b)y$.

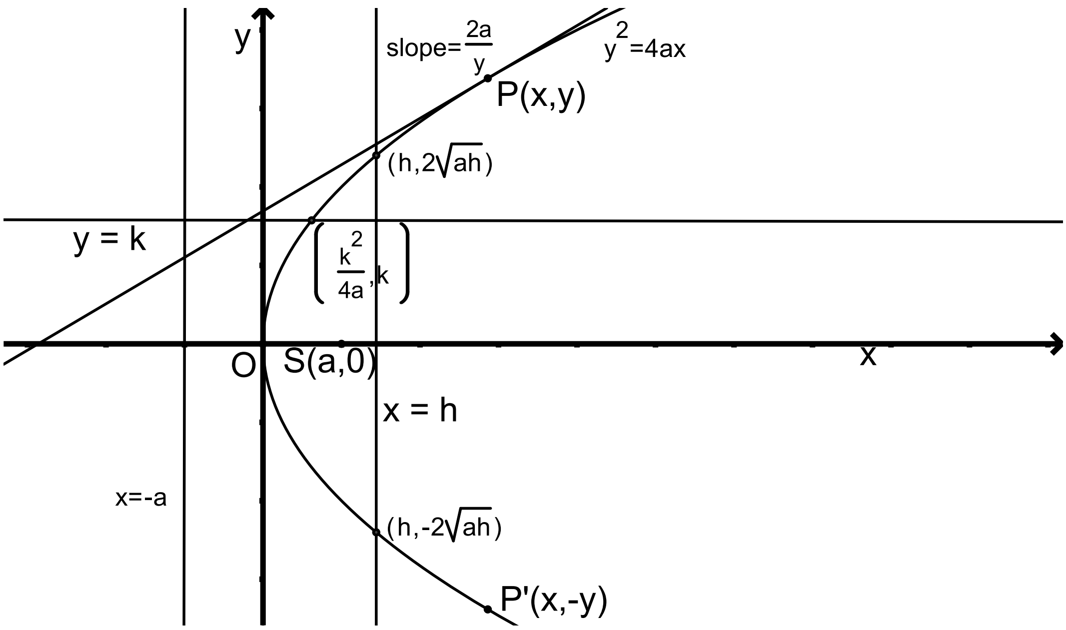

Figure 23: Shape of the parabola (Geogebra file)

8.2 Parametric form. To introduce a parametric form for the parabola we first find, in terms of a, m, the value of c which makes the line $y = mx + c$ a tangent to the parabola $y^2 = 4ax$, and obtain the coordinates of the point of contact.

Starting with

$$y = mx + c \quad ,$$

multiply both sides by $4a$:

$$4ay = m \times 4ax + 4ac$$

and substituting from $y^2 = 4ax$ and rearranging gives

$$my^2 - 4ay + 4ac = 0 \quad . \tag{1}$$

The line will be a tangent if equation (1) has equal roots (see **Figure 24**), i.e.

$$(-4a)^2 = 16mac \; , \text{ i.e. } \; c = \frac{a}{m} \quad .$$

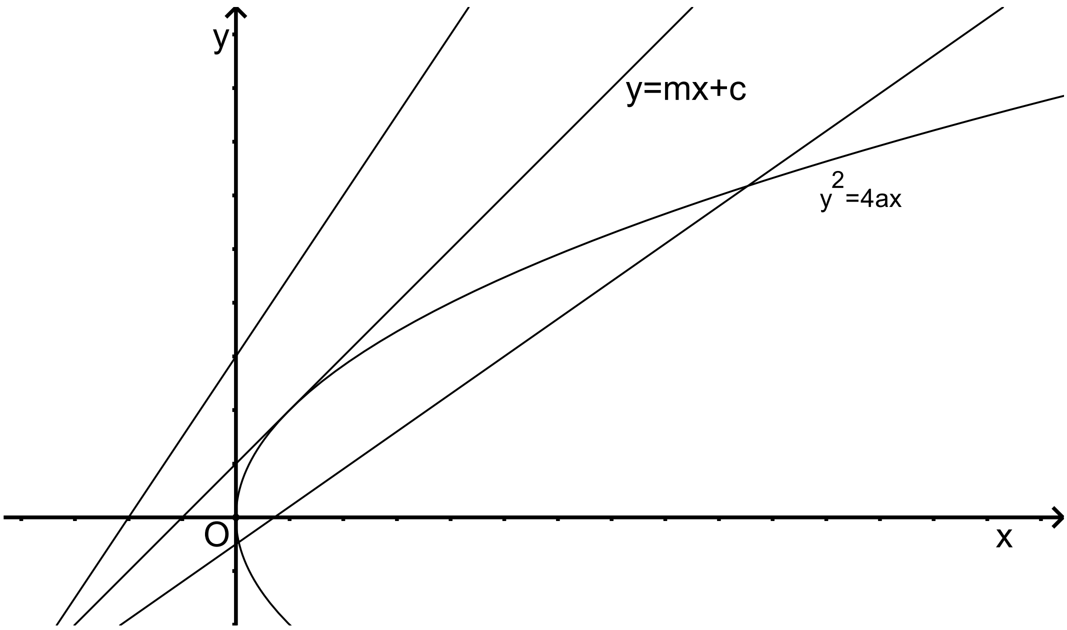

Figure 24: Intersection of a straight line with the parabola (Geogebra file)

When the roots of equation (1) are equal, they will be given by half the sum of the roots (see §2.1):

$$y = \frac{1}{2} \times \frac{4a}{m} = \frac{2a}{m}$$

and thus

$$x = \frac{y^2}{4a} = \left(\frac{2a}{m}\right)^2 \times \frac{1}{4a} = \frac{a}{m^2} \quad .$$

Therefore the point of contact is $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$. Hence the equation of a general tangent to $y^2 = 4ax$ may now be written

$$y = mx + \frac{a}{m} \; , \quad (m \neq 0) \quad .$$

This last result leads us to a very useful way of representing a point on the parabola. Substituting $t = 1/m$, we see that the tangent

$$y = \frac{x}{t} + at$$

of gradient $m = 1/t$ touches the parabola at $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right) = (at^2, 2at)$. Since the tangent was a general one, we have shown that any point on the parabola $y^2 = 4ax$ may be written $(at^2, 2at)$ (see **Figure 25**).
The equations $x = at^2$, $y = 2at$ are called the **parametric equations** of the parabola $y^2 = 4ax$.

