A mathematician's miscellany, and an apology

Paul Glaister

Throughout my career I have been very fortunate to be able to work on many interesting problems in mathematics, some of which have appeared in the public domain in the form of publications (455 at the time of writing!) in journals etc.[1]

The three main areas of study I have pursued are:

- a numerical analysis and computational fluid dynamics, including the development and analysis of numerical schemes for the solution of problems arising in applied mathematics;
- b mathematics and science education, at both school and university, and the school-university interface;
- c teaching and learning, primarily in higher education.

If there is any impact of this work it is, in part, indicated by the numbers of citations of my work by other authors, referencing and using this work, e.g. on Google Scholar[2] I have been cited 1364 times and on Mendeley[3] there have been 18959 downloads of a subset of my publications which are in the ScienceDirect[4] database and made available through Scopus[5].

The impact of work in areas (b) and (c) is mainly through a national and international audience of mathematics and science education practitioners in schools, colleges and universities, using my work for enhancement, enrichment and hopefully enjoyment.

In the categories in (a) authors of the 12 most recent papers citing my work have been working on:

- sea wave energy using an oscillating water column as an alternative, renewable energy source, which is sustainable and with no impact on environmental pollution;
- a finite volume scheme for the solution of a multi-component gas flow model in a pipe on non-flat topography;
- development of an in vitro methodology capable of use in commercial testing laboratories for measuring the human ingestion bio accessibility of polyaromatic hydrocarbons (PAHs) in soil;
- an explicit homogeneous conservative quasi-acoustic scheme for the numerical solution of one-dimensional shallow-water equations with an uneven bottom
- monotone, second-order accurate numerical scheme is presented for solving the differential form of the adjoint shallow-water equations in generalized two-dimensional coordinates;
- air-water interactions within storm water systems during rapid inflow conditions;
- verification, validation and uncertainty quantification in thermal-hydraulics analysis;
- implicit second-order accurate spatial scheme for steady-state thermal-hydraulic simulations of the two-phase two-fluid six-equation model for use in the nuclear energy industry;
- le Châtelier's Principle applied to model Strong Acid-Strong Base titrations;
- continuous adjoint method for steady-state two-phase flow simulations;
- compressible flow at high pressure with a linear equation of state;

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[1] http://centaur.reading.ac.uk/view/creators/90000233.html
[2] https://scholar.google.com/citations?hl=en&user=vm3zUvUAAAAJ
[3] https://www.mendeley.com/profiles/paul-glaister/stats/
[4] https://www.sciencedirect.com/search?authors=glaister%20p&show=25&sortBy=relevance
[55] https://www.scopus.com/authid/detail.uri?authorId=7003342589

- coexistence of two important communication techniques, non-orthogonal multiple access (NOMA - a key enabling technology in next-generation wireless networks due to its superior spectral efficiency) and mobile edge computing (MEC).

some of which refer to work I did more than 30 years ago!

Of all the areas I have worked on, though, some of the most enjoyable and memorable pursuits have been when posing, and exploring, a variety of mathematical problems that lend themselves to relatively elementary mathematics and which are accessible to students in schools and colleges. Having said that, I discovered recently that the one at the very bottom of the list above on wireless technology uses some results I have included later and which *are* accessible to (mainly) A level students, although there are a couple which are very relevant for Core Maths students. The last problem is the most challenging of them all – you have been warned!

I have decided to put this document together to share some of this miscellany of problems, all taken from category (b) above, including some of the findings in them. This is also an apology, and all in the spirit of the infamous and prestigious mathematicians and collaborators G H Hardy[6] and J E Littlewood[7], but at a somewhat more **modest** level!

I believe all the ideas are accessible to many pre-university students, in contrast to many of my other publications that are most definitely not, although clearly some researchers have made good use of those too!

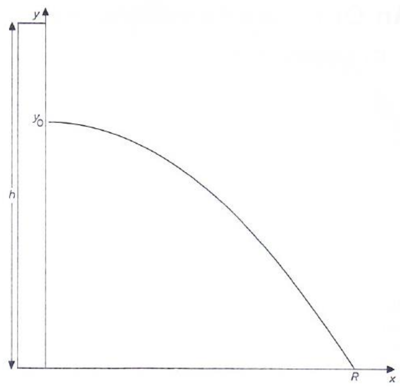
I hope you and your students enjoy exploring at least some of the ideas here.

For further details about my background and interests please see my website[8], bio[9] and CV[10].

[6] https://www.amazon.co.uk/Mathematicians-Apology-G-H-Hardy/dp/1466402695
[7] https://www.amazon.co.uk/Littlewoods-Miscellany-John-Littlewood/dp/052133702X
[8] https://www.paulglaister.org/
[9] https://www.paulglaister.org/about-bio/
[10] http://www.personal.reading.ac.uk/~smsglais/CV_Paul_Glaister.htm

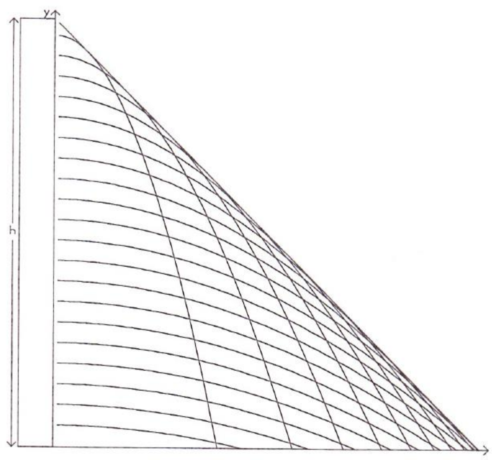
An alternative projectile problem

Consider a cylindrical vessel which is filled to a height h. A hole is drilled in the vessel so the water spurts out, as shown.



Assuming the height of the water is kept fixed (by having a source flowing into it), determine:

- the equation for the path of water (the trajectory);
- the distance from the base of the vessel where the water lands, and the height of the hole for which this distance is greatest;
- the distance travelled by the water (the length of the trajectory), and the height of hole for which this distance is greatest; and
- the 'envelope' of all such trajectories (shown in the figure), where points 'above' this will not get wet, i.e the 'curve of safety'.



It is possible to witness this in practice by holding an 'oscillating water sprinkler (shown below) in a vertical position.

Note that the direction of the initial velocity of the water as it leaves the vessel will be horizontal. The magnitude of the velocity - the speed - is determined by Toricelli's theorem which says that this speed will be determined by conservation of energy of a drop of water falling freely under gravity from rest at the surface of the water in the vessel.

How do the results above change if air resistance is taken into account?

Arithmetic progressions in number sequences

Noting that:

$$\begin{aligned}&4+5+6=7+8\\&9+10+11+12=13+14+15\end{aligned}\ .$$

What other instances are there of these relationships?

In general we have that:

$$\begin{aligned}&(n-1)^2+((n-1)^2+1)+\cdots+((n-1)^2+(n-1))\\&=((n-1)^2+n)+((n-1)^2+n+1)+\cdots+((n-1)^2+2n-2)\ ,\ n=2,3,\ldots\end{aligned}\ .$$

What about sums of squares?

We have

$$\begin{aligned}&10^2+11^2+12^2=13^2+14^2\\&21^2+22^2+23^2+24^2=25^2+26^2+27^2\end{aligned}\ .$$

In general we have that:

$$\begin{aligned}&\left((2n-1)(n-1)\right)^2+\left((2n-1)(n-1)+1\right)^2+\cdots+\left((2n-1)(n-1)+n-1\right)^2\\&=\left((2n-1)(n-1)+n\right)^2+\left((2n-1)(n-1)+n+1\right)^2+\cdots+\left((2n-1)(n-1)+2n-2\right)^2\ ,\ n=2,3,\ldots\end{aligned}\ .$$

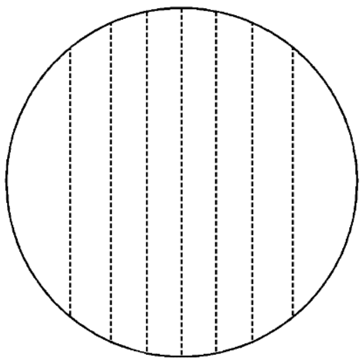
For example with $n=2$ and $n=5$ we have

$3^2+4^2=5^2$ and $36^2+37^2+38^2+39^2+40^2=41^2+42^2+43^2+44^2$.

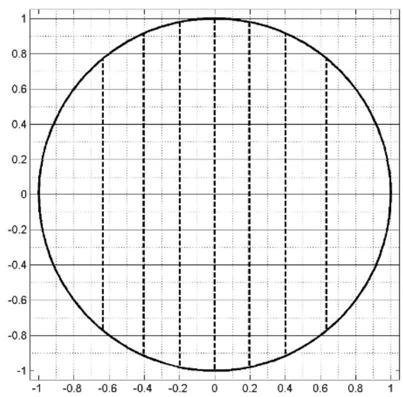
Cake cutting

Where should I cut a circular cake so that each piece is of equal size?

Here is what it would like for 8 pieces of cake in the shape of a cylinder:



which can easily be verified with the addition of a grid:



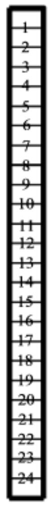
and counting the squares.

Work out where the cake needs to be cut when more than two pieces of equal size (i.e. volume of cake) are required.

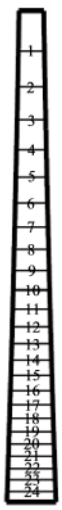
Candle divisions

An Advent candle has divisions marked for each day in the lead up to Christmas and ideally (assuming a constant rate of burning of the wax by volume) we would want the candle burning for the same length of time each day until the next division is reached.

The divisions will be equally-spaced if the candle is cylindrical, as shown:



but some candles are more interesting in shape, so where should the divisions be placed if the candle is a truncated square-based or circle-based (i.e a cone) pyramid shown on the left?



Or maybe curved in shape, say in the form of a logarithmic/exponential function shown on the right?

Card shuffling for beginners

If you take a standard pack of 52 cards, ordered as follows:

2♠3♠4♠5♠6♠7♠8♠9♠10♠J♠Q♠K♠A♠2♣3♣4♣5♣6♣7♣8♣9♣10♣J♣Q♣K♣A♣2♥3♥4♥5♥6♥7♥8♥9♥10♥J♥Q♥K♥A♥2♦3♦4♦5♦6♦7♦8♦9♦10♦J♦Q♦K♦A♦

and place them in 4 piles, as shown:

2♠	2♣	2♥	2♦
3♠	3♣	3♥	3♦
4♠	4♣	4♥	4♦
5♠	5♣	5♥	5♦
6♠	6♣	6♥	6♦
7♠	7♣	7♥	7♦
8♠	8♣	8♥	8♦
9♠	9♣	9♥	9♦
10♠	10♣	10♥	10♦
J♠	J♣	J♥	J♦
Q♠	Q♣	Q♥	Q♦
K♠	K♣	K♥	K♦
A♠	A♣	A♥	A♦

then an 'out riffle shuffle' takes the top card from each pile, working from left to right, and then back to the left hand pile, continuing in the same way until all 52 cards have been collected. If you place them again in 4 piles then this is what you will get:

2♠	5♣	8♥	J♦
2♣	5♥	8♦	Q♠
2♥	5♦	9♠	Q♣
2♦	6♠	9♣	Q♥
3♠	6♣	9♥	Q♦
3♣	6♥	9♦	K♠
3♥	6♦	10♠	K♣
3♦	7♠	10♣	K♥
4♠	7♣	10♥	K♦
4♣	7♥	10♦	A♠
4♥	7♦	J♠	A♣
4♦	8♠	J♣	A♥
5♠	8♣	J♥	A♦

Further out riffle shuffles continue in the same way.

How many out riffle shuffles are needed before the pack is restored to its original order?

An 'in riffle shuffle' takes the top card from each pile, working from *right* to *left* instead, and then back to the right hand pile, continuing in the same way until all 52 cards have been collected.

How many in riffle shuffles are needed before the pack is restored to its original order?

What happens with 13 piles of 4 cards (so each pile starts off with the same denomination as shown below:

2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠	A♠
2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣	A♣
2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥	A♥
2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦	A♦

How many out and in riffle shuffles are needed before the pack is restored to its original order?

What about 2 piles of 26 cards, or 26 piles of 2 cards, and so on?

What about packs of different numbers of cards?

There is some very interesting mathematics associated with this problem. Students could also investigate this by using technology to explore further.

For the examples quoted above the numbers of riffle shuffles required to restore the pack to its original order are shown:

n piles	of m cards	out riffle shuffles	in riffle shuffles
4	13	4	26
13	4	4	13
2	26	8	52
26	2	8	52

The pairings of results for out riffle shuffles is no coincidence and is related to the following:

What do you notice in the following tables?

k	1	2	3	4	5	6	7	8	9	10	11	12
2^k (mod 5)	2	4	3	1	2	4	3	1	2	4	3	1
3^k (mod 5)	3	4	2	1	3	4	2	1	3	4	2	1

k	1	2	3	4	5	6	7	8	9	10	11	12
2^k (mod 9)	2	4	8	7	5	1	2	4	8	7	5	1
5^k (mod 9)	5	7	8	4	2	1	5	7	8	4	2	1

The general result is $\{k : n^k \equiv 1 \bmod (nm-1)\} = \{k : m^k \equiv 1 \bmod (nm-1)\}$. The least number of out riffle shuffles to restore a pack of nm cards placed in n piles each of m cards to its original order is the least value of k for which $n^k \equiv 1 \bmod (nm-1)$. [Note $4^{13} \equiv 13^4 \bmod 51$.]

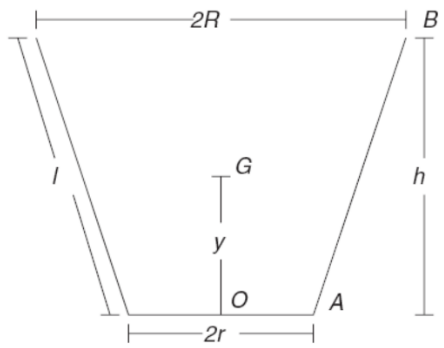
One interesting example is with 24 cards where: $2^{11} = 3^{11} = 4^{11} = 6^{11} = 8^{11} = 12^{11} = 1 \bmod 23$, and 11 is the least integer for which each of these occur. Thus the least number of out riffle shuffles to restore a pack of 24 to its original order using $2, 3, 4, 6, 8, 12$ piles each of $12, 8, 6, 4, 3, 2$ cards, respectively, is 11, regardless of the division. This would be a good starting point for investigating this problem using, say, just:

A♠2♠3♠4♠5♠6♠ A♣2♣3♣4♣5♣6♣ A♥2♥3♥4♥5♥6♥ A♦2♦3♦4♦5♦6♦

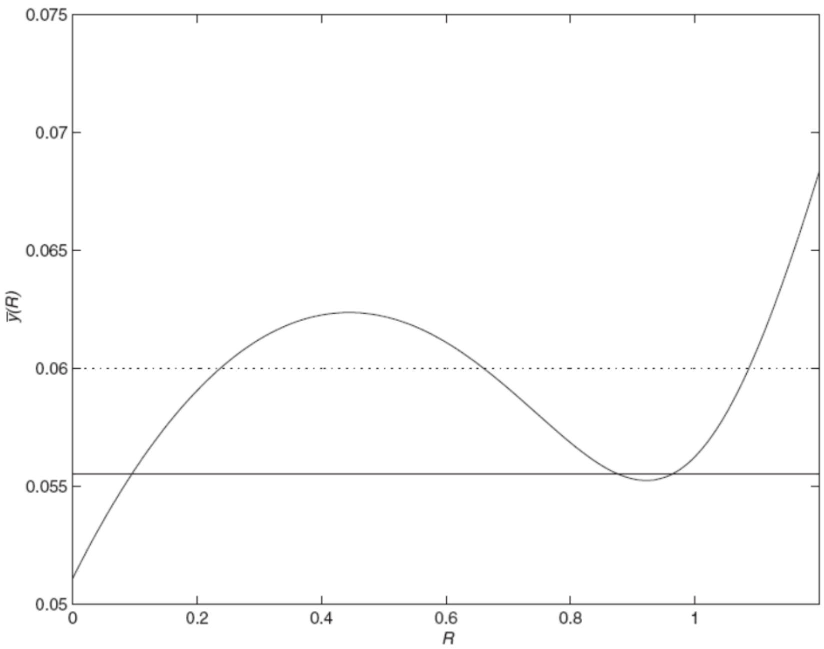
Centres of mass

Consider the centre of mass of objects comprising a truncated hollow cone with a base but open at the top and made of uniform material.