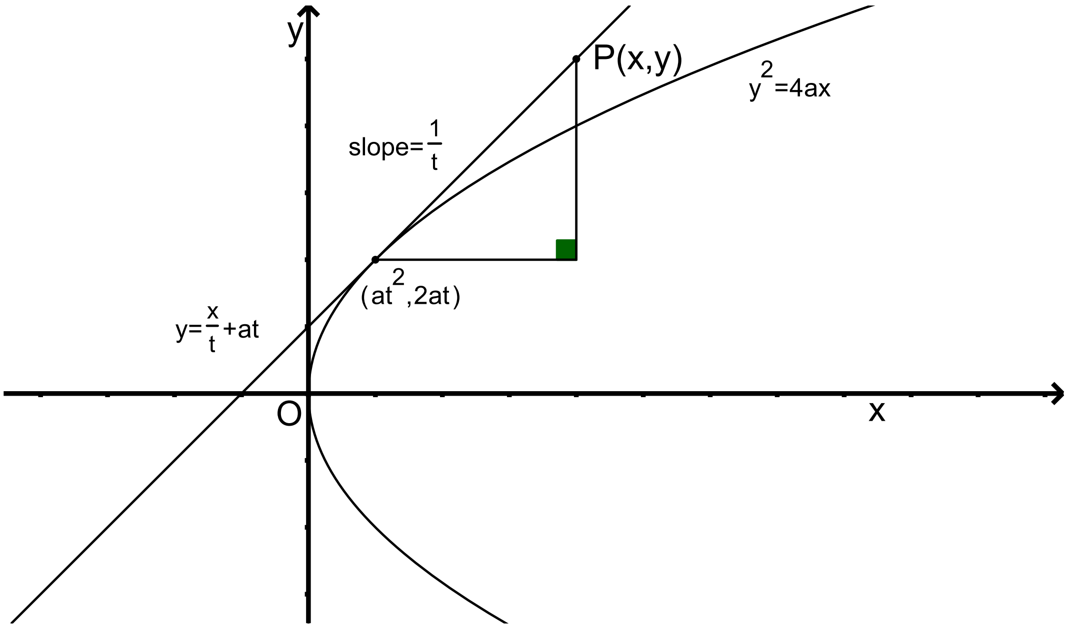

Figure 25: Parametric form (Geogebra file)

It is a straightforward matter to verify, by substitution, that $(at^2, 2at)$ always lies on the parabola $y^2 = 4ax$ by noting that $y^2 = (2at)^2 = 4a^2t^2 = 4a \times at^2 = 4ax$.

8.3 Equation of tangent. We have found the equation of the tangent at $(at^2, 2at)$, but a more direct method is as follows.

To find the gradient at $(at^2, 2at)$ using the parametric form,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy/dt}{dx/dt} \quad .$$

However, since $y = 2at, x = at^2$, we have

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t} \quad .$$

The equation of the tangent is now obtained (see **Figure 26**):

$$\frac{y - 2at}{x - at^2} = \frac{1}{t}$$

i.e.

$$x - ty + at^2 = 0 \quad .$$

Note that the gradient of the tangent to the parabola $y^2 = 4ax$ at the general point with coordinates $(at^2, 2at)$ is $1/t$.

In Cartesian form we can determine the equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) as $yy_1 = 2a(x + x_1)$, as follows.

Differentiating both sides of $y^2 = 4ax$ with respect to x, to find the gradient:

$$2y\frac{dy}{dx} = 4a \quad .$$

Therefore at (x_1, y_1), $\dfrac{dy}{dx} = \dfrac{4a}{2y_1} = \dfrac{2a}{y_1}$, and the tangent is (see **Figure 27**)

$$\frac{y - y_1}{x - x_1} = \frac{2a}{y_1}$$

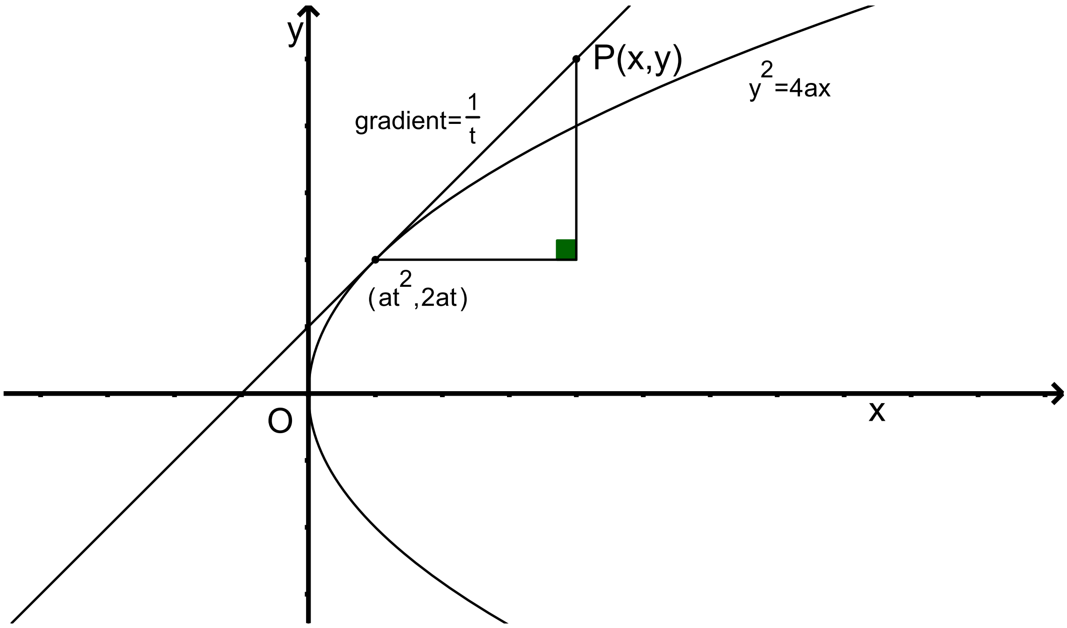

Figure 26: gradient of tangent to the parabola (Geogebra file)

i.e.