Suppose the radius of the base is fixed, with a value of r, say, and the height, h, is also fixed, how does the height of the centre of mass vary with the radius, R, of the (open) top?



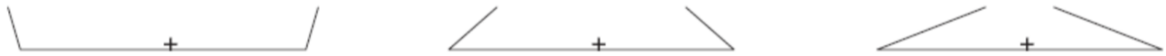
The figure shows the graph of the height of the centre of mass above the base, $\overline{y}(R)$, as a function of R in the case $r=1$, $h=0\cdot3$, showing how this varies, and that there could be three such objects with the same location of centre of mass.



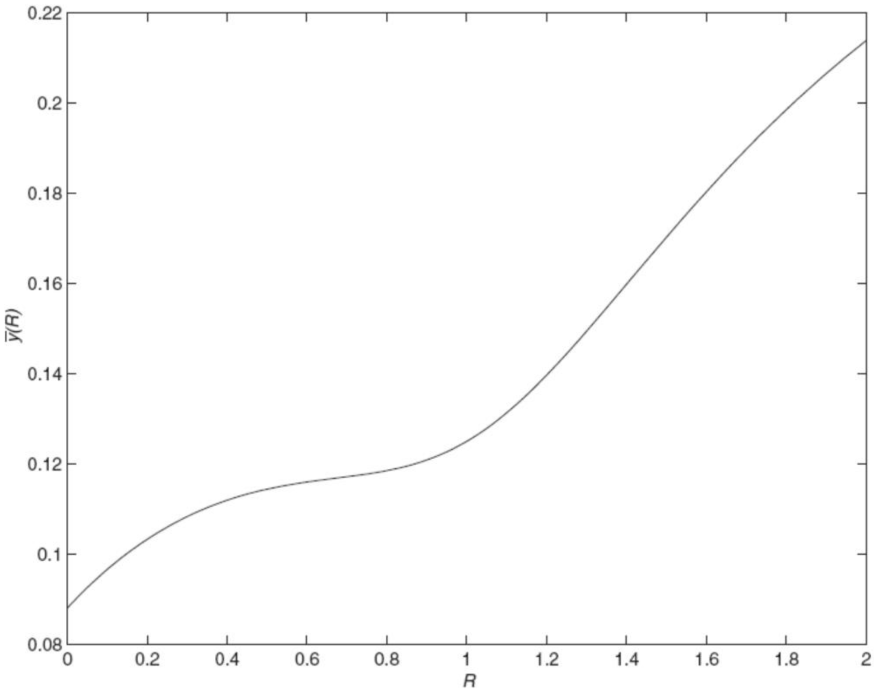
The figure:

shows one example of this, corresponding to the solid line in the figure above.

Another example, corresponding to the dotted line in the figure above, is shown:



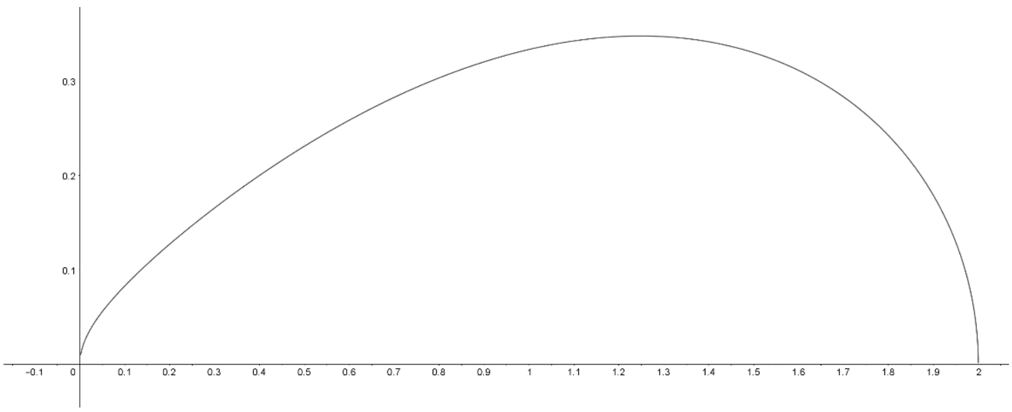
If we increase the height to, say, $h=0\cdot5$, then this situation doesn't occur, but we see that the graph has two oblique points of inflexion:



Now consider the alternative problem where the radius of the base is again fixed, say $r=1$, and instead of h being fixed it is the (slant) height, l, that is fixed, say $l=1$.

How does the height of the centre of mass vary with the radius, R, of the (open) top in this case?

The figure shows the variation in the location of the centre of mass:



Show that the maximum height can be found from the solution of a cubic equation which has one and only one real root in the interval $R\in(0,2)$.

Dated November 11 2020

A collapsing arctan series?

Here is an example of a series which doesn't, on the face of it, appear to be one which 'collapses':

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}+\tan^{-1}\frac{1}{21}+\tan^{-1}\frac{1}{31}+\tan^{-1}\frac{1}{43}+\tan^{-1}\frac{1}{57}+\cdots=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{1+n+n^2}\right) \quad .$$

Using appropriate technology multiply the partial sums:

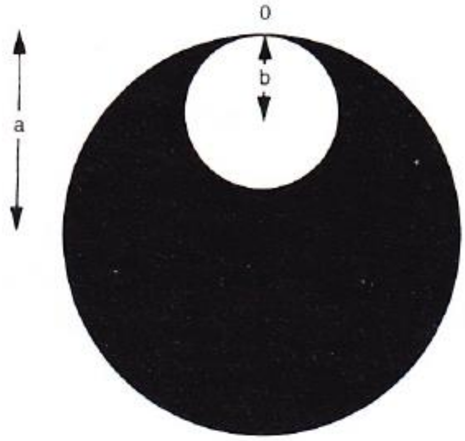
$$\tan^{-1}\frac{1}{3}, \qquad \tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}, \qquad \tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}, \ldots$$

by 4 to see if you can guess what the sum is.

Can you prove your guess using the 'collapsing series' idea?

Dynamical earrings - 24 carat mathematics

The earring shown is formed of a circular disk with a hole cut out. Have you ever observed earrings like this oscillating when worn? The frequency of oscillation will depend on the position of the centre of mass.

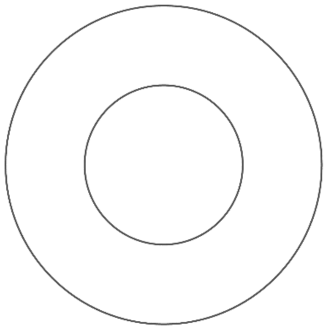


Determine the location of the centre of mass of the earring shown, and work out the ratio $\dfrac{a}{b}$ of the radii when the centre of mass is at the *edge* of the cut out disk, which is the case for the earring shown below:

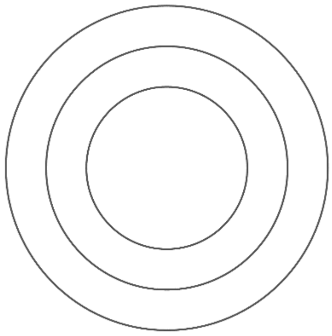
(The solution is $\dfrac{a}{b} = \frac{1}{2}(1+\sqrt{5}) \approx 1 \cdot 618$, the Golden Ratio!).

Equicentric patterns

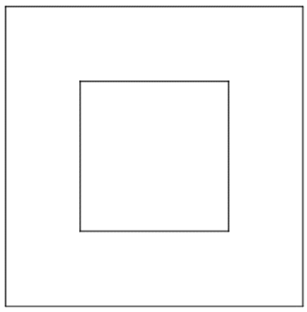
With two concentric circles shown, what is the locus of all points that are equidistant from these?



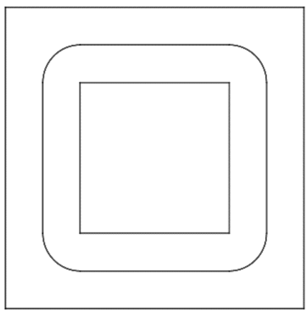
Clearly the solution is a further concentric circle, as shown:



What is the locus of all points equidistant from the two concentric (same centre) squares shown?



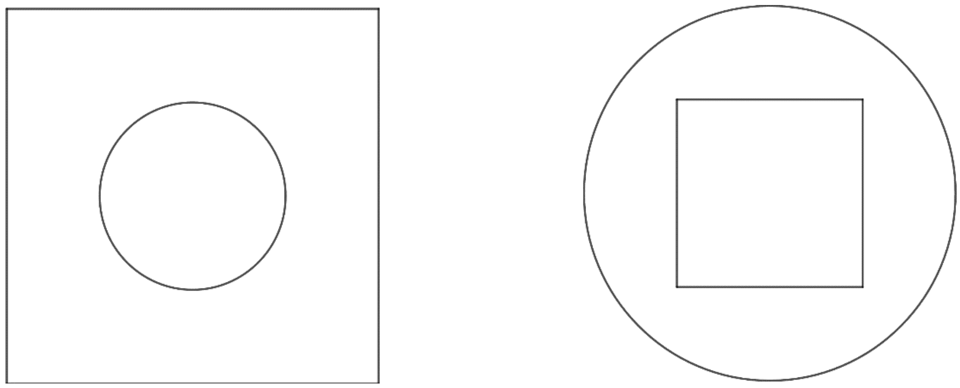
This time the solution is less obvious.



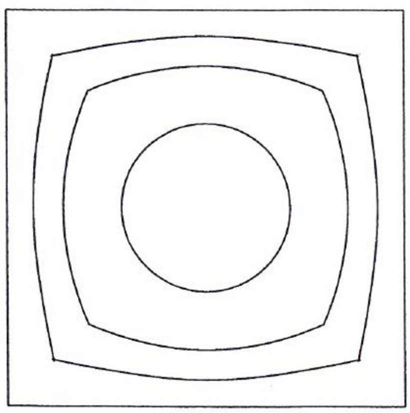
We say that the locus shown, comprising four line segments and four arcs of circles, is equicentric (equal distance and same centre) to the two squares.

What is the curve that is equicentric to the new locus and the original squares?

With the arcs of circles shown this suggests the problem of determining the curve that is equicentric to the circle and square that are concentric:

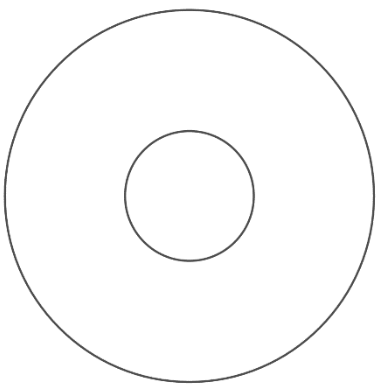


For the left hand pair, successive equicentric curves are shown

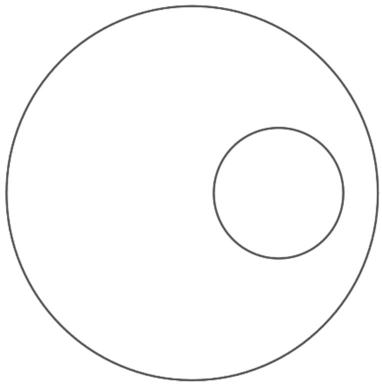


What are the equations of these curves?

Returning to concentric circles:



what happens to the equicentric curve – the middle-sized circle – when the circles are no longer concentric:



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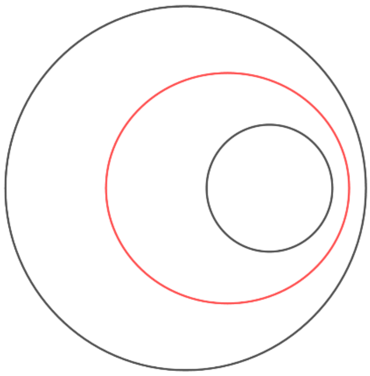
What is the shape of the equicentric curve:



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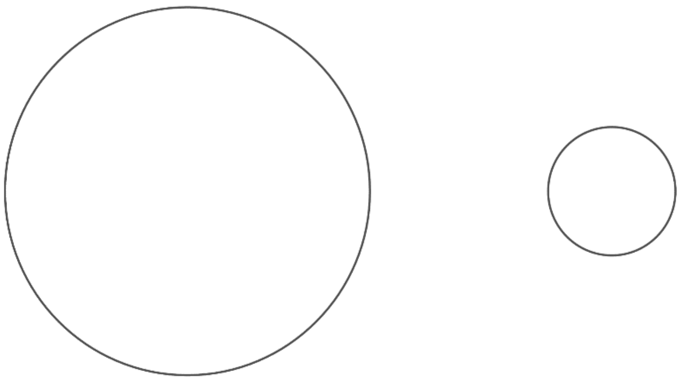
What happens when the circles are separated so the smaller one is no longer inside the larger one:



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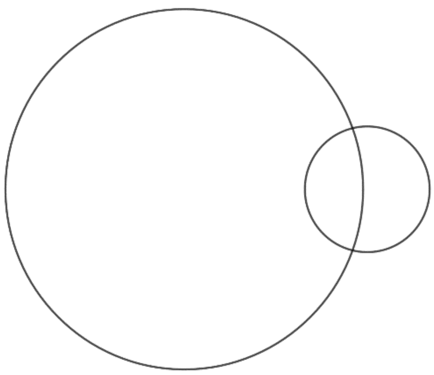
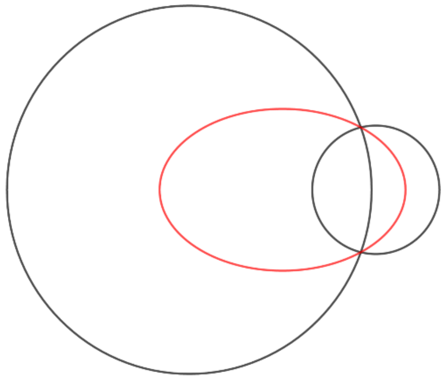
What is the equation of the new equicentric curve?