$$y_1 y = 2ax - 2ax_1 + y_1^2 \quad .$$

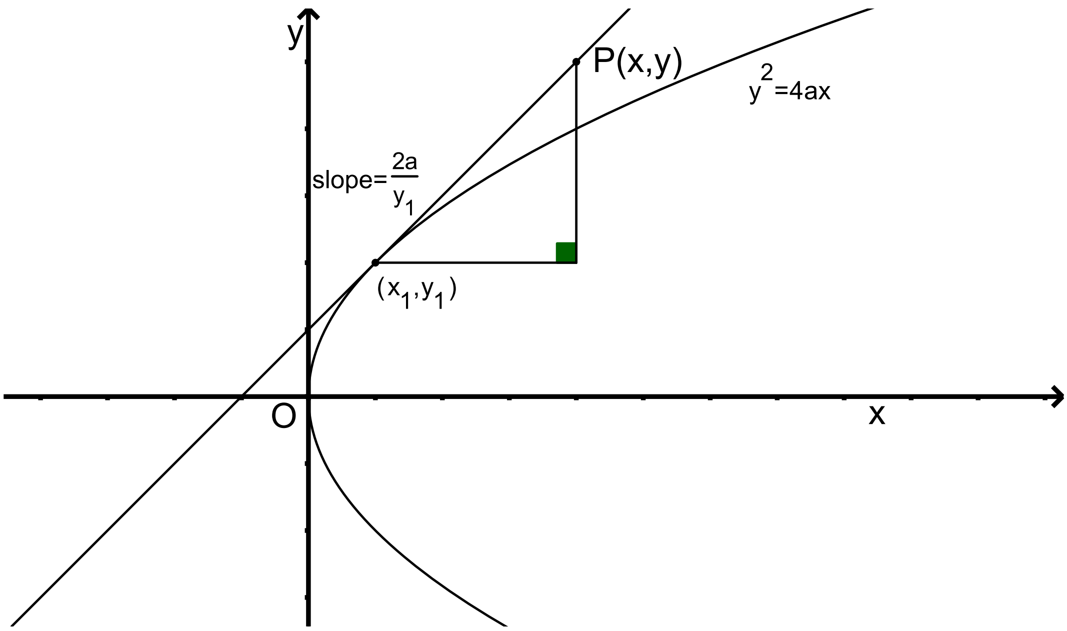

Figure 27: Cartesian form of tangent to the parabola (Geogebra file)

However (x_1, y_1) lies on the parabola, so $y_1^2 = 4ax_1$, and hence

$$y_1 y = 2ax - 2ax_1 + 4ax_1$$

i.e.

$$yy_1 = 2a(x + x_1) \quad .$$

8.4 Equation of normal. To determine the parametric form for the equation of the normal at the point $(at^2, 2at)$ we first note that the gradient of the tangent is $1/t$ and hence the gradient of the normal, which is perpendicular to the tangent, is $\dfrac{-1}{1/t} = -t$.

The equation of the normal is now obtained (see **Figure 28**):

$$\frac{y - 2at}{x - at^2} = -t$$

i.e.

$$y + tx - at^3 - 2at = 0$$

or

$$y + tx = at^3 + 2at \ .$$

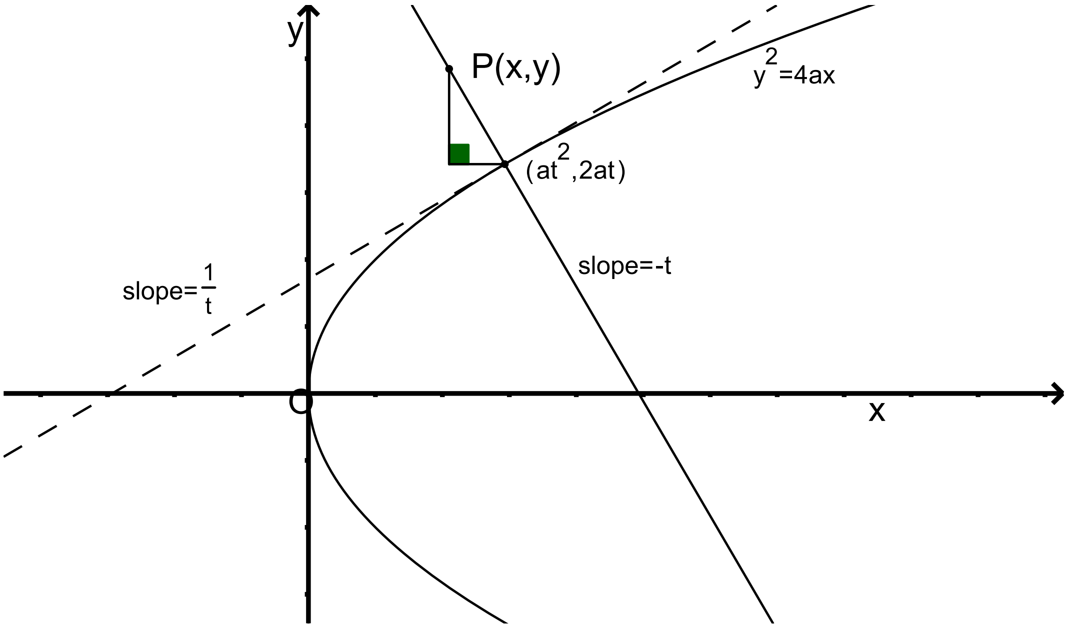

Figure 28: gradient of normal to the parabola (Geogebra file)

Note that the gradient of the normal to the parabola $y^2 = 4ax$ at the general point with coordinates $(at^2, 2at)$ is $-t$.

In Cartesian form we can determine the equation of the normal to the parabola $y^2 = 4ax$ at (x_1, y_1) as $2a(y - y_1) + y_1(x - x_1) = 0$, as follows.

From above the gradient of the tangent at the point (x_1, y_1) is $\dfrac{2a}{y_1}$, and hence the gradient of the normal is $\dfrac{-1}{2a/y_1} = \dfrac{-y_1}{2a}$.

Therefore at (x_1, y_1), $\dfrac{dy}{dx} = \dfrac{-y_1}{2a}$, and the normal is (see **Figure 29**)

$$2a(y - y_1) + y_1(x - x_1) = 0 \ .$$

8.5 Equation of chord. We now determine the equation of the chord joining the points at $\mathrm{P}_1(at_1^2, 2at_1)$, $\mathrm{P}_2(at_2^2, 2at_2)$.

The gradient of the chord $\mathrm{P}_1\mathrm{P}_2$ is

$$\begin{aligned} \frac{2at_2 - 2at_1}{at_2^2 - at_1^2} &= \frac{2a(t_2 - t_1)}{a(t_2 - t_1)(t_1 + t_2)} \\ &= \frac{2}{t_1 + t_2} \ . \end{aligned}$$

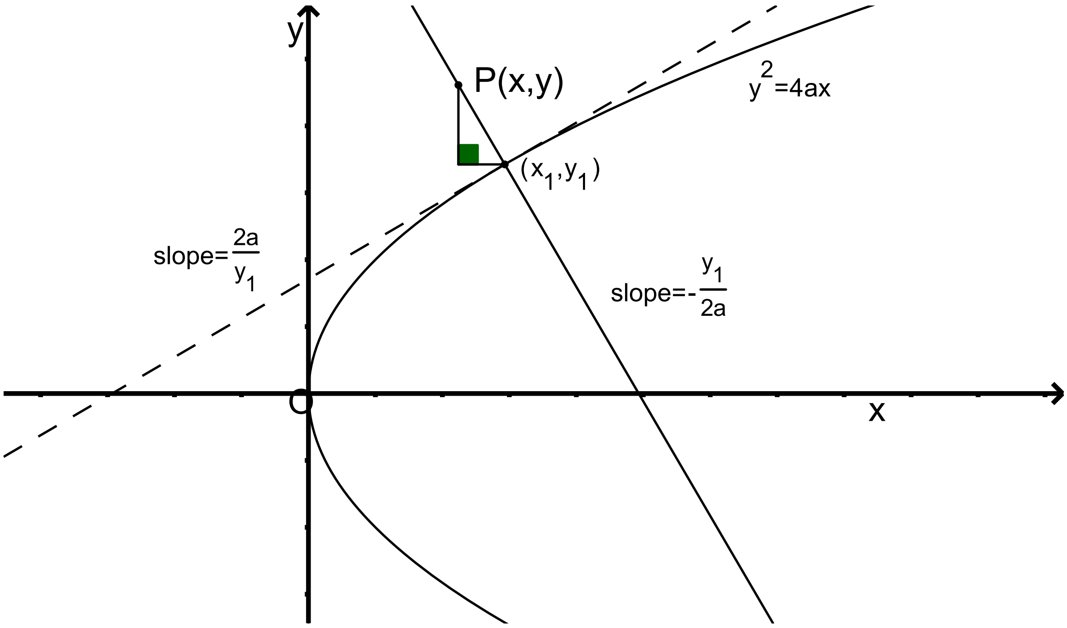

Figure 29: Cartesian form of normal to the parabola (Geogebra file)