What about when the circles intersect



?

What is the equicentric curve this time:



?

Is this the complete picture for this case?

Fermi estimation and Brexit

This is particularly relevant to Core Maths.

Fermi estimates or 'back of an envelope' calculations are often used to test quickly and informally the accuracy of statements which are made in the press. These calculations can often save time and money for both individuals and for companies (see [1], for example).

The concept and use of Fermi estimation has increased in popularity in a very significant way in Post 16 mathematics in England through the introduction of the new Core Maths qualifications, all of which feature Fermi estimation.

An example where, arguably, the stakes could not be higher can be found in the archives of the Treasury Select Committee from 23 May 2018.

Appearing before the Committee to give evidence were the Chief Executive & Permanent Secretary, Jon Thompson, and the Deputy Chief Executive & Second Permanent Secretary, Jim Harra, at HM Revenue and Customs (HMRC).

The subject of the parliamentary session was 'The UK's economic relationship with the European Union' and concerned the two options for a customs plan after Brexit: (i) 'maximum facilitation' (so-called 'max fac' which seeks to use technology to avoid a hard border) and (ii) 'customs partnership' under which Britain would remain part of the EU customs area and collect tariffs on behalf of the EU.

The senior civil servants outlined the relative likely costs of these alternative options which could form an ideal starter to a lesson on Fermi estimation. The relevant extract from the session can be accessed via the links in [2].

References

1. Fermi estimates, STEM Learning, https://www.stem.org.uk/resources/elibrary/resource/36077/fermi-estimates (accessed May 26 2018).

2. The UK's economic relationship with the European Union, Treasury Select Committee, May 23 2018, https://goo.gl/d9E1dC/ or https://t.co/AUMyCfRVQW/ https://parliamentlive.tv/event/index/066a04fe-51d7-4dcf-a9c3-20849bad75e8/ or https://parliamentlive.tv/event/index/066a04fe-51d7-4dcf-a9c3-20849bad75e8?in=14:54:50&out=14:59:05 (Start Time: 14:54:50; End Time 14:59:05) or https://tinyurl.com/ybkd46ve (audio clip) or https://tinyurl.com/y9jfe4eo (video clip) (accessed May 26 2018).

Fibonacci and Fermat meet Pell and Pythagoras

The series

$$t+t^2+t^3+\cdots,\quad -1<t<1$$

is a positive integer whenever $t=\dfrac{n}{n+1}$ for some positive integer n, e.g.

$$\tfrac{1}{2}+\left(\tfrac{1}{2}\right)^2+\left(\tfrac{1}{2}\right)^3+\cdots=1 \qquad \text{and } \tfrac{2}{3}+\left(\tfrac{2}{3}\right)^2+\left(\tfrac{2}{3}\right)^3+\cdots=2$$

since the sum is $\dfrac{t}{1-t}$.

But what about if we multiply the terms in the series by the Fibonacci numbers $1,1,2,3,5,8,\ldots$, i.e.

$$F_1t+F_2t^2+F_2t^3+\cdots \qquad ?$$

Here we need to show that the sum is $\dfrac{t}{1-t-t^2}$ and then seek values of t for which this is a positive integer (Note that the series only converges for $-\frac{1}{2}(\sqrt{5}-1)<t<\frac{1}{2}(\sqrt{5}-1)$.)

Along the way you will use the Pythagorean triples $m^2-n^2, 2mn, m^2+n^2$, where $m>n\geq 1$ are positive integers.

You will also need to consider solutions of the Fermat-Pell equation $x^2-5y^2=4$ for positive integers x,y . For this try the first few values $y=1,2,\ldots$ to see which ones give a solution where $5y^2+4$ is a perfect square.

You should find that the first three solutions are

$$F_1\tfrac{1}{2}+F_2\left(\tfrac{1}{2}\right)^2+F_3\left(\tfrac{1}{2}\right)^3+\cdots=2$$

$$F_1\tfrac{3}{5}+F_2\left(\tfrac{3}{5}\right)^2+F_3\left(\tfrac{3}{5}\right)^3+\cdots=15$$

$$F_1\tfrac{8}{13}+F_2\left(\tfrac{8}{13}\right)^2+F_3\left(\tfrac{8}{13}\right)^3+\cdots=104 \qquad .$$

What do you notice?

In general the only solutions to the posed problem are

$$F_1\left(\tfrac{F_{2i}}{F_{2i+1}}\right)+F_2\left(\tfrac{F_{2i}}{F_{2i+1}}\right)^2+F_3\left(\tfrac{F_{2i}}{F_{2i+1}}\right)^3+\cdots=F_{2i}F_{2i+1}\,,\quad i=1,2,\ldots$$

(Note that the results here are related to the result that a positive integer n is a Fibonacci number if and only if either $5n^2+4$ or $5n^2-4$ is a perfect square.)

Fibonacci, Freddie and Fermat the frog

Freddie the frog is heading towards his pond and travels $\frac{1}{2}$ metre on his first jump but only half this distance on his second, and so on, so that in all subsequent jumps he travels a distance which is one half that of his previous jump. This means that the total distance he travels is

$$S = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 1 \quad .$$

Freddie likes to experiment with other jumps, and by multiplying each of his individual jumps by $1, 2, 3, \ldots$ he notices that he travels twice the distance, since

$$P = \tfrac{1}{2} \times 1 + \tfrac{1}{4} \times 2 + \tfrac{1}{8} \times 3 + \cdots = 2$$

i.e.

$$\tfrac{1}{2} \times 1 + \tfrac{1}{4} \times 2 + \tfrac{1}{8} \times 3 + \cdots = 2\left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots\right) \qquad .$$

When Fibonacci passes by he asks what would happen if instead he multiplied each of the individual steps by *his* numbers: $1, 1, 2, 3, 5, 8, 13, 21, \cdots$.

So what is the distance he travels? This is the same as the last case since

$$T = \tfrac{1}{2} F_1 + \tfrac{1}{4} F_2 + \tfrac{1}{8} F_3 + \cdots = 2$$

so

$$\tfrac{1}{2} F_1 + \tfrac{1}{4} F_2 + \tfrac{1}{8} F_3 + \cdots = 2\left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots\right) = \tfrac{1}{2} \times 1 + \tfrac{1}{4} \times 2 + \tfrac{1}{8} \times 3 + \cdots \qquad .$$

With the *tribonacci* sequence, namely $1, 1, 1, 3, 5, 9, 17, 31, \cdots$, we get

$$\tfrac{1}{2} G_1 + \tfrac{1}{4} G_2 + \tfrac{1}{8} G_3 + \cdots = 3\left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots\right) = 3 \quad .$$

What about jumping forwards and backwards? This time

$$U = F_1 - \tfrac{1}{2} F_2 + \tfrac{1}{4} F_3 - \tfrac{1}{8} F_4 + \cdots = \tfrac{4}{5}$$

and

$$V = F_1 - \tfrac{1}{2} 2F_2 + \tfrac{1}{4} 3F_3 - \tfrac{1}{8} 4F_4 + \cdots$$

as well, i.e.

$$F_1 - \tfrac{1}{2} 2F_2 + \tfrac{1}{4} 3F_3 - \tfrac{1}{8} 4F_4 + \cdots = F_1 - \tfrac{1}{2} F_2 + \tfrac{1}{4} F_3 - \tfrac{1}{8} F_4 + \cdots \quad .$$

What about

$$W = \tfrac{1}{2} F_1 + \tfrac{1}{4} 2F_2 + \tfrac{1}{8} 3F_3 + \cdots \qquad ?$$

Here $W = 10$ and thus

$$5\left(\tfrac{1}{2}F_1+\tfrac{1}{4}F_2+\tfrac{1}{8}F_3+\cdots\right)=\tfrac{1}{2}F_1+\tfrac{1}{4}2F_2+\tfrac{1}{8}3F_3+\cdots=5\left(\tfrac{1}{2}\times 1+\tfrac{1}{4}\times 2+\tfrac{1}{8}\times 3+\cdots\right) \quad .$$

Then Fermat turns up and asks them to try:

$$F_2-\tfrac{1}{2}2F_3+\tfrac{1}{4}3F_4-\cdots \quad .$$

Where does this get them?

(Nowhere!)

Forget series (sums), try products with technology

Technology is good for experimenting with finite series (or sums). Common examples being the arithmetic progression

$1+2+3+\cdots+n$ whose sum is $\frac{1}{2}n(n+1)$,

the geometric progression

$1+2+2^2+\cdots+2^{n-1}$ whose sum is 2^n-1 .

Sums of squares, cubes, alternating series, etc. Even Fibonacci numbers defined by $F_{n+2}=F_{n+1}+F_n$, where $F_1=F_2=1$, i.e. the sequence $1,1,2,3,5,8,\ldots$,) can also be found

$$1^2+2^2+3^2+\cdots+n^2=\tfrac{1}{6}n(n+1)(2n+1)$$

$$1^3+2^3+3^3+\cdots+n^3=\tfrac{1}{4}n^2(n+1)^2 \quad .$$

Even Fibonacci numbers defined by $F_{n+2}=F_{n+1}+F_n$, where $F_1=F_2=1$, i.e. the sequence $1,1,2,3,5,8,\ldots$ can also be investigated (the sum of each is not given as that is your challenge to find them!

$$F_1+F_2+\cdots+F_n$$

$$F_1+F_3+\cdots+F_{2n-1}$$

$$F_2+F_4+\cdots+F_{2n}$$

$$F_1-F_2+\cdots+(-1)^{n-1}F_n$$

$$F_1-F_3+\cdots+(-1)^{n-1}F_{2n-1}$$

$$F_1F_2+F_3F_4+\cdots+F_{2n-1}F_{2n}$$

$$F_1F_2+F_3F_4+\cdots+F_{2n}F_{2n+1}$$

$$F_3+F_6+\cdots+F_{3n}$$

You can also try the same with the Lucas numbers which are like the Fibonacci numbers but start with $1,3$: $L_{n+2}=L_{n+1}+L_n$, where $L_1=1, L_2=3$, i.e. the sequence $1,3,4,7,11,18,\ldots$. In this case also try the product:

$$L_2L_4\cdots L_{2n} \quad .$$

The extension to infinite series, and their possible convergence, by looking at the limit of a finite series can also be profitably explored using a spreadsheet or similar technology.

Examples here could be the geometric progression

$$1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n} \quad \text{which converges to } 2 \text{ as } n\to\infty$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2} \quad \text{which converges to } \frac{\pi^2}{6} \text{ as } n\to\infty$$

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\cdots+(-1)^{n-1}\frac{1}{n^2} \quad \text{which converges to } \frac{\pi^2}{12} \text{ as } n\to\infty$$

$$1-\frac{1}{2}+\frac{1}{3}-\cdots+(-1)^{n-1}\frac{1}{n} \quad \text{which converges to } \ln 2 \text{ as } n\to\infty$$

$$1-\frac{1}{3}+\frac{1}{5}-\cdots+(-1)^{n-1}\frac{1}{2n-1} \quad \text{which converges to } \frac{\pi}{4} \text{ as } n\to\infty$$

or the *divergence* of the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

where for any given positive number x there is a value of n for which this sum is larger than x.

For a finite series one looks at a selection of values of n; for an infinite series one looks at the corresponding finite series and continues to add more terms, hopefully getting closer to a fixed value - the limit.

There are also many opportunities to experiment with finite and infinite *products* (one of which we have mentioned above: $L_2L_4\cdots L_{2n}$).

Starting with

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$$

what do you notice as n increases? This converges to $\frac{1}{2}$ as $n\to\infty$. Can you prove this? It is quite straightforward (the 'difference of two squares' is quite useful here.) We will assume that this result has been proved.

Now try:

$$\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\cdots\left(1-\frac{1}{(2n+1)^2}\right)$$

Can you guess what this converges to as $n\to\infty$?

The answer is $\frac{\pi}{4}$, but how could we prove this?

To explore this we first note that:

$$\begin{aligned}\frac{1}{2}&=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)\cdots\\&=\left[\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\right]\left[\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\cdots\right]\\&=\left[\left(1-\frac{(1/2)^2}{1^2}\right)\left(1-\frac{(1/2)^2}{2^2}\right)\cdots\right]\left[\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\cdots\right]\end{aligned}\tag{*}$$

so if we could calculate

$$\left(1-\frac{(1/2)^2}{1^2}\right)\left(1-\frac{(1/2)^2}{2^2}\right)\cdots$$

then the result above would tell us what $\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\cdots$ is.

Define

$$f(x)=\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\cdots$$

we see that $f(1/2)$ is the product we would need to be able to determine the product we are seeking. (Note too that $f(1)$ contains of one of the product terms in (*)).

Try estimating the product $f(1/2)=\left(1-\frac{(1/2)^2}{1^2}\right)\left(1-\frac{(1/2)^2}{2^2}\right)\cdots$ using your technology. (If the value of $\frac{\pi}{4}$ above is correct then you should find that $f(1/2)=\frac{2}{\pi}$.)

So what might $f(x)$ represent?

First we note that $f(-1)=f(1)=f(-2)=f(2)=\cdots=0$, so $f(x)$ has zeros at all the integers except $x=0$. What familiar function has this property?

The first one that springs to mind is $\sin(\pi x)$ which has roots at $x=0,\pm1,\pm2,\ldots$. Conversely, since $\sin(\pi x)$ has zeros at $x=0,\pm1,\pm2,\ldots$ it is reasonable to assume that

$$\sin(\pi x)=Ax\left(1-\frac{x}{1}\right)\left(1+\frac{x}{1}\right)\left(1-\frac{x}{2}\right)\left(1+\frac{x}{2}\right)\cdots$$

where A is to be found.

It is clear that the right hand side is zero at the same values of x as $\sin(\pi x)$. To determine A we could substitute in a particular value of x. Setting $x=0$ is not very helpful; however,

$$\frac{\sin(\pi x)}{x}=A\left(1-\frac{x}{1}\right)\left(1+\frac{x}{1}\right)\left(1-\frac{x}{2}\right)\left(1+\frac{x}{2}\right)\cdots$$

and hence

$$\lim_{x\to 0}\frac{\sin(\pi x)}{x} = A\left(1-\frac{0}{1}\right)\left(1+\frac{0}{1}\right)\left(1-\frac{0}{2}\right)\left(1+\frac{0}{2}\right)\cdots = A \qquad .$$

The limit on the left hand side is readily found

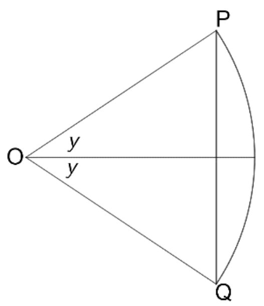
$$\lim_{x\to 0}\frac{\sin(\pi x)}{x} = \pi\lim_{x\to 0}\frac{\sin(\pi x)}{\pi x} = \pi\left(\lim_{y\to 0}\frac{\sin(y)}{y}\right) = \pi\times 1 = \pi$$

from the well-known result $\lim_{y\to 0}\frac{\sin(y)}{y} = 1$, and thus $A=\pi$ and so

$$f(x) = \left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\cdots = \left(1-\frac{x}{1}\right)\left(1+\frac{x}{1}\right)\left(1-\frac{x}{2}\right)\left(1+\frac{x}{2}\right)\cdots = \frac{\sin(\pi x)}{\pi x}$$

giving $f(x) = \frac{\sin(\pi x)}{\pi x}$.