Therefore the equation of the chord is (see **Figure 30**)

$$\frac{y-2at_1}{x-at_1^2}=\frac{2}{t_1+t_2}$$

i.e.

$$\begin{aligned} 2x-(t_1+t_2)y &= 2at_1^2-(t_1+t_2)\times 2at_1 \\ &= -2at_1t_2 \end{aligned}$$

and hence

$$x-\tfrac{1}{2}(t_1+t_2)y+at_1t_2=0 \quad .$$

Note, as $t_2 \to t_1$, the chord approaches the tangent at t_1, giving $x-t_1y+at_1^2=0$, which is the equation of the tangent at $(at_1^2, 2at_1)$.

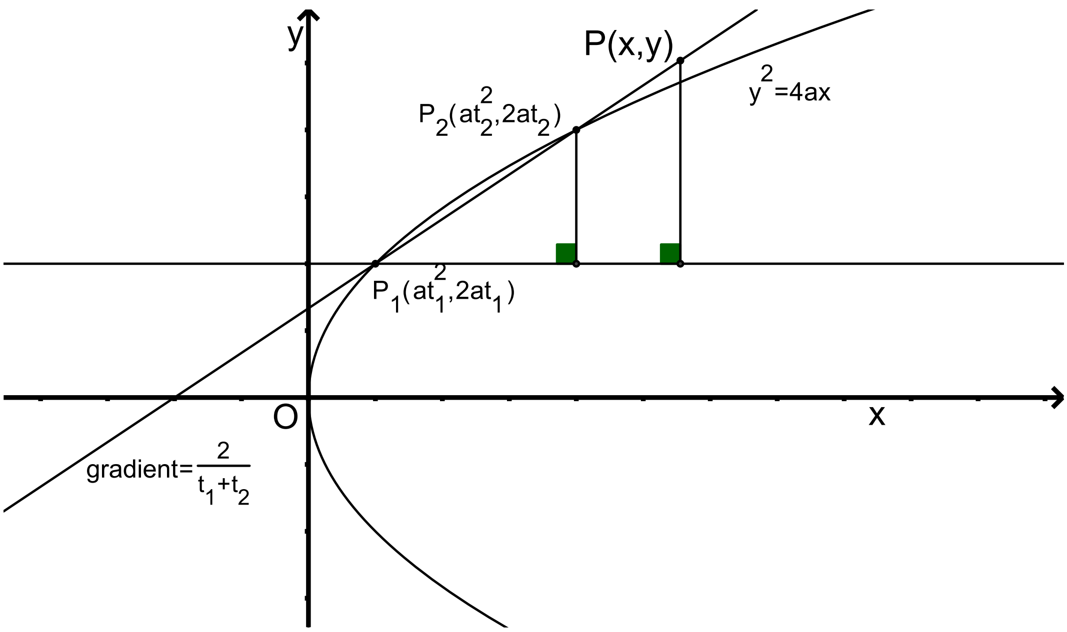

Figure 30: a chord to the parabola (Geogebra file)

8.6 Some definitions. Any chord of a parabola passing through the focus is called a **focal chord**. Recall that the axis of symmetry is called the **axis of the parabola**. The focal chord perpendicular to the axis is called the **latus rectum**. To find the length of the latus rectum of the parabola $y^2 = 4ax$, substitute $x = a$:

$$y^2 = 4a^2$$

i.e.

$$y = \pm 2a \ .$$

Hence the length of the latus rectum is $4a$ (see **Figure 31**).

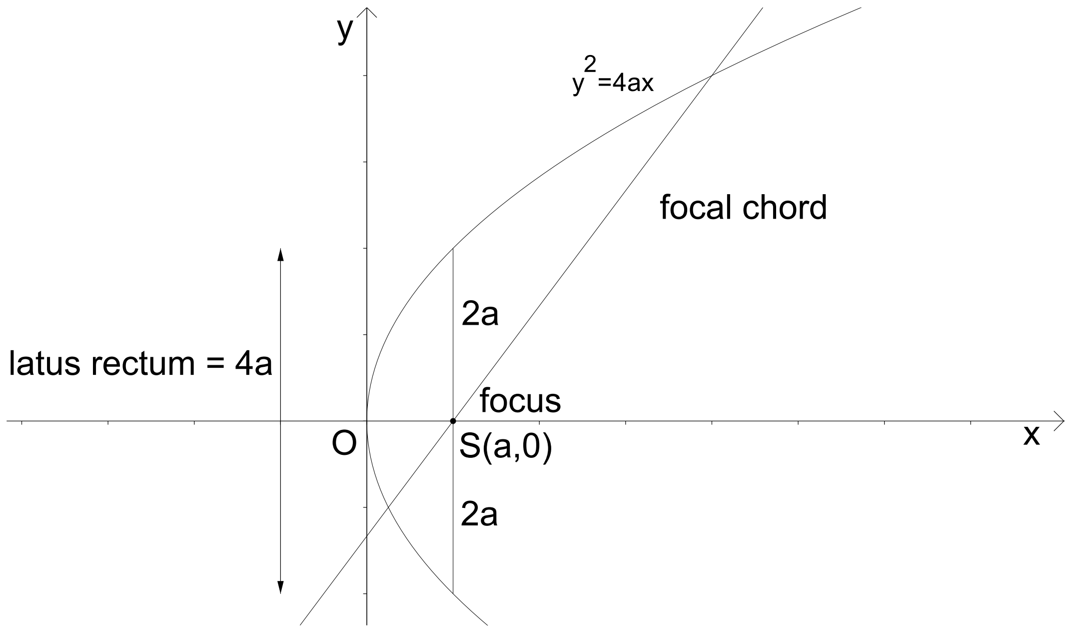

Figure 31: Focal Chord and Latus Rectum (Geogebra file)

8.7 The point of intersection of tangents, normals and the equation of the chord of contact.

We now show that the coordinates of the point of intersection R(X, Y) of the tangents at $(ap^2, 2ap)$ and $(aq^2, 2aq)$ are

$$X = apq \ \text{ and } \ Y = a(p+q) \ .$$

First X and Y satisfy both equations for the tangents (see **Figure 32**), i.e.

$$X - pY + ap^2 = 0 \qquad (i)$$
$$X - qY + aq^2 = 0 \qquad (ii)$$

Subtracting $(i) - (ii)$ gives

$$(q-p)Y + a(p^2 - q^2) = 0 \qquad (iii)$$

and factorising using $p^2 - q^2 \equiv (p-q)(p+q)$ means that we can write (iii) as

$$(q-p)Y + a(p-q)(p+q) = 0 \qquad (iv)$$

i.e.

$$(q-p)Y - a(q-p)(p+q) = 0 \qquad (v)$$

Dividing (v) through by $p - q$ (assuming P and Q are two different points so $p \neq q$), gives

$$Y = a(p+q) \qquad (vi)$$

and substituting Y from (vi) into (i) gives

$$\begin{aligned} X &= pY - ap^2 \\ &= ap(p+q) - ap^2 \\ &= apq \qquad (vii) \end{aligned}$$

giving

$$X = apq \text{ and } Y = a(p+q)$$

as the coordinates of the point of intersection of the tangents at P and Q.

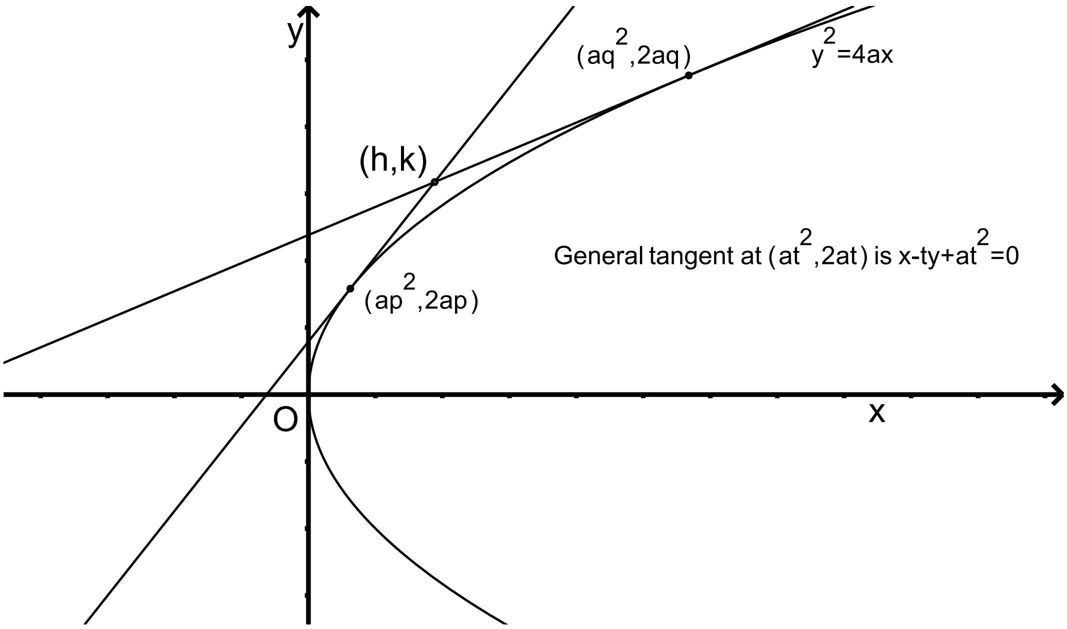

Figure 32: Point of intersection of tangents (Geogebra file)

An alternative way of looking at this problem is as follows.

Suppose that (h, k) lies on the tangent at $(at^2, 2at)$, whose equation is

$$x - ty + at^2 = 0$$

then

$$h - tk + at^2 = 0 \ .$$

This is a quadratic equation in t, so that there are at most two tangents from a point (h, k) to a parabola (see **Figure 32**). If the roots of the equation are p and q, the usual sum and product formulae give

$$p + q = k/a, \quad pq = h/a \ .$$

Thus the tangents to the parabola at the points with parameters p and q meet in the point (h, k) where

$$h = apq, \quad k = a(p+q),$$

that is, in the point

$$(apq, a(p+q)) \ .$$

The same piece of algebra generates the equation of the chord joining the point of contact of tangents drawn to the parabola from the point (h, k). This line is called the **chord of contact**;

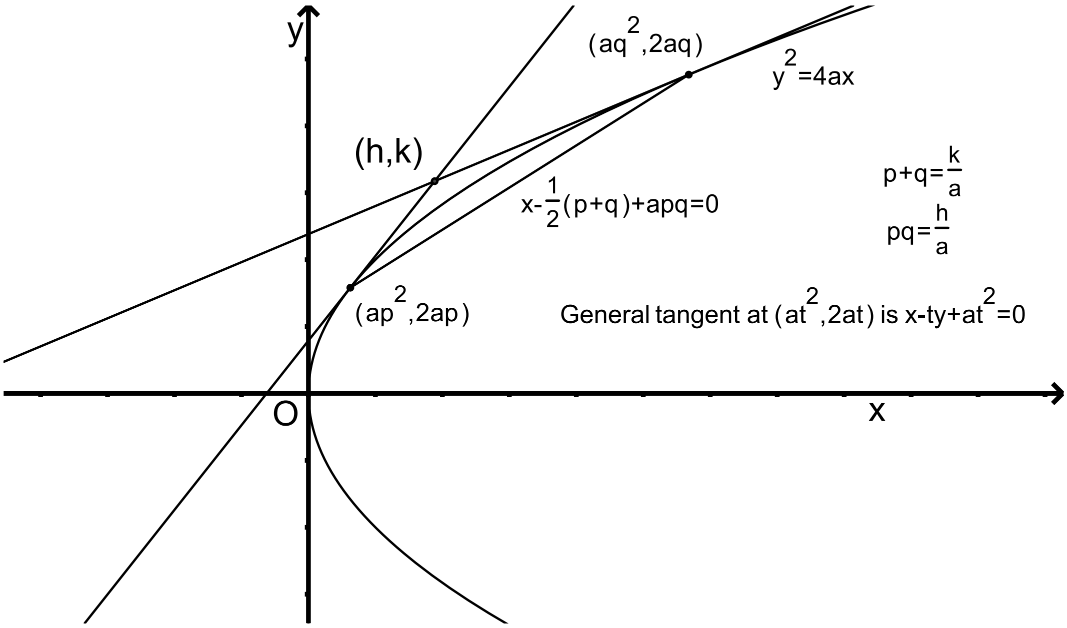

Figure 33: Chord of contact and polar line (Geogebra file)

it is also called the **polar line** of (h, k) with respect to the parabola (see **Figure 33**).