(Note from the diagram:



with PQ the arc of a circle of radius 1 subtending an angle $2y$ (radians) at the centre, O, and $\overline{\mathrm{PQ}}$ the line segment from P to Q, then clearly $\lim_{y\to 0}\frac{\overline{\mathrm{PQ}}}{\mathrm{PQ}} = 1$; however, $\mathrm{PQ} = 2y$ and $\overline{\mathrm{PQ}} = 2\sin(y)$, so $\lim_{y\to 0}\frac{2\sin(y)}{2y} = 1$, i.e. $\lim_{y\to 0}\frac{\sin(y)}{y} = 1$.)

I am sure that the representation for $\sin(\pi x) = \pi x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\cdots$ will not be familiar students; however, the fact that the right hand side has the same zeros as $\sin(\pi x)$ should convince them. Indeed, in contrast to the usual Taylor/Maclaurin series expansion

$$\sin(\pi x) = \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots$$

the *product* should appeal for this very reason. A graph of the partial products of $\pi x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\cdots$ soon verifies the result.

The relation with the previous products is now apparent. With $x=\frac{1}{2}$ we obtain

$$\left(1-\frac{(1/2)^2}{1^2}\right)\left(1-\frac{(1/2)^2}{2^2}\right)\cdots = f(1/2) = \frac{\sin\left(\frac{\pi}{2}\right)}{\pi\left(\frac{1}{2}\right)} = \frac{2}{\pi}$$

so from (*)

$$\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\cdots = \frac{1/2}{2/\pi} = \frac{\pi}{4}$$

as indicated before. (Note that this is the well-known Wallis's product for successive approximations of: $\pi = 4\times\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{5}\right)\cdots = 4\times\frac{2}{3}\times\frac{4}{3}\times\frac{3}{5}\times\frac{7}{5}\cdots$. Use this result to find successive approximations of π .)

We can also independently obtain our first result since

$$\lim_{x\to 1}\frac{\sin(\pi x)}{1-x} = \pi 2\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots$$

$$\begin{aligned}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots &= \frac{1}{2\pi}\lim_{x\to 1}\frac{\sin(\pi x)}{1-x} = \frac{1}{2\pi}\lim_{z\to 0}\frac{\sin\left(\pi(1-z)\right)}{z} = \frac{1}{2\pi}\lim_{y\to 0}\frac{\sin(\pi z)}{z} \\ &= \frac{1}{2}\lim_{z\to 0}\frac{\sin(\pi z)}{\pi z} = \frac{1}{2}\lim_{y\to 0}\frac{\sin(y)}{y} = \frac{1}{2}\times 1 = \frac{1}{2}\end{aligned}.$$

There is one further approximation for π obtained by setting $x=\frac{1}{4}$, i.e.

$$\left(1-\frac{(1/4)^2}{1^2}\right)\left(1-\frac{(1/4)^2}{2^2}\right)\cdots = \frac{\sin\left(\frac{\pi}{4}\right)}{\pi\frac{1}{4}} = \frac{4}{\sqrt{2}\pi}$$

so

$$\frac{4}{\sqrt{2}\pi} = \left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1+\frac{1}{8}\right)\left(1-\frac{1}{12}\right)\left(1+\frac{1}{12}\right)\cdots = \frac{3}{4}\times\frac{5}{4}\times\frac{7}{8}\times\frac{9}{8}\times\frac{11}{12}\times\frac{13}{12}\cdots$$

and hence

$$\pi = \frac{4}{\sqrt{2}} = \frac{4}{3}\times\frac{4}{5}\times\frac{8}{7}\times\frac{9}{9}\times\frac{12}{11}\times\frac{12}{13}\cdots \quad .$$

Now have a look at

$$\left(1-\frac{2^2}{3^2}\right)\left(1-\frac{2^2}{5^2}\right)\left(1-\frac{2^2}{7^2}\right)\cdots$$

using technology and also prove that whose value is $\frac{1}{3}$. The corresponding function to be considered here is

$$g(x)=\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{3^2}\right)\left(1-\frac{x^2}{5^2}\right)\cdots \quad .$$

A similar analysis to that above shows that $g(x)=\cos\left(\frac{\pi x}{2}\right)$, i.e.

$$g(x)=\cos\left(\frac{\pi x}{2}\right)=\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{3^2}\right)\left(1-\frac{x^2}{5^2}\right)\cdots$$

by noticing that $g(x)$ has zeros at $x=1,3,5,\ldots$.

Determine $\lim_{x\to 1}\frac{g(x)}{1-x}$. What does it tell you? ($\left(1-\frac{2^2}{3^2}\right)\left(1-\frac{2^2}{5^2}\right)\left(1-\frac{2^2}{7^2}\right)\cdots=\frac{1}{3}$).

Show that

$$\left(1-\frac{3^2}{5^2}\right)\left(1-\frac{3^2}{7^2}\right)\cdots=\frac{3\pi}{32}$$

$$\left(1-\frac{4^2}{5^2}\right)\left(1-\frac{4^2}{7^2}\right)\cdots=\frac{3}{35}$$

$$\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{6^2}\right)\left(1-\frac{1}{9^2}\right)\cdots=\frac{3\sqrt{3}}{2\pi}$$

Fraction fireworks

Noticing that $\frac{81}{90}=0\cdot 1\times(90-81)$, what other values satisfy a similar relationship $\frac{m}{n}=0\cdot 1\times(n-m)$ for $n>m$?

Other ones are $\frac{5}{10}=0\cdot 1\times(10-5)$, $\frac{9}{15}=0\cdot 1\times(15-9)$, and $\frac{32}{40}=0\cdot 1\times(40-32)$ (these 4 are the only solutions).

What about solutions of $\frac{m}{n}=0\cdot 01\times(n-m)$ for $n>m$? This time there are 7 solutions.

What about solutions of $\frac{m}{n}=0\cdot 1\times(m-n)$ for $m>n$? This time there are 9 unique solutions, 8 of which occur in 'pairs', e.g. $\frac{49}{35}=0\cdot 1\times(49-35)$ and $\frac{49}{14}=0\cdot 1\times(49-14)$. (The odd one out is $\frac{40}{20}=0\cdot 1\times(40-20)$, which is its 'own' pair.)

One interesting example is $\frac{121}{11}=\frac{11^2}{11}=11=0\cdot 1\times 110=0\cdot 1\times(121-11)$.

Moving to the case $\frac{m}{n}=0\cdot 01\times(m-n)$ for $m>n$, one similarly interesting solution is $\frac{10201}{101}=\frac{101^2}{101}=101=0\cdot 01\times 1010=0\cdot 01\times(10201-101)$ which is the example above with a '0' inserted between every non-zero digit and between the '1' and the decimal point.

Further generalisations?

A generalised algebraic identity bites Pythagoras

The identity

$$(a^2-b^2)^2+(2ab)^2=(a^2+b^2)^2$$

can be used to generate Pythagorean triples that are integer solutions of

$$x^2+y^2=z^2 \quad ,$$

e.g. with $a=2, b=1$ we have $3^2+4^2=5^2$.

What about solutions of the three-dimensional version:

$$x^2+y^2+z^2=d^2 \qquad ?$$

Here we 'spot' (!):

$$\left(a[a+b]\right)^2+\left(b[a+b]\right)^2+\left(ab\right)^2=\left(a^2+b^2+ab\right)^2 \quad ,$$

e.g. with $a=2, b=1$ we have $2^2+3^2+6^2=7^2$, and more generally with $b=1$ and $a=n$, for some integer n:

$$n^2+(n+1)^2+\left(n[n+1]\right)^2=\left(1+n+n^2\right)^2$$

e.g. with $n=3$ we have $3^2+4^2+12^2=13^2$, which will be familiar as it combines $3^2+4^2=5^2$ and $5^2+12^2=13^2$.

For another example take $a=3, b=2$ giving $6^2+10^2+15^2=19^2$.

What about a four-dimensional version?

This time we 'spot' (again!) that:

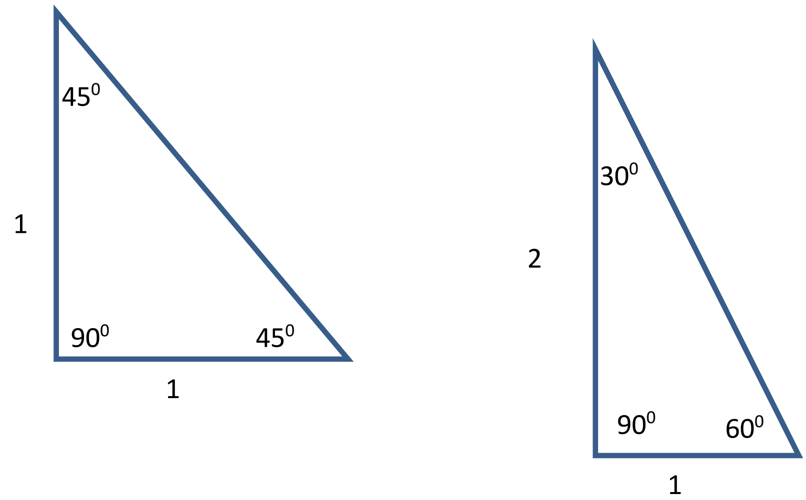
$$\left(a[a+b+c]\right)^2+\left(b[a+b+c]\right)^2+\left(c[a+b+c]\right)^2+\left(ab+bc+ca\right)^2=\left(a^2+b^2+c^2+ab+bc+ca\right)^2 \ ,$$

e.g. with $a=3, b=2, c=1$ we have $6^2+11^2+12^2+18^2=25^2$, and with $a=3, b=-2, c=1$ we have $2^2+4^2+5^2+6^2=9^2$.

The $n-$dimensional version is easy now!

Generalising Alexander's angles

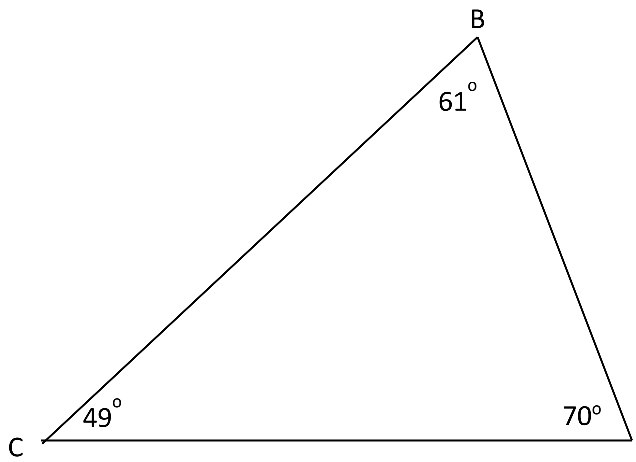
In the right-angled triangles shown the ratio of two sides to two of the angles is the same value:



i.e. $\frac{45^\circ}{45^\circ} = \frac{1}{1}$ and $\frac{60^\circ}{30^\circ} = \frac{2}{1}$.

Are there any other right-angled triangles for which this happens?

What about more general triangles?



In the $49^\circ - 61^\circ - 70^\circ$ triangle shown we have $\frac{\mathrm{BC}}{\mathrm{AB}} = \frac{\sin \angle\mathrm{CAB}}{\sin \angle\mathrm{BCA}}$, and with these angles we therefore have $\frac{\mathrm{BC}}{\mathrm{AB}} = \frac{\sin 70^\circ}{\sin 49^\circ} \approx 1\cdot 2451$. However $\frac{61^\circ}{49^\circ} \approx 1\cdot 2449$, so that to within approximately 2×10^{-4}, we have $\frac{\sin 70^\circ}{\sin 49^\circ} = \frac{61^\circ}{49^\circ}$, i.e.

$$\frac{\mathrm{BC}}{\mathrm{AB}} = \frac{\angle\mathrm{CAB}}{\angle\mathrm{BCA}} .$$

If we make very small adjustments to the 49° and 61° angles this expression becomes exact.

For some other examples consider the $30^\circ - 50^\circ - 100^\circ$, $10^\circ - 34^\circ - 136^\circ$ and $50^\circ - 58^\circ - 72^\circ$ triangles.

Handling data projectors

Faced with two tables of data relating to possible screen sizes for a range of 'projector-screen' distances for two different models of data projector, as shown:

Table 1 Projector A

Screen size in metres	Projector-screen distance in metres
1.0	1.2– 1.4
2.0	2.3– 2.8
2.5	2.9– 3.6
3.8	4.4– 5.4
5.1	5.9– 7.2
6.3	7.3– 9.0
7.5	8.8–10.7

Table 2 Projector B

Screen size in metres	Projector-screen distance in metres
0.8	1.2– 1.5
1.0	1.6– 2.0
1.5	2.4– 2.9
2.0	3.3– 3.9
2.5	4.1– 4.9
3.0	4.9– 5.9
3.8	6.1– 7.3
4.6	7.3– 8.8
5.1	8.1– 9.8
6.1	9.8–11.7

how do you set about the task of deciding which of these will give the range of screen sizes for your requirements when the 'projector-screen' distance you have is fixed, e.g. if you have two locations this is to be used in where the distances are $2 \cdot 25\ \mathrm{m}$ and $4\ \mathrm{m}$, and possibly with upper and lower constraints?

A couple of simple graphs come to the rescue:

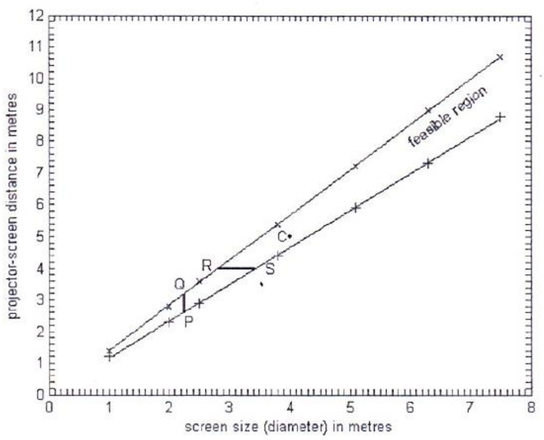


Fig. 1 Graphs of data for Projector A

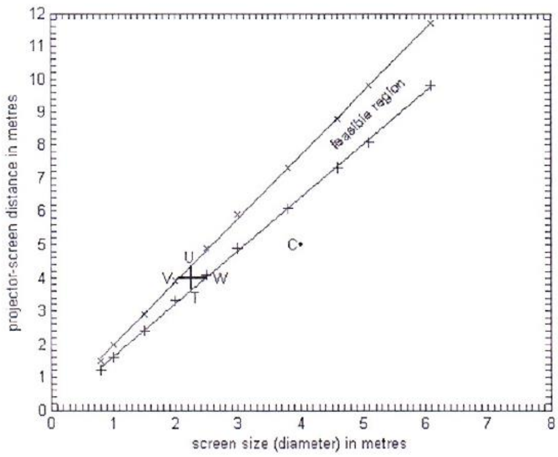
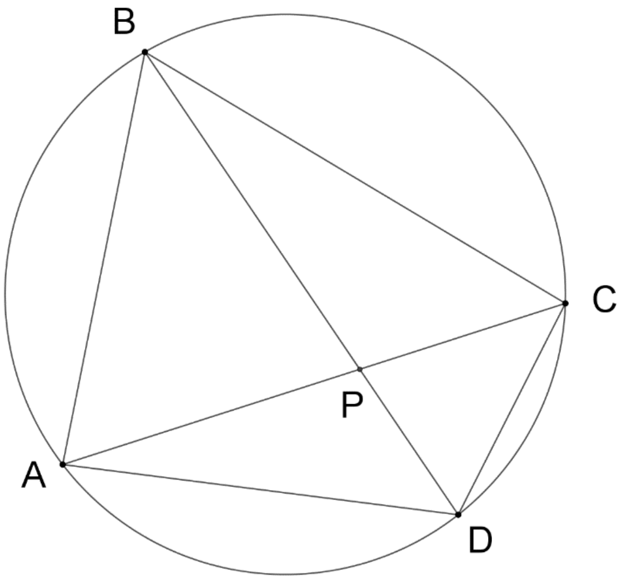


Fig. 2 Graphs of data for Projector B

Intersecting chords

Students who have been exposed to some Euclidean geometry can be shown the following diagram



and asked to deduce $\frac{\text{AB.BC}}{\text{CD.AD}}=\frac{\text{BP}}{\text{DP}}$ $\frac{(\text{CD})^2}{(\text{AB})^2}=\frac{\text{CP.DP}}{\text{AP.BP}}$.

Along the way they will need to use similar triangles to deduce, for example, $\frac{\text{AP}}{\text{DP}}=\frac{\text{BP}}{\text{CP}}=\frac{\text{AB}}{\text{CD}}$ leading to the 'intersecting chords theorem' $\text{AP.CP}=\text{BP.DP}$, which is very useful in physics when determining the wavelength of light using Newton's rings (see http://www.schoolphysics.co.uk/age16-19/Wave%20properties/Interference/text/Newton's_rings/index.html, for example.)

Inverse tan land

The result

$$\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$$

is well-known, but what other integers $1\le a<b<c$ is

$$\tan^{-1}a+\tan^{-1}b+\tan^{-1}c=\pi \quad ?$$

(There are none – prove it.)

What about integer values of a,b,c,d where $1\le a<b<c<d$ for which

$$\tan^{-1}a+\tan^{-1}b+\tan^{-1}c+\tan^{-1}d=k\pi$$

for some positive integer k ?

(There are none – prove it.)

What about:

$$\tan^{-1}a+\tan^{-1}b+\tan^{-1}c+\tan^{-1}d+\tan^{-1}e=k\pi \qquad .$$

There are solutions, and all solutions where values are less than or equal to 30 are shown in the table:

a	b	c	d	e
1	2	4	23	30
1	2	5	13	21
1	2	7	8	18
1	3	4	7	13
1	3	5	7	8

and the value of k is 2, so for the last solution in the table we have

$$\tan^{-1}1+\tan^{-1}3+\tan^{-1}5+\tan^{-1}7+\tan^{-1}8=2\pi \qquad .$$

We also see from the table that some solutions have values in common, so for example from the first two rows we have that

$$\tan^{-1}4+\tan^{-1}23+\tan^{-1}30=\tan^{-1}5+\tan^{-1}13+\tan^{-1}21$$

Notice that all the solutions in the table start with 1. The first question that arises is whether there are solutions without 1 as a member? For example, is there one starting with 2 ? Is the last one as given in the table the 'smallest' one in the sense that the sum of the values is least? Are there infinitely many solutions and, if not, what is the 'largest' one, and which one has the largest first member? Are there solutions where the values are consecutive, or just even, or just odd, or members of well-known sequences, such as squares, Fibonacci numbers, and so on?

What about

$$\tan^{-1}a+\tan^{-1}b+\tan^{-1}c+\tan^{-1}d+\tan^{-1}e+\tan^{-1}f=k\pi$$

(No solutions – prove it.)

This begs the questions as to whether any of the *even numbered angle cases* has a solution, and also whether all *odd numbered angle cases* always have at least one solution.

What about

$$\tan^{-1} a + \tan^{-1} b + \tan^{-1} c + \tan^{-1} d + \tan^{-1} e + \tan^{-1} f + \tan^{-1} g = k\pi \quad ?$$

There are solutions, and those for which the values are less than or equal to 30 are shown in the table:

a	b	c	d	e	f	g
1	2	7	18	21	23	30
1	2	12	13	17	18	21
1	3	5	7	21	23	30
1	3	5	12	13	17	21
1	3	7	8	12	17	18
1	4	5	7	8	23	30
1	4	5	8	12	13	17
2	3	4	5	7	8	13

and the value of k is 3. We now also see that there is a solution (the last one in the table) where 1 is *not* a member, for which we have

$$\tan^{-1} 2 + \tan^{-1} 3 + \tan^{-1} 4 + \tan^{-1} 5 + \tan^{-1} 7 + \tan^{-1} 8 + \tan^{-1} 13 = 3\pi \quad .$$

As an aside we also see from the solutions in rows 1 and 3 that the following must be true:

$$\tan^{-1} 2 + \tan^{-1} 18 = \tan^{-1} 3 + \tan^{-1} 5$$

and from rows 6 and 7 that:

$$\tan^{-1} 7 + \tan^{-1} 23 + \tan^{-1} 30 = \tan^{-1} 12 + \tan^{-1} 13 + \tan^{-1} 17 \quad .$$

What about the eight angle case?

For the nine angle case here are two of the solutions:

$$\tan^{-1} 1 + \tan^{-1} 5 + \tan^{-1} 7 + \tan^{-1} 8 + \tan^{-1} 12 + \tan^{-1} 13 + \tan^{-1} 17 + \tan^{-1} 18 + \tan^{-1} 21 = 4\pi$$
$$\tan^{-1} 2 + \tan^{-1} 3 + \tan^{-1} 4 + \tan^{-1} 7 + \tan^{-1} 8 + \tan^{-1} 12 + \tan^{-1} 13 + \tan^{-1} 17 + \tan^{-1} 18 = 4\pi$$

from which we also see that

$$\tan^{-1} 2 + \tan^{-1} 3 + \tan^{-1} 4 = \tan^{-1} 1 + \tan^{-1} 5 + \tan^{-1} 21 \quad .$$

Solutions of *even numbered cases* could be formed if there are two solutions of the corresponding case with half as many angles which have *no* values in common. For example, if we can find two such solutions of the *five angle case* we can add these to form a solution of the *ten angle case* where the sum will be 4π . Unfortunately none of those in the first table have this property. Similar remarks apply to the second table above in respect of the *seven angle case* and *fourteen angle case*, unfortunately.

This latter remark provokes one final question : are there instances of two solutions in any case which have *no* values in common?

Logarithmic series

From

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots \quad , \quad -1 < x \le 1$$

we find (with $x=1$ $x=-\frac{1}{2}$):

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \cdots = \ln 2 \qquad \text{and} \qquad 1\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2^2} + \tfrac{1}{3}\cdot\tfrac{1}{2^3}\cdots = \ln 2 \qquad ,$$

and since

$$\tfrac{1}{2} + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \cdots = 1$$

we find

$$\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \cdots\right)\left(\tfrac{1}{2} + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \cdots\right) = 1\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2^2} + \tfrac{1}{3}\cdot\tfrac{1}{2^3}\cdots \quad .$$

Are there other series for which this holds, i.e.

$$(a_1 - a_2 + a_3 - \cdots)(b_1 + b_2 + b_3 + \cdots) = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots \qquad ?$$

Other interesting series one can find just from the above expansion are

$$2\left(\tfrac{1}{1}\cdot\tfrac{1}{3^1} + \tfrac{1}{3}\cdot\tfrac{1}{3^3} + \tfrac{1}{5}\cdot\tfrac{1}{3^5}\cdots\right) + \cdots = \ln 2$$

$$2\left(\tfrac{1}{1}\cdot\tfrac{1}{2^1} + \tfrac{1}{3}\cdot\tfrac{1}{2^3} + \tfrac{1}{5}\cdot\tfrac{1}{2^5}\cdots\right) + \cdots = \ln 3 \qquad .$$

Also we have

$$1\cdot\left(\tfrac{1}{2} + \tfrac{1}{3}\right) + \tfrac{1}{2}\cdot\left(\tfrac{1}{2^2} + \tfrac{1}{3^2}\right) + \tfrac{1}{3}\cdot\left(\tfrac{1}{2^3} + \tfrac{1}{3^3}\right) + \cdots = \ln 3 \qquad ,$$

and more generally

$$1\cdot\left(\tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n}\right) + \tfrac{1}{2}\cdot\left(\tfrac{1}{2^2} + \tfrac{1}{3^2} + \cdots + \tfrac{1}{n^2}\right) + \tfrac{1}{3}\cdot\left(\tfrac{1}{2^3} + \tfrac{1}{3^3} + \cdots + \tfrac{1}{n^3}\right) + \cdots = \ln n \quad , \quad n \ge 2 \qquad .$$

With $n=4$ we therefore have

$$1\cdot\left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}\right) + \tfrac{1}{2}\cdot\left(\tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2}\right) + \tfrac{1}{3}\cdot\left(\tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{4^3}\right) + \cdots = 2\left(\left(1\cdot\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\cdot\tfrac{1}{2^2}\right) + \left(\tfrac{1}{3}\cdot\tfrac{1}{2^3}\right) + \cdots\right)$$

and generally (with $n = 2^N$, $N = 1, 2, \ldots$):

$$\begin{aligned} &1\cdot\left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + \tfrac{1}{2^N}\right) + \tfrac{1}{2}\cdot\left(\tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \cdots + \tfrac{1}{2^{2N}}\right) + \tfrac{1}{3}\cdot\left(\tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{4^3} + \cdots + \tfrac{1}{2^{3N}}\right) + \cdots \\ &\quad = N\left(\left(1\cdot\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\cdot\tfrac{1}{2^2}\right) + \left(\tfrac{1}{3}\cdot\tfrac{1}{2^3}\right) + \cdots\right) \quad , \quad N = 1, 2, \ldots \end{aligned} \qquad .$$

Mathematician versus machine

Given the function

$$f(x) = \frac{x}{1+x^2}$$

how would you evaluate $f^{(n)}(0)$ and $f^{(n)}(1)$, the $n\text{th}$ derivative of $f(x)$ evaluated at $x=0$ and $x=1$, respectively, for any natural number n ?

There is a great temptation to use a computer algebra system (CAS), for example in Matlab® the commands:

```
>>syms x

>>vpa(subs(diff(x/(1+x*x),3),x,1))
```

will evaluate $f^{(3)}(1)$, but what about larger values of n, say $10, 100, \ldots$? What about $n = 10^6$? Try it! If it manages to do this, what patterns do you spot for $f^{(n)}(0)$ and $f^{(n)}(1)$? What does your CAS give for a general n, if anything?

Mathematics to the rescue…

…rearrange the expression to $(1+x^2)\,f(x) = x$ and differentiate $n \geq 2$ times, from which you can show that

$$(1+x^2)\,f^{(n)}(x) + 2nxf^{(n-1)}(x) + n(n-1)\,f^{(n-2)}(x) = 0\,,\; n \geq 2 \quad .$$

Together with 'starting values': $f^{(0)}(0) = 0,\; f^{(1)}(0) = 1,\; f^{(0)}(1) = \frac{1}{2},\; f^{(1)}(1) = 0$, this expression can then be used to generate the following:

$$f^{(2k)}(0) = 0,\; f^{(2k+1)}(0) = (-1)^k (2k+1)!\,,\; k = 0,1,\ldots$$

$$f^{(4k+1)}(1) = 0,\; f^{(4k+2)}(1) = (-1)^{k+1}\frac{(4k+2)!}{2^{2k+2}}$$

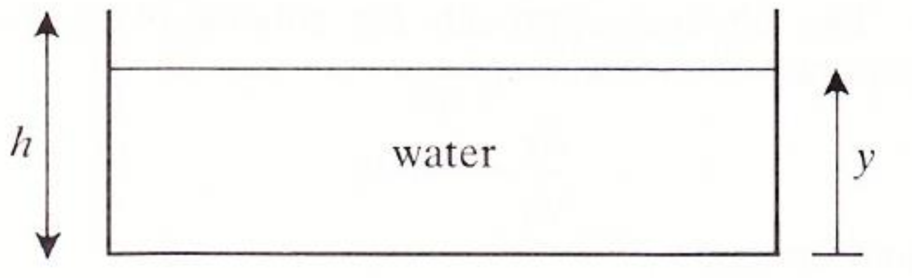
$$f^{(4k+3)}(1) = (-1)^k \frac{(4k+3)!}{2^{2k+2}},\; f^{(4k)}(1) = (-1)^k \frac{(4k)!}{2^{2k+1}}\,,\; k = 0,1,\ldots \quad .$$

which can also be proved using mathematical induction.

How is your CAS getting on with its calculation?

Mathematics for the bath

How long does it take for the last half of a bath to empty compared with the first half?

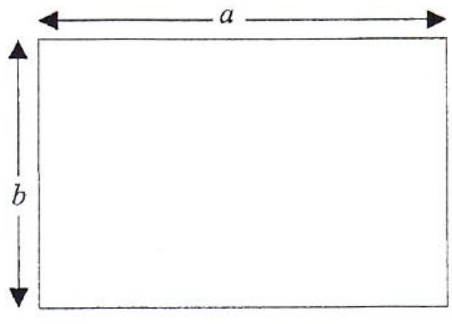


i.e with $y(0) = h$, $y(t_{\frac{1}{2}h}) = \frac{1}{2}h$ and $y(t_h) = 0$, what is $\dfrac{t_h - t_{\frac{1}{2}h}}{t_{\frac{1}{2}h}}$?

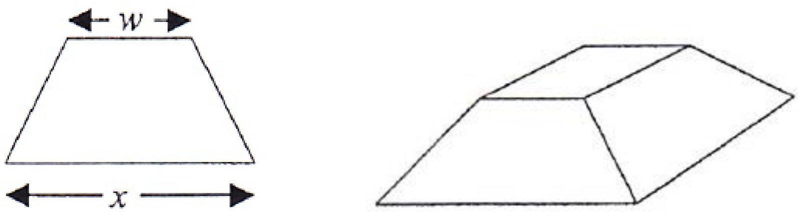
Toricelli's theorem (conservation of energy for a drop of water falling freely under gravity from rest at the surface of the bath water determines the speed of that drop of water as it leaves the bath through the plughole) predicts this ratio as $1+\sqrt{2}$, i.e. it takes around $2\cdot 4$ times as long for the last half to empty compared with the first half. Thus the last half takes about 70% of the total time for the bath to empty. Is this borne out in practice?