The equation of the chord joining the points with parameters p and q is

$$x - \tfrac{1}{2}(p+q)y + apq = 0 \quad .$$

If the tangents at these points meet in (h, k), we know already that

$$p + q = k/a, \;\; pq = h/a \quad .$$

Substituting these values in the equation of the chord, we find that the equation of the chord of contact is

$$x - (k/2a)y + h = 0$$

i.e.

$$ky = 2a(x+h) \quad .$$

From the same equation

$$h - kt + at^2 \;\; = 0$$

which states that the tangent at $(at^2, 2at)$ passes through (h, k), we can deduce the condition on a point in the plane for there to be a tangent through it. The equation has real and distinct roots in t provided

$$k^2 > 4ah \quad .$$

We say that points of the plane through which there are two tangents to the parabola are *outside* the parabola, and points through which there are no tangents are *inside* (see **Figure 34**).

Thus we see that the parabola divides the plane into two regions:

(i) *outside* the parabola when $y^2 > 4ax$,

(ii) *inside* the parabola when $y^2 < 4ax$.

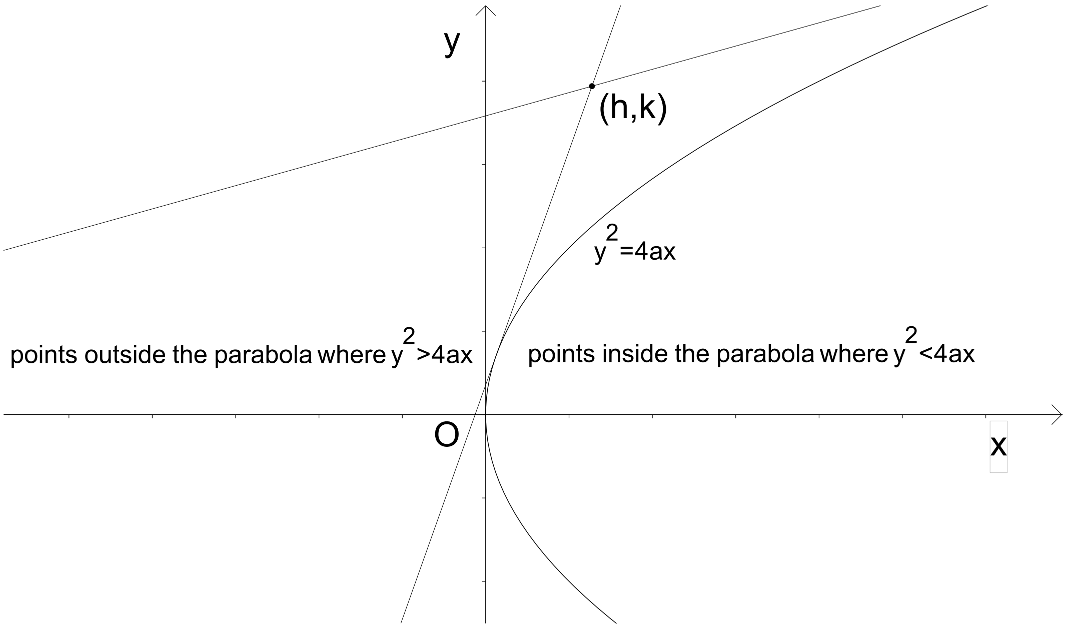

Figure 34: Points inside and outside the Parabola (Geogebra file)

Between these two regions there is the boundary, the parabola itself, where $y^2 = 4ax$.

We now show that the coordinates of the point of intersection R(X, Y) of the normals at $(ap^2, 2ap)$ and $(aq^2, 2aq)$ are

$$X = a(p^2 + pq + q^2 + 2) \text{ and } Y = -apq(p + q) \ ,$$

which can be re-written as

$$X = a(a((p + q)^2 - pq + 2)) \text{ and } Y = -apq(p + q) \ .$$

First X and Y satisfy both equations for the normals (see **Figure 35**), i.e.

$$pX + Y = ap^3 + 2ap \qquad (i)$$
$$qX + Y = aq^3 + 2aq \qquad (ii)$$

Subtracting $(i) - (ii)$ gives

$$(p - q)X = a(p^3 - q^3) + 2a(p - q) \qquad (iii)$$

Factorising using $p^3 - q^3 \equiv (p - q)(p^2 + pq + q^2)$ means that when we divide (iii) by $p - q$ (assuming P and Q are two different points), we have

$$\begin{aligned} X &= a(p^2 + pq + q^2) + 2a \\ &= a(p^2 + q^2 + pq + 2) \qquad (iv) \end{aligned}$$

which can also be written as

$$X = a\left((p + q)^2 - pq + 2\right) \qquad (v)$$

using $(p + q)^2 \equiv p^2 + q^2 + 2pq$.

Finally, from (i) and (iv)

$$\begin{aligned} Y &= ap^3 + 2ap - ap(p^2 + q^2 + pq + 2) \\ &= a(p^3 + 2p - p^3 - pq^2 - p^2q - 2p) \\ &= a(-pq^2 - p^2q) \\ &= -apq(p+q) \qquad (vi) \end{aligned}$$

(vi) and either (iv) or (v) give the coordinates of the point of intersection of the normals at P and Q.

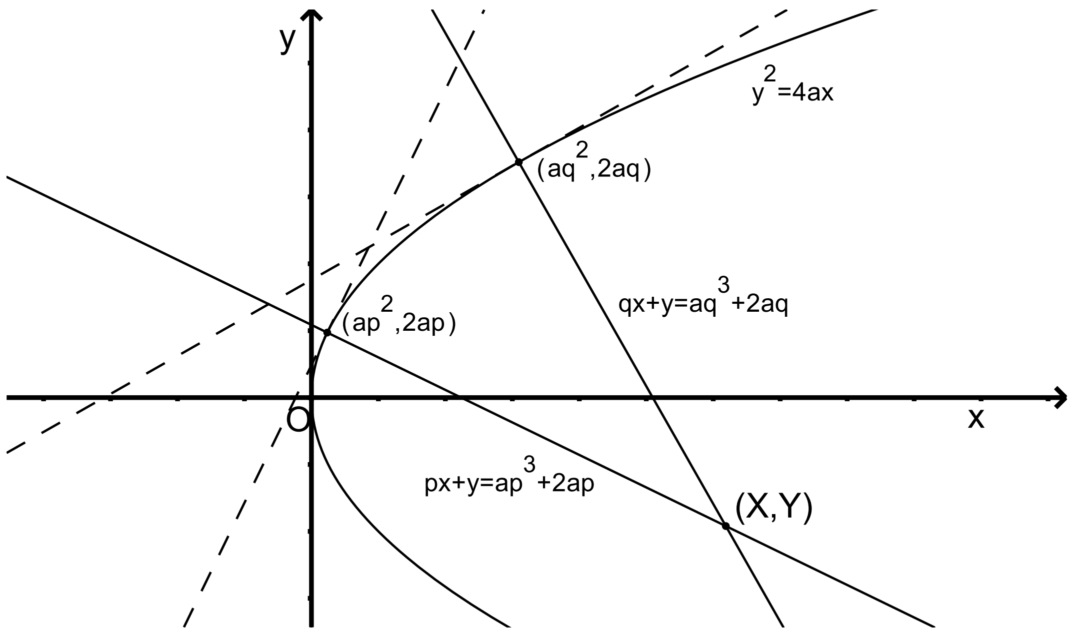

Figure 35: Point of intersection of normals (Geogebra file)

8.8 Solving problems about loci.

A problem with which coordinate geometry is particularly well suited to deal is that of finding a locus. The problem is very often as follows: P is a point which can vary on some curve C, and another point, R, say, is constructed from P and any fixed points and lines which may be given. As a result of the variation of P, R will also vary, and the problem is to find this locus of R.

The curve C which is given may be any curve. We will illustrate this first in our examples by taking it as the parabola $y^2 = 4ax$. The variable point P on the parabola we take as $(ap^2, 2ap)$, and we notice that when P varies the parameter p will vary also. The problem is then tackled in three stages:

(i) Find the coordinates (X, Y) of R, in terms of p and any fixed numbers (usually at least the a involved in $y^2 = 4ax$).

(ii) Eliminate p between the expressions for X and Y, thus finding a relation which is satisfied by the coordinates of R for all values of p, and so for all positions of P.

(iii) The relationship between X and Y (which can be rewritten as x and y, respectively) represents the locus of R.

This in outline is the method adopted, though in particular circumstances modifications are needed. Frequently R depends on two points P and Q on the parabola, and so on the parameters p and q. These are connected by some equation which comes from some condition on P and Q, for example, PQ passing through a fixed point. Again, in some more involved examples, stage (i) of this method need not be carried out in full.

Another modification arises when the varying of R results from some variation other than P moving on the parabola. For example, R might depend on a variable point on a fixed line, or on the parabola which itself varies. In each case the important thing is to decide what causes the variation of the point (X, Y) whose locus is required.

8.9 Conormal points. We have already shown that from a point not on the curve there are either two or no tangents to a parabola. We now investigate the corresponding result for normals.

The equation of the normal to a parabola at a point $(at^2, 2at)$ is

$$tx + y = at^3 + 2at \quad .$$

If this passes through a fixed point (h, k),

$$th + k = at^3 + 2at$$

i.e.

$$at^3 + t(2a - h) - k = 0 \quad .$$

This is a cubic equation in t. We deduce that there are at most three normals to a parabola from a point. The points where the normals meet the parabola are called **conormal** points on the curve (see **Figure 36**).

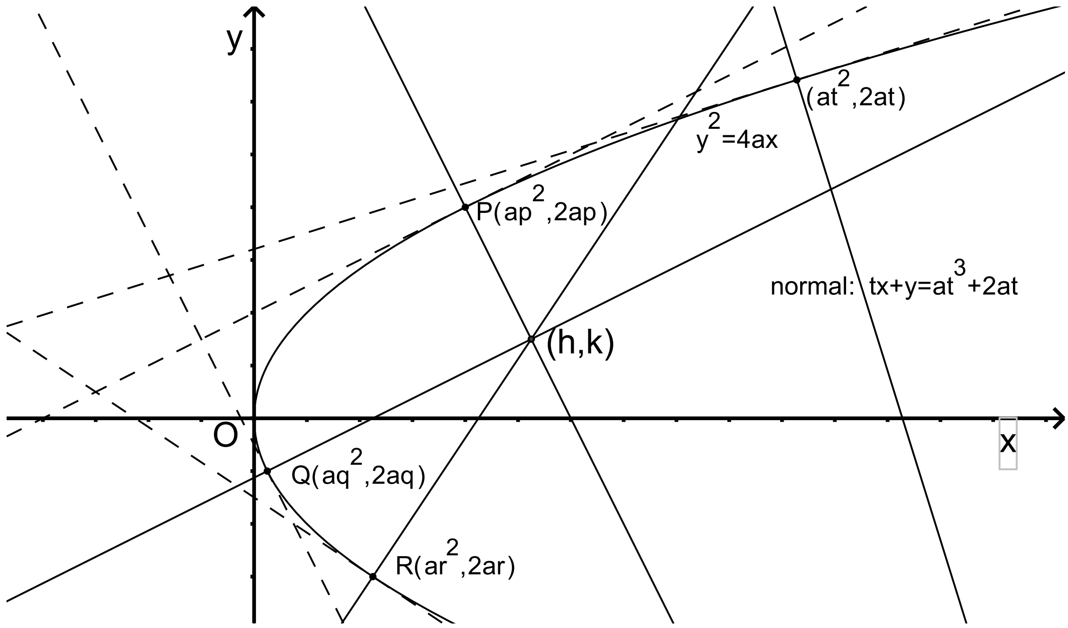

Figure 36: Conormal points (Geogebra file)

If the parameters of three conormal points are p, q and r, then these are the roots of the cubic equation in t. The result on the sum and products of the roots (see §2.2) gives

$$p + q + r = 0 \quad , \quad qr + rp + pq = \frac{2a - h}{a} \quad , \quad pqr = k/a \quad , \tag{1}$$

so we have the result that, if the normals at the points with parameters p, q and r are **concurrent** (i.e. they meet at the same point), then

$$p + q + r = 0 \quad .$$

Conversely, if $p+q+r=0$, then the normals at the points p, q and r are concurrent. To see why this is the case, suppose that two of the normals, say at p and q, meet in (h,k), then there is a third normal through this point, and if the parameter of its point of contact is r', then $p+q+r'=0$. It follows that $r=r'$.

Results about conormal points can usually be derived from the three equations (1) found above.

For example, eliminating r, which is equal to $-(p+q)$, we find that the point of intersection of the normals at p and q is given by

$$pq+(p+q)r = pq-(p+q)^2 = 2-h/a, \tag{1}$$

and

$$pq \times -(p+q) = k/a \tag{2}$$

i.e. the normals intersect in the point (h,k) given by (1) and (2). Therefore

$$(h,k) = \big(a((p+q)^2-pq+2), -apq(p+q)\big) = \big(a(p^2+q^2+pq+2), -apq(p+q)\big) \quad ,$$

the result found earlier.