Maths woodwork it out

Given the rectangular piece of wood shown:



I wish to cut out four identical trapezia as follows:

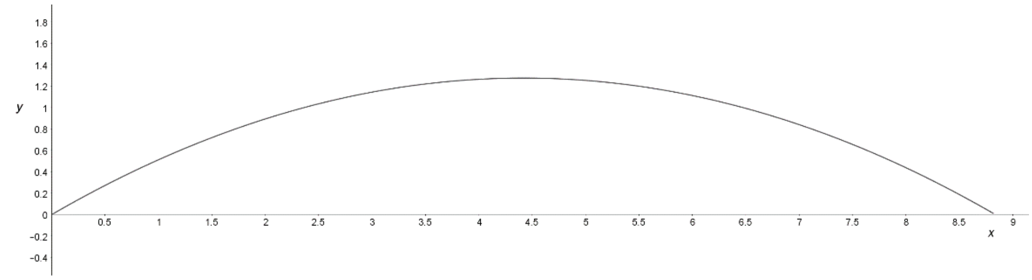


to construct the truncated pyramid shown, open at the top and bottom, in such a way that the dimensions of the upper opening, a square of side w, are *fixed*, and the area of the lower opening (a square of side x, to be determined) is the maximum possible.

How do I achieve this?

Maximum length and area of flight

If a ball is thrown with the same speed U at varying angles θ the trajectory is that of a parabola, as shown



which is of the form

$$y = x\tan\theta - \frac{g\sec^2\theta}{2U^2}x^2$$

using the horizontal and vertical displacement expressions

$$x(t) = Ut\cos\theta \quad , \quad y(t) = Ut\sin\theta - \tfrac{1}{2}gt^2$$

where g is the constant acceleration due to gravity and t denotes the time after release.

The 'horizontal range' can be shown to be $R = \frac{U^2}{g}\sin 2\theta$, which is when the ball is level with the point it was thrown from, occurring when $t = \frac{2U}{g}\sin\theta$, and the horizontal range attains its maximum value (over all θ) of $R_{\max} = \frac{U^2}{g}$ for $\theta = 45^\circ$.

The ball reaches its highest point at $x = \frac{1}{2}R$ and $y = h = \frac{U^2}{2g}\sin^2\theta$ when $t = \frac{U}{g}\sin\theta$, and h attains its maximum value (over all θ) of $\frac{U^2}{2g}$ for $\theta = 90^\circ$, i.e. vertical projection.

The 'time of flight' is how long the ball takes to attain its horizontal range, i.e. $t = \frac{2U}{g}\sin\theta$, and this also attains its maximum value (over all θ) (of value $t = \frac{2U}{g}$) for $\theta = 90^\circ$, i.e. the ball that is thrown vertically is in the air for the longest time.

Here are two slightly more challenging problems.

1. The **length** of the arc of a trajectory (from the point of projection to attaining the horizontal range) is given by

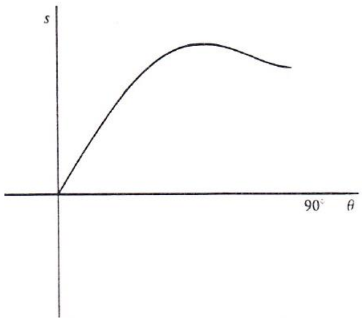
$$s(\theta)=\int_0^{\frac{U^2}{g}\sin 2\theta}\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}}dx$$

where the equation for the trajectory $y(x)$ is given above.

We find that

$$s(\theta)=\frac{U^2}{g}\left(\sin\theta+\cos^2\theta\sinh^{-1}(\tan\theta)\right)$$

a graph of which is shown:



indicating that there is an angle of projection for which the distance travelled by the ball attains a maximum value (over all θ). Show that this maximum value is approximately $\frac{6U^2}{5g}$ attained for $\theta\approx 56^\circ$ and the time of flight is approximately $\frac{5U}{3g}$.

[A standard problem in projectiles is to determine the angles of projection for which the ball passes through a given point. If this point has coordinates (X,Y) then $Y=X\tan\theta-\frac{g(1+\tan^2\theta)}{2U^2}X^2$, from which there are either two, one or zero possible angles of projection, depending on whether (X,Y) is beneath, on, or above the “parabola of safety” $x=\frac{2U^2}{g}\left(\frac{U^2}{2g}-y\right)$. We see from the graph above that there are also either two, one, or zero angles of projection for which the ball travels a given distance s.]

2. The **area** swept out by the arc of a trajectory (from the point of projection to attaining the horizontal range) is given by

$$A(\theta)=\int_0^{\frac{U^2}{g}\sin 2\theta}y\,dx$$

where the equation for the trajectory $y(x)$ is again given above.

We find that

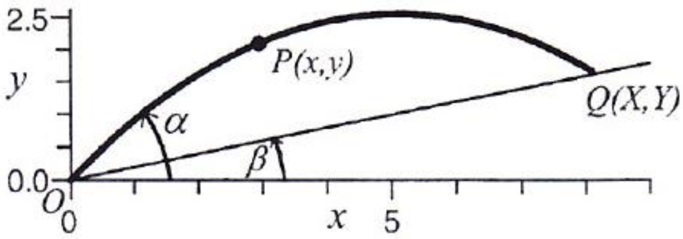
$$A(\theta)=\frac{2U^4}{3g^2}\sin^3\theta\cos\theta$$

which attains a maximum value (over all θ) of $\dfrac{\sqrt{3}U^4}{8g^2}$ for $\theta = 60^\circ$. The time of flight in this case is $\dfrac{\sqrt{3}U}{g}$.

(Note that this is exactly 3 times the area swept out by a ball thrown at an angle of $\theta = 30^\circ$, and the time of flight is $\sqrt{3}$ times s times as long.)

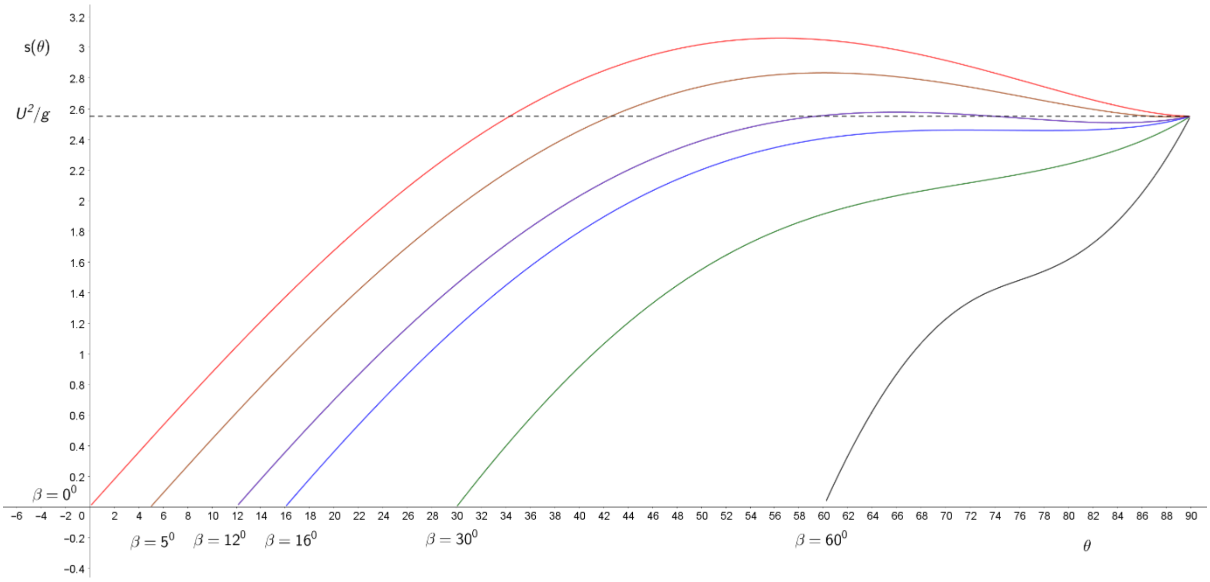
What are the corresponding results when there is an air resistance proportional to $(\text{speed})^n$ for $n > 0$, e.g. a linear ($n = 1$) and a quadratic ($n = 2$) law of resistance?

Consider now the more general problem of projection up an inclined plane as shown.



What is the corresponding expression for the 'length of flight' and 'area swept out' in terms of U, α, β, g, and what are the maximum values of these over all α? (Hint the maximum area swept out occurs when $\tan^{-1}\left(2\tan\beta + \sqrt{3+\tan^2\beta}\right)$).

Further, the figure below



shows the distance travelled as a function of angle of projection for different angles of slope.

We see that for some angles of slope there are either three, two one or zero angles of projection for which the distance travelled, s, is a given value.

We also see that there are some angles of slope for which the maximum distance travelled is greater than $\dfrac{U^2}{g}$, one angle of slope for which the maximum distance travelled is $\dfrac{U^2}{g}$ and which is achieved for two angles of projection, and other angles of slope where the maximum distance travelled is $\frac{U^2}{g}$ and only occurs in the case of vertical projection.

What are the corresponding results when there is an air resistance proportional to $(\text{speed})^n$ for $n > 0$, e.g. a linear ($n = 1$) and a quadratic ($n = 2$) law of resistance?

Mismat(c)hmaking

Divide a class – even numbers only allowed – into two equal sets, A and B, and assign each student in set A with one of the n consecutive integers $\{1, 2, \ldots, n\}$ and assign each student in set B one of the next n consecutive integers, i.e. $\{n+1, n+2, \ldots, 2n\}$. Then set them off trying to 'pair-up', one from each set, in a matching in which each pair doesn't have anything 'in common', i.e. for each pair their numbers, m, n, have *no* common prime factor (or divisor), i.e. their highest common factor $\mathrm{HCF}(m,n)=1$. (We refer to numbers with this property as co-prime.)

Is this always possible regardless of the size of the class?

One obvious matching worth trying in this case is where the first person in A is paired with the last person in B, the second person in A paired with the second to last person in B, and so on, giving the pairs $\{1, 2n\}, \{2, 2n-1\}, \ldots, \{n, n+1\}$, which we refer to as the *reverse* pairing.

The table below shows the matchings that exist for the first few class sizes. The first row denotes the set A and the second row denotes the set B, written so that the pairings are then denoted by the columns.

For the first case with $n=2$ there is only one matching, the *reverse* one $\{1,2\},\{4,3\}$, in which the highest common factor in each pair is clearly 1. The only other possible pairing is $\{1,3\},\{2,4\}$ which is *not* a matching because the second pair have 2 as a common factor.

For the second case $n=3$ there are $3!=6$ possible pairings, but only one of these gives a matching, and this is again the *reverse* pairing.

For the next case $n=4$ there are $4!=24$ possible pairings, only two of which give a matching, and *neither* of these is the *reverse* pairing $\{1,8\},\{2,7\},\{3,6\},\{4,5\}$ because the third pair have 3 as a common factor.

For the final case $n=5$ in the last table there are $5!=120$ possible pairings, only 6 of which give a matching, but this time the *reverse* pairing is again a matching.

$n=2$	1	2
	4	3

$n=3$	1	2	3
	6	5	4

$n=4$	1	2	3	4
	6	7	8	5
	1	2	3	4
	6	5	8	7

$n=5$	1	2	3	4	5
	10	9	8	7	6
	1	2	3	4	5
	10	7	8	9	6
	1	2	3	4	5
	8	9	10	7	6
	1	2	3	4	5
	8	7	10	9	6
	1	2	3	4	5
	6	9	10	7	8
	1	2	3	4	5
	6	7	10	9	8

Three interesting questions that these results suggest are:

- For what values of n is no matching possible?
- When there is a matching how many matchings are possible in terms of n (out of a possible $n!$ pairings)?
- For what values of n are the *reverse* pairing a matching?

The table shows the number of matchings for the first few values of n

n	2	3	4	5	6	7	8	9	10	11
no of matchings	1	1	2	6	12	44	132	504	2016	10368

The values of $n \le 50$ for which the reverse pairing *is* a matching are as follows:

2, 3, 5, 6, 8, 9, 11, 14, 15, 18, 20, 21, 23, 26, 29, 30, 33, 35, 36, 39, 41, 44, 48, 50 .

What do you notice about these numbers? Write down the sequence $2n+1$. What do you notice now?

These all have the property that $2n+1$ is prime. So you may ask the question as to why $2n+1$ being prime means that all pairs:

1	2	3	….	….	….	….	n-2	n-1	n
2n	2n-1	2n-2	….	….	….	….	n+3	n+2	n+1

are coprime?

What about if all these pairs are co-prime – does this imply that $2n+1$ is prime?

(These numbers appear in the following problem: A king invites n couples to sit around a round table with $2n+1$ *seats. For each couple, the king decides a prescribed distance* d *between* 1 *and* n *which the two spouses have to be seated from each other (distance* d *means that they are separated by exactly* $d-1$ *chairs). There is a solution for every choice of the distances* $d=1,2,\ldots,n$ *if and only if 2n + 1 is a prime number.)*

Finally, what about posing the original question for matchings for other integer assignments, for example where A is assigned the integers $\{1,2,\ldots,n\}$ and B is assigned any n consecutive positive integers?

Oh no, my series has collapsed

Faced with the challenge of evaluating

$$\sum_1^\infty \frac{1}{k(k+1)}$$

we could express the left hand side in terms of partial fractions $\frac{1}{k(k+1)} \equiv \frac{A}{k} + \frac{B}{k+1}$, to give $A = 1,\ B = -1$, so that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

and then

$$\sum_1^N \frac{1}{k(k+1)} = \sum_1^N \frac{1}{k} - \frac{1}{k+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{N} - \frac{1}{N+1} = 1 - \frac{1}{N+1} \to 1 \text{ as } N \to \infty$$

with the series 'collapsing'.

Alternatively, we could spot that

$$\frac{1}{k(k+1)} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

with the same result.

Can you do the same for

$$\sum_1^\infty \frac{1}{k(k+1)(k+2)}, \sum_1^\infty \frac{1}{k(k+2)}, \sum_1^\infty \frac{k^2+1}{k(k+1)(k+3)(k+6)}, \sum_1^\infty \frac{k}{(2k+1)(2k-1)(2k-5)} \qquad ?$$

Or more generally other series of the form $\sum_1^\infty \frac{f(k)}{g(k)}$ where $f(k), g(k)$ are polynomials in k (where $\deg(g(k)) - \deg(f(k)) \geq 2$ for convergence)?

For the first one we could again use partial fractions to give:

$$\frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}\right)$$

so

$$\sum_1^N \frac{1}{k(k+1)(k+2)} = \tfrac{1}{2}\sum_1^N \frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}$$
$$= \tfrac{1}{2}\left(\begin{array}{l} \frac{1}{1}-\frac{2}{2}+\frac{1}{3}+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{3}-\frac{2}{4}+\frac{1}{5}+\frac{1}{4}-\frac{2}{5}+\frac{1}{6}+\cdots \\ \cdots+\frac{1}{N-3}-\frac{2}{N-2}+\frac{1}{N-1}+\frac{1}{N-2}-\frac{2}{N-1}+\frac{1}{N}+\frac{1}{N-1}-\frac{2}{N}+\frac{1}{N+1}+\frac{1}{N}-\frac{2}{N+1}+\frac{1}{N+2} \end{array} \right)$$
$$= \tfrac{1}{2}\left(\frac{1}{2} + \left(\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\tfrac{1}{N+1}\right)(1-2+1) - \frac{1}{N+1} + \frac{1}{N+2} \right)$$
$$= \tfrac{1}{2}\left(\frac{1}{2} - \frac{1}{N+1} + \frac{1}{N+2} \right) \to \frac{1}{4} \text{ as } N \to \infty$$

So there is some ‘collapsing’ because of the alternating sums of binomial coefficients (from Pascal’s triangle) giving $1-2+1=0$. Can you show that $\sum_1^\infty \frac{1}{k(k+1)(k+2)\cdots(k+n)} = \frac{1}{nn!}$, $n=1,2,\ldots$ in the same way?

Use technology to convince yourself of these results.

Turning now to the other three examples, the second one is quite easy too but having used partial fractions we require a few more terms to spot the pattern:

$$\sum_1^N \frac{1}{k(k+2)} = \tfrac{1}{2}\sum_1^N \frac{1}{k} - \frac{1}{k+2}$$
$$= \tfrac{1}{2}\left(\begin{array}{l} \frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+\frac{1}{5}-\frac{1}{7}+\frac{1}{6}-\frac{1}{8}+\frac{1}{7}-\frac{1}{9}+\cdots \\ \cdots+\frac{1}{N-4}-\frac{1}{N-2}+\frac{1}{N-3}-\frac{1}{N-1}+\frac{1}{N-2}-\frac{1}{N}+\frac{1}{N-1}-\frac{1}{N+1}+\frac{1}{N}-\frac{1}{N+2} \end{array} \right).$$
$$= \tfrac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} \right) \to \frac{3}{4} \text{ as } N \to \infty$$

What about the other two, though, as these have numerators which change, so what approach is needed now?

So long as $g(k)$ has zeros that differ by integers then this can be done, i.e. when $g(k)$ can be written as the product of unique linear factors:

$$g(k) = a(k-a_1)(k-a_2)\cdots(k-a_n)$$

where $\left|a_i - a_j\right| \in \mathbb{N}$ for all $i \neq j$.

We return to the first of the other three examples $\sum_1^\infty \frac{1}{k(k+1)(k+2)}$ and express:

$$\frac{1}{k(k+1)(k+2)} \equiv \frac{A}{k(k+1)} + \frac{B}{(k+1)(k+2)} \equiv \frac{A(k+2)+Bk}{k(k+1)(k+2)}$$

i.e. $A=\tfrac{1}{2}, B=-\tfrac{1}{2}$, so that

$$\sum_1^N \frac{1}{k(k+1)(k+2)}$$

$$= \sum_1^N \left(\frac{\frac{1}{2}}{k(k+1)} - \frac{\frac{1}{2}}{(k+1)(k+2)} \right)$$
$$= \sum_1^N \big(G(k) - G(k+1) \big) = G(1) - G(N+1)$$

where

$$G(k) = \frac{\frac{1}{2}}{k(k+1)}$$

so

$$\sum_1^\infty \frac{1}{k(k+1)(k+2)} = \lim_{N\to\infty} \sum_1^N \frac{1}{k(k+1)(k+2)} = \lim_{N\to\infty} G(1) - G(N+1)$$
$$= G(1) = \frac{1}{4}$$

as before.

Now use technology to convince yourself this is true.

For the more complex example $\sum_1^\infty \frac{k^2+1}{k(k+1)(k+3)(k+6)}$ we need a little more ingenuity, namely 'filling in the missing terms':

$$\frac{k^2+1}{k(k+1)(k+3)(k+6)} = \frac{(k^2+1)(k+2)(k+4)(k+5)}{k(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)}$$

and then we need to express the numerator:

$$(k^2+1)(k+2)(k+4)(k+5) \equiv A + Bk + Ck(k+1) + Dk(k+1)(k+2) + Ek(k+1)(k+2)(k+3)$$
$$+ Fk(k+1)(k+2)(k+3)(k+4)$$

for A, B, C, D, E, F to be found, which we do by setting the left-hand-side as $F(k) \equiv (k^2+1)(k+2)(k+4)(k+5)$, and then substituting $k = 0, -1, -2, -3, -4, -5$, or equating like powers of k, we get equations for A, B, C, D, E, F whose solution is $A = 40, B = 16, C = -4, D = -2, E = 1, F = 1$, and so

$$\sum_1^N \frac{k^2+1}{k(k+1)(k+3)(k+6)}$$

$$=\sum_1^N \frac{\left(A+Bk+Ck(k+1)+Dk(k+1)(k+2)+Ek(k+1)(k+2)(k+3)+Fk(k+1)(k+2)(k+3)(k+4)\right)}{k(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)}$$

$$=\sum_1^N \left(\begin{array}{l}\dfrac{A}{k(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)}+\dfrac{B}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)} \\ +\dfrac{C}{(k+2)(k+3)(k+4)(k+5)(k+6)}+\dfrac{D}{(k+3)(k+4)(k+5)(k+6)}+\dfrac{E}{(k+4)(k+5)(k+6)}+\dfrac{F}{(k+5)(k+6)}\end{array}\right)$$

$$=\sum_1^N \left(\begin{array}{l}\dfrac{\frac{1}{6}A}{k(k+1)(k+2)(k+3)(k+4)(k+5)}-\dfrac{\frac{1}{6}A}{(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)} \\ +\dfrac{\frac{1}{5}B}{(k+1)(k+2)(k+3)(k+4)(k+5)}-\dfrac{\frac{1}{5}B}{(k+2)(k+3)(k+4)(k+5)(k+6)} \\ +\dfrac{\frac{1}{4}C}{(k+2)(k+3)(k+4)(k+5)}-\dfrac{\frac{1}{4}C}{(k+3)(k+4)(k+5)(k+6)} \\ +\dfrac{\frac{1}{3}D}{(k+3)(k+4)(k+5)}-\dfrac{\frac{1}{3}D}{(k+4)(k+5)(k+6)} \\ \dfrac{\frac{1}{2}E}{(k+4)(k+5)}-\dfrac{\frac{1}{2}E}{(k+5)(k+6)}+\dfrac{\frac{1}{1}F}{(k+5)}-\dfrac{\frac{1}{1}F}{(k+6)}\end{array}\right)$$

$$=\sum_1^N \left(G(k)-G(k+1)\right)=G(1)-G(N+1)$$

where

$$G(k)=\frac{\frac{1}{6}A}{k(k+1)(k+2)(k+3)(k+4)(k+5)}+\frac{\frac{1}{5}B}{(k+1)(k+2)(k+3)(k+4)(k+5)}$$
$$+\frac{\frac{1}{4}C}{(k+2)(k+3)(k+4)(k+5)}+\frac{\frac{1}{3}D}{(k+3)(k+4)(k+5)}+\frac{\frac{1}{2}E}{(k+4)(k+5)}+\frac{\frac{1}{1}F}{(k+5)}$$

so

$$\sum_1^\infty \frac{k^2+1}{k(k+1)(k+3)(k+6)}=\lim_{N\to\infty}\sum_1^N \frac{k^2+1}{k(k+1)(k+3)(k+6)}=\lim_{N\to\infty} G(1)-G(N+1)$$
$$=G(1)=\frac{1}{6!}\left(\tfrac{1}{6}A+\tfrac{1}{5}B+\tfrac{1}{2}C+2D+12E+120F\right)=\frac{1019}{5400}$$

Now use technology to convince yourself this is true.

What do you get if you try using partial fractions for $\displaystyle\sum_1^\infty \frac{k^2+1}{k(k+1)(k+3)(k+6)}$ instead, or maybe even for the simpler problem $\displaystyle\sum_1^\infty \frac{1}{k(k+1)(k+3)(k+6)}$?

Here you will get:

$$\sum_1^N \frac{1}{k(k+1)(k+3)(k+6)}=\sum_1^N \frac{1}{90}\left(\frac{5}{k}-\frac{9}{k+1}+\frac{5}{k+3}-\frac{1}{k+6}\right)$$
$$=\frac{1}{90}\left(\begin{array}{l}\frac{5}{1}-\frac{9}{2}+\frac{5}{4}-\frac{1}{7}+\frac{5}{2}-\frac{9}{3}+\frac{5}{5}-\frac{1}{8}+\frac{5}{3}-\frac{9}{4}+\frac{5}{6}-\frac{1}{9}+\frac{5}{4}-\frac{9}{5}+\frac{5}{7}-\frac{1}{10}+\frac{5}{5}-\frac{9}{6}+\frac{5}{8}-\frac{1}{11}+\cdots \\ \cdots+\frac{5}{k}-\frac{9}{k+1}+\frac{5}{k+3}-\frac{1}{k+6}+\frac{5}{k}-\frac{9}{k+1}+\frac{5}{k+3}-\frac{1}{k+6}+\frac{5}{k}-\frac{9}{k+1}+\frac{5}{k+3}-\frac{1}{k+6}+\frac{5}{k}-\frac{9}{k+1}+\frac{5}{k+3}-\frac{1}{k+6}\end{array}\right)$$

and it is not obvious that the series will 'collapse'.

However, if we continue:

$$\sum_{1}^{N}\frac{1}{k(k+1)(k+3)(k+6)}=\frac{1}{90}\sum_{1}^{N}\left(\frac{5}{k}-\frac{9}{k+1}+\frac{5}{k+3}-\frac{1}{k+6}\right)=\frac{1}{90}\sum_{1}^{N}\left(\frac{5}{k}-\frac{5}{k+1}-\frac{4}{k+1}+\frac{5}{k+3}-\frac{1}{k+6}\right)$$
$$=\frac{1}{90}\sum_{1}^{N}\left(\frac{5}{k}-\frac{5}{k+1}-\frac{4}{k+1}+\frac{4}{k+3}+\frac{1}{k+3}-\frac{1}{k+6}\right)$$
$$=\frac{1}{90}\begin{pmatrix}\frac{5}{1}-\frac{5}{2}+\frac{5}{2}-\frac{5}{3}+\cdots+\frac{5}{N-1}-\frac{5}{N}+\frac{5}{N}-\frac{5}{N+1}\\ -\frac{4}{2}+\frac{4}{4}-\frac{4}{3}+\frac{4}{5}-\frac{4}{4}+\frac{4}{6}-\frac{4}{5}+\frac{4}{7}-\cdots-\frac{4}{N-2}+\frac{4}{N}-\frac{4}{N-1}+\frac{4}{N+1}-\frac{4}{N}+\frac{4}{N+2}-\frac{4}{N+1}+\frac{4}{N+3}\\ \frac{1}{4}-\frac{1}{7}+\frac{1}{5}-\frac{1}{8}+\frac{1}{6}-\frac{1}{9}+\frac{1}{7}-\frac{1}{10}+\frac{1}{8}-\frac{1}{11}+\frac{1}{9}-\frac{1}{12}+\frac{1}{10}-\frac{1}{13}+\frac{1}{11}-\frac{1}{14}+\cdots+\frac{1}{N-4}-\frac{1}{N-1}+\frac{1}{N-3}\\ -\frac{1}{N}+\frac{1}{N-2}-\frac{1}{N+1}+\frac{1}{N-1}-\frac{1}{N+2}+\frac{1}{N}-\frac{1}{N+3}+\frac{1}{N+1}-\frac{1}{N+4}+\frac{1}{N+2}-\frac{1}{N+5}+\frac{1}{N+3}-\frac{1}{N+6}\end{pmatrix}$$
$$\frac{1}{90}\begin{pmatrix}\frac{5}{1}-\frac{5}{N+1}\\ -\frac{4}{2}-\frac{4}{3}+\frac{4}{N+2}+\frac{4}{N+3}\\ +\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{N+4}-\frac{1}{N+5}-\frac{1}{N+6}\end{pmatrix}\to\frac{137}{5400}\text{ as }N\to\infty$$

we see again that the series clearly 'collapses'.

Alternatively we can write as

$$\sum_{1}^{N}\frac{1}{k(k+1)(k+3)(k+6)}=\frac{1}{90}\sum_{1}^{N}\left(\frac{5}{k}-\frac{5}{k+1}-\frac{4}{k+1}+\frac{4}{k+3}+\frac{1}{k+3}-\frac{1}{k+6}\right)$$
$$=\frac{1}{90}\sum_{1}^{N}\big(G(k)-G(k+1)+H(k)-H(k+2)+I(k)-I(k+3)\big)$$
$$=\frac{1}{90}\begin{pmatrix}G(1)-G(2)+G(2)-G(3)+\cdots+G(N-2)-G(N-1)+G(N-1)-G(N)\\ +H(1)-H(3)+H(2)-H(4)+H(3)-H(5)+\cdots+H(N-2)-H(N)+H(N-1)-H(N+1)+H(N)-H(N+2)\\ +I(1)-I(4)+I(2)-I(5)+I(3)-I(6)+I(4)-I(7)+\cdots\\ \cdots+I(N-3)-I(N)+I(N-2)-I(N+1)+I(N-1)-I(N+2)+I(N)-I(N+3)\end{pmatrix}$$
$$=\frac{1}{90}\begin{pmatrix}G(1)-G(N)+H(1)+H(2)-H(N+1)-H(N+2)\\ +I(1)+I(2)+I(3)-I(N+1)-I(N+2)-I(N+3)\end{pmatrix}$$

where

$$G(k)=\frac{1}{90}\left(\frac{5}{k}\right),\ H(k)=-\frac{1}{90}\left(\frac{4}{k+1}\right),\ I(k)=\frac{1}{90}\left(\frac{1}{k+3}\right),$$

so

$$\sum_{1}^{\infty}\frac{1}{k(k+1)(k+3)(k+6)}=\lim_{N\to\infty}\sum_{1}^{N}\frac{1}{k(k+1)(k+3)(k+6)}=\lim_{N\to\infty}\frac{1}{90}\begin{pmatrix}G(1)-G(N)+H(1)+H(2)-H(N+1)-H(N+2)\\ +I(1)+I(2)+I(3)-I(N+1)-I(N+2)-I(N+3)\end{pmatrix}$$
$$=G(1)+H(1)+H(2)+I(1)+I(2)+I(3)=\frac{137}{5400}$$

as before.

What about trying $\sum_{1}^{\infty} \frac{k}{(2k+1)(2k-1)(2k-5)}$ using one of the techniques above.

Use technology to experiment/convince yourself of your result.

Use technology to experiment with $\sum_{1}^{\infty} \frac{k^2+1}{k(k+1)(k+3)}$ and see what happens. What happens if you use the technique above on this problem?