12. Equation of the chord of contact and point of intersection of tangents for an ellipse, hyperbola and parabola.

The standard ellipse can be written as $x^2/a^2+y^2/b^2=1$, and the equation of the tangents at the points $\mathrm{P}(x_1,y_1)=\mathrm{P}(a\cos\theta, b\sin\theta)$ and $\mathrm{Q}(x_2,y_2)=\mathrm{Q}(a\cos\phi, b\sin\phi)$ can be written as

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad , \quad \frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1 \tag{1}$$

or

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \quad , \quad \frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1 \quad . \tag{2}$$

If we wish to find the coordinates of the point of intersection $\mathrm{T}(h,k)$ of these tangents we might consider solving the pair of simultaneous linear equations (arising from substituting $(x,y)=(h,k)$ in (2))

$$\frac{h}{a}\cos\theta+\frac{k}{b}\sin\theta=1 \quad , \quad \frac{h}{a}\cos\phi+\frac{k}{b}\sin\phi=1 \tag{3}$$

for h, k, which is possible using some algebra. However, what we might do instead is first observe that from the point $\mathrm{T}(h,k)$ one can draw two tangents to the ellipse, which meet it at (x_1,y_1) and (x_2,y_2), where, from (1)

$$\frac{hx_1}{a^2}+\frac{ky_1}{b^2}=1 \quad , \quad \frac{hx_2}{a^2}+\frac{ky_2}{b^2}=1 \quad . \tag{4}$$

Now equations (4) can viewed as expressing the fact that each of the points (x_1,y_1) and (x_2,y_2) lie on the line

$$\frac{hx}{a^2}+\frac{ky}{b^2}=1 \tag{5}$$

and hence this is the **chord of contact** or **polar line** of (h,k). However, the equation of this chord can also be written as

$$bx\cos\tfrac{1}{2}(\phi+\theta)+ay\sin\tfrac{1}{2}(\phi+\theta)=ab\cos\tfrac{1}{2}(\phi-\theta) \quad . \tag{6}$$

Comparing (5) and (6) we have

$$\frac{h/a^2}{b\cos\frac{1}{2}(\phi+\theta)} = \frac{k/b^2}{ay\sin\frac{1}{2}(\phi+\theta)} = \frac{1}{ab\cos\frac{1}{2}(\phi-\theta)}$$

i.e.

$$h = \frac{a\cos\frac{1}{2}(\phi+\theta)}{\cos\frac{1}{2}(\phi-\theta)} \ , \ k = \frac{b\sin\frac{1}{2}(\phi+\theta)}{\cos\frac{1}{2}(\phi-\theta)}$$

as the coordinates of the point of intersection of tangents at $\mathrm{P}(a\cos\theta, b\sin\theta)$ and $\mathrm{Q}(a\cos\phi, b\sin\phi)$. This result clearly applies in the special case of a circle where $b = a$.

Similarly for the standard hyperbola, which can be written as $x^2/a^2 - y^2/b^2 = 1$, where this time the equation of the tangents at the points $\mathrm{P}(x_1, y_1) = \mathrm{P}(a\sec\theta, b\tan\theta)$ and $\mathrm{Q}(x_2, y_2) = \mathrm{Q}(a\sec\phi, b\tan\phi)$ can be written as

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \ , \ \frac{xx_2}{a^2} - \frac{yy_2}{b^2} = 1 \tag{7}$$

or

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1 \ , \ \frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1 \ . \tag{8}$$

Again, to find the coordinates of the point of intersection $\mathrm{T}(h, k)$ of these tangents we might consider solving the pair of simultaneous linear equations

$$\frac{h}{a}\sec\theta - \frac{k}{b}\tan\theta = 1 \ , \ \frac{h}{a}\sec\phi - \frac{k}{b}\tan\phi = 1 \tag{9}$$

for h, k. Alternatively, again observing that from the point $\mathrm{T}(h, k)$ one can draw two tangents to the hyperbola, which meet it at (x_1, y_1) and (x_2, y_2), where, from (7)

$$\frac{hx_1}{a^2} - \frac{ky_1}{b^2} = 1 \ , \ \frac{hx_2}{a^2} - \frac{ky_2}{b^2} = 1 \ . \tag{10}$$

Viewing equations (10) as expressing the fact that each of the points (x_1, y_1) and (x_2, y_2) lie on the line

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1 \tag{11}$$

this is the **chord of contact** or **polar line** of (h, k). However, equation (11) is also the equation of the chord through P and Q, which can be written as

$$bx\cos\tfrac{1}{2}(\phi-\theta) - ay\sin\tfrac{1}{2}(\phi+\theta) = ab\cos\tfrac{1}{2}(\phi+\theta) \ . \tag{12}$$

Comparing (11) and (12) we have

$$\frac{h/a^2}{b\cos\frac{1}{2}(\phi-\theta)} = \frac{k/b^2}{ay\sin\frac{1}{2}(\phi+\theta)} = \frac{1}{ab\cos\frac{1}{2}(\phi+\theta)}$$

i.e.

$$h = \frac{a\cos\frac{1}{2}(\phi-\theta)}{\cos\frac{1}{2}(\phi+\theta)} \ , \ k = \frac{b\sin\frac{1}{2}(\phi+\theta)}{\cos\frac{1}{2}(\phi+\theta)}$$

as the coordinates of the point of intersection of tangents at $\mathrm{P}(a\sec\theta, b\tan\theta)$ and $\mathrm{Q}(a\sec\phi, b\tan\phi)$.

Finally, for the standard parabola, which can be written as $y^2 = 4ax$, this time the equation of the tangents at the points $\mathrm{P}(x_1, y_1) = \mathrm{P}(ap^2, 2ap)$ and $\mathrm{Q}(x_2, y_2) = \mathrm{Q}(aq^2, 2bq)$ can be written as

$$yy_1 = 2a(x + x_1) \ , \ yy_2 = 2a(x + x_2) \tag{13}$$

or

$$x - py + ap^2 = 0 \ \ , \ \ x - qy + aq^2 = 0 \ \ . \tag{14}$$

Again, to find the coordinates of the point of intersection T(h, k) of these tangents we might consider solving the pair of simultaneous linear equations

$$h - pk + ap^2 = 0 \ \ , \ \ h - qk + aq^2 = 0 \tag{15}$$

for h, k. Alternatively, again observing that from the point T(h, k) one can draw two tangents to the parabola, which meet it at (x_1, y_1) and (x_2, y_2), where, from (13)

$$ky_1 = 2a(h + x_1) \ \ , \ \ ky_2 = 2a(h + x_2). \tag{16}$$

Viewing equations (16) as expressing the fact that each of the points (x_1, y_1) and (x_2, y_2) lie on the line

$$ky = 2a(h + x) \ \ , \ \ \text{i.e. } 2ax - ky + 2ah = 0 \tag{17}$$

which is the **chord of contact** or **polar line** of (h, k). However, equation (17) is also the equation of the chord through P and Q, which can be written as

$$x - \tfrac{1}{2}(p + q)y + apq = 0 \ \ . \tag{18}$$

Comparing (17) and (18) we have

$$\frac{2a}{1} = \frac{k}{\frac{1}{2}(p + q)} = \frac{2ah}{apq}$$

i.e.

$$h = apq \ \ , \ \ k = a(p + q)$$

as the coordinates of the point of intersection of tangents at P$(ap^2, 2ap)$ and Q$(aq^2, 2aq)$.