Optimisation problems in geometry

A fixed length is divided into two parts, one forming a circle and the other a square. Show that the combined area is a *minimum* when

$$\frac{\text{area of circle}}{\text{area of square}} = \frac{\text{perimeter of circle}}{\text{perimeter of square}}$$

A fixed area is divided into two parts, one forming a circle and the other a square. Show that the combined perimeter is a *maximum* also when

$$\frac{\text{area of circle}}{\text{area of square}} = \frac{\text{perimeter of circle}}{\text{perimeter of square}}$$

A fixed surface area is divided into two parts, one forming a sphere and the other a cube. Show that the combined surface area is a *minimum* when

$$\frac{\text{volume of sphere}}{\text{volume of cube}} = \frac{\text{surface area of sphere}}{\text{surface area of cube}}$$

A fixed volume is divided into two parts, one forming a sphere and the other a cube. Show that the combined volume is a *maximum* also when

$$\frac{\text{volume of sphere}}{\text{volume of cube}} = \frac{\text{surface area of sphere}}{\text{surface area of cube}}$$

Are these results also true for hyperspheres and hypercubes in $n-$ dimensions for $n > 3$, noting that the results above are for $n = 2, 3$.

Oscillations of a falling spring

Two masses are attached to either end of a light (negligible mass) spring. The spring hangs vertically in equilibrium with the upper end attached to a fixed point. The system is then released from rest (by detaching the upper mass from the fixed point) with both masses remaining fixed to the ends of the spring.

Determine the resulting motion of each mass.

Outliers – least squares and an alternative

A standard problem in statistics is to 'fit' a straight line to a set of data, and the technique usually recommended to students is that of *least squares*.

Unfortunately, least squares is not resistant to outliers, meaning that 'suspect' points which lie outside the range of the others can have an undue influence on the estimate of the slope and intercept. This is primarily because *all* data values directly influence the least squares estimates.

Given that the identification of outliers from the data is not always a straightforward matter, with sound arguments required to justify the removal of points from the data, it is clearly desirable to have available a simple line-fitting technique which *is* resistant to outliers, i.e. one for which the slope and intercept estimates are not sensitive to the presence of outliers.

There are techniques which are as equally simple to apply as least squares and which seek to minimise the effect that outliers can have.

Least squares

In the usual linear regression situation, given a set of n observed values (x_i, y_i), $i = 1,2,\ldots,n$, the standard least squares estimates for the slope (m) and intercept (c) are

$$\hat{m} = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\,\bar{y}}{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2} \quad , \quad \hat{c} = \bar{y} - \hat{m}\,\bar{x} \quad \text{where} \quad \bar{x} = \tfrac{1}{n}\sum_{i=1}^{n} x_i \quad , \quad \bar{y} = \tfrac{1}{n}\sum_{i=1}^{n} y_i \quad .$$

As discussed above, if the data contains outliers then these will appear in the formulae above and can have an undesirable effect on the estimates. Least squares is therefore not resistant to outliers because the number of points which can be outliers before the estimates are affected is zero.

A method that is resistant to outliers because the underlying construction is based on medians is due to Theil. We now give a brief description of this method, and a variation of it.

Theil's incomplete method

In this method the estimate of the slope is calculated from the median of pairwise slopes. When $n = 2N$ is even, for each pair of points (x_i, y_i), (x_{N+i}, y_{N+i}), $i = 1,2,\ldots,N$, the slope of the straight line through these is calculated from

$$m_i = \frac{y_{N+i} - y_i}{x_{N+i} - x_i} \quad , \quad i = 1,2,\ldots,N$$

i.e. the pairwise slopes are based on the first points, second points, etc. in each half of the data. The estimates of the slope and intercept are then given by

$$\hat{m} = \text{median}(m_i)$$

$$\hat{c} = \text{median}(c_i) = \text{median}(y_i - \hat{m}\,x_i)$$

For an odd number of points the middle point is omitted when calculating the pairwise slopes. It is a straightforward matter to use technology to calculate $\hat{m}$ and $\hat{c}$ as we shall show shortly. (Note that the median is the middle value when an odd number of values is put in ascending order, and the mean of the two middle values when an even number of values is put in ascending order.)

Theil's complete method

There is a version of Theil's method in which $\hat{m} = \text{median}\left(\dfrac{y_j - y_i}{x_j - x_i}\right)$ for $1 \le i, j \le N$, i.e. <u>all</u> pairwise slopes are determined, and $\hat{c}$ is as in the incomplete method. Clearly more calculations are required for the complete method.

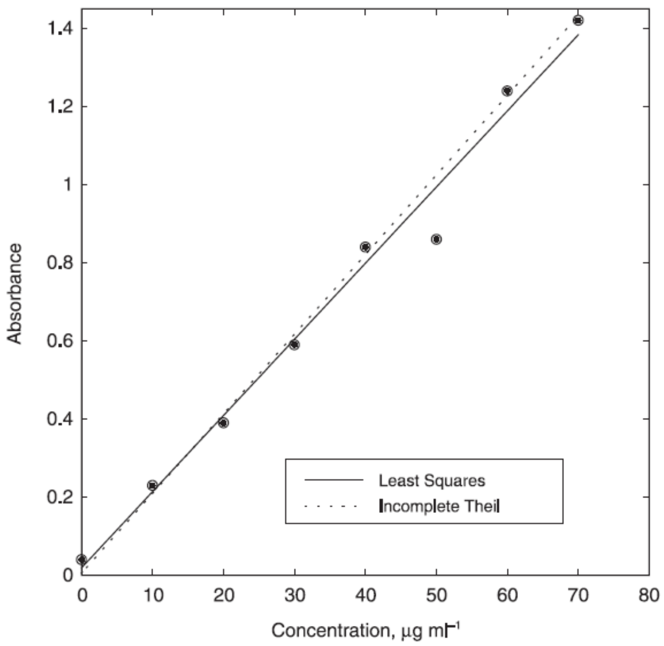
Example

Both versions of Theil's method are more resistant to outliers because the nature of the median means that the outlier need not directly affect the estimates. By way of an illustrative example consider the data in the table:

x_i	0	10	20	30	40	50	60	70
y_i	$0\cdot04$	$0\cdot23$	$0\cdot39$	$0\cdot59$	$0\cdot84$	$0\cdot86$	$1\cdot24$	$1\cdot42$

and depicted as ⊙ in the figure, along with the straight line fits using the least squares and Theil's incomplete estimates. (The data represents the results obtained in a calibration experiment.)

Theil's complete method gives a straight line which is indistinguishable by eye from the one obtained using the incomplete version, and is therefore not shown. It is clear that the estimates from Theil's methods, which are $\hat{m} = 0\cdot0204$, $\hat{c} = 0\cdot00563$ (all values are rounded to three significant figures) for the incomplete version give best fit lines which do not appear to be affected by the outlying data point. This is in contrast to the least squares line, whose estimates are $\hat{m} = 0\cdot0195$ and $\hat{c} = 0\cdot0192$, which *is* affected by this point. Thus the point $(x_6, y_6) = (50, 0\cdot86)$, the suspected outlier, has affected the least squares estimates, and this line is 'pulled towards' this point, unlike Theil's best fit lines. It is obvious that lines of fit which are unduly influenced by such points may give poor estimates for extrapolated values at extremes of the data range.

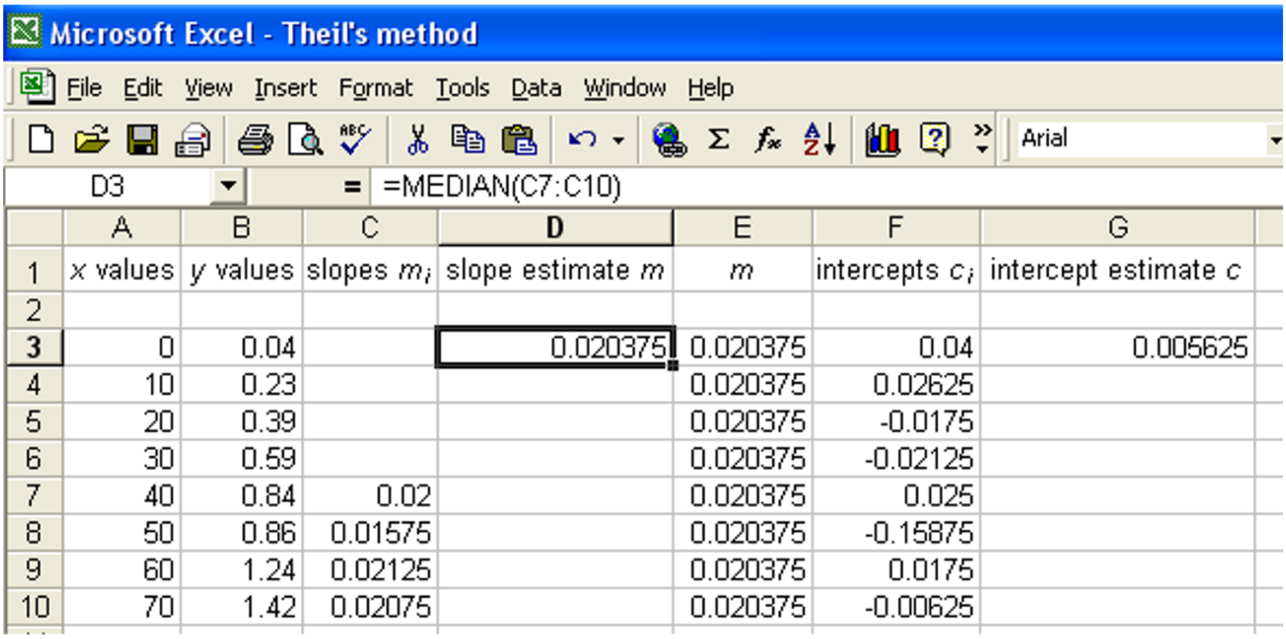


Spreadsheet

Create your own spreadsheet, say, to explore how good Theil’s method is.

Put the x values in A3 to A10 and the y values in B3 to B10. The formula (B7-B3)/(A7-A3) is put in C7, representing m_1, and this is ‘replicated’ using the ‘drag and drop facility’ to complete the entries in C8 to C10 representing m_2, m_3, m_4. The formula MEDIAN(C7:C10) is put in D3 to calculate the slope estimate $\hat{m}$. This value is then copied and then pasted (as a value and not as a formula) in E3 to E10. The formula B3-E3*A3 is put in cell F3 and then replicated down to cell F10, and finally the formula MEDIAN(F3:F10) is put in G3 to calculate the intercept estimate $\hat{c}$.

Not surprisingly it is more complicated to explain than to carry out!



Microsoft Excel - Theil's method

File Edit View Insert Format Tools Data Window Help

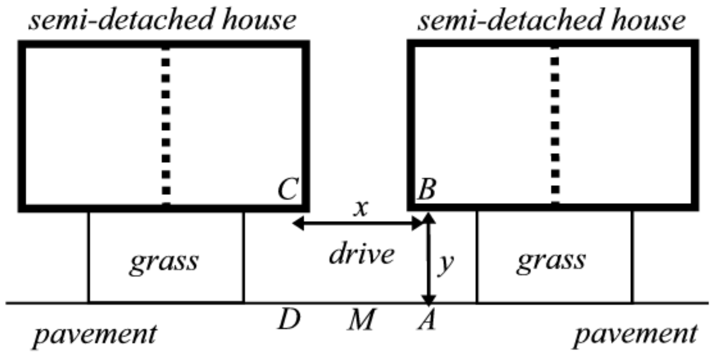
D3 = =MEDIAN(C7:C10)

	A	B	C	D	E	F	G
1	x values	y values	slopes m_i	slope estimate m	m	intercepts c_i	intercept estimate c
2							
3	0	0.04		0.020375	0.020375	0.04	0.005625
4	10	0.23			0.020375	0.02625	
5	20	0.39			0.020375	-0.0175	
6	30	0.59			0.020375	-0.02125	
7	40	0.84	0.02		0.020375	0.025	
8	50	0.86	0.01575		0.020375	-0.15875	
9	60	1.24	0.02125		0.020375	0.0175	
10	70	1.42	0.02075		0.020375	-0.00625	

Next time you use least squares make a comparison of results with those determined from one of Theil’s methods.

Paper round maths

A typical paper round for the delivery of free newspapers to every house in a long road involves wheeling a trolley comprising a bag full of newspapers. Suppose the houses are semi-detached with front gardens of grass on which one is not supposed to walk, but share a common concrete drive, as shown in the figure.



Assuming that the trolley is wheeled, in turn, to each pair of adjacent semi-detached houses sharing a drive, there are clearly three obvious strategies for delivering two newspapers to the two letterboxes.

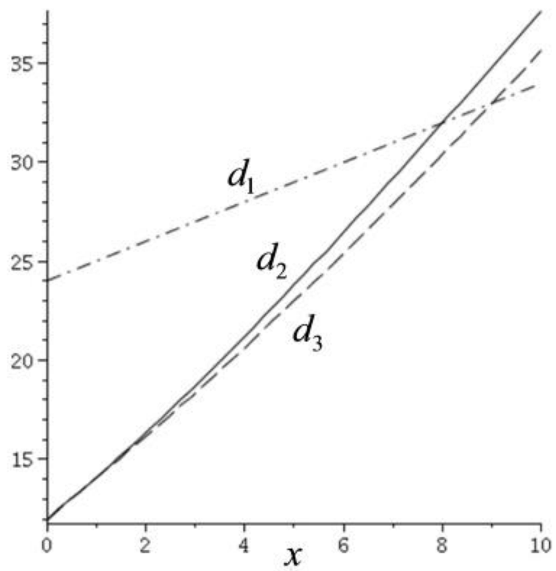
Strategy 1 In the most obvious strategy we leave the trolley at *A*, pick up one newspaper, deliver to *B*, walk back to the trolley at *A*, walk to *D* with the trolley, pick up another newspaper, deliver to *C*, walk back to the trolley at *D*, and then on to the next houses sharing a drive.

Strategy 2 Here we leave the trolley at *A*, pick up two newspapers, deliver first to *B*, walk to *C*, deliver the other newspaper, return directly to the trolley at *A*, walk to *D* with the trolley, and then on to the next houses sharing a drive.

Strategy 3 For our final strategy we leave the trolley at the mid-point *M*, pick up two newspapers, deliver first to *B*, walk to *C*, deliver the other newspaper, return directly to the trolley at *M*, walk to *D* with the trolley, and then on to the next houses sharing a drive.

So the key question is which is the 'best' strategy to use? In our case that means the one that involves the least amount of walking!

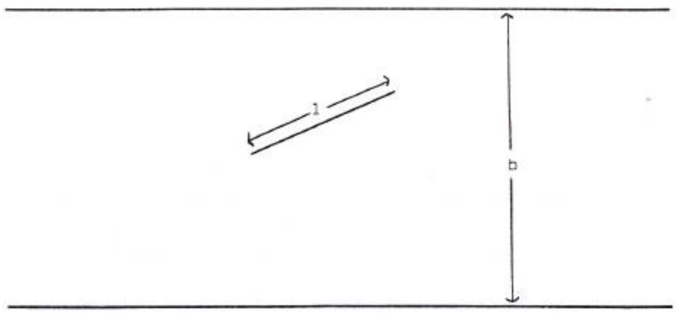
Show that the three distances (for a fixed distance y) as a function of x are as in the figure. What do you deduce?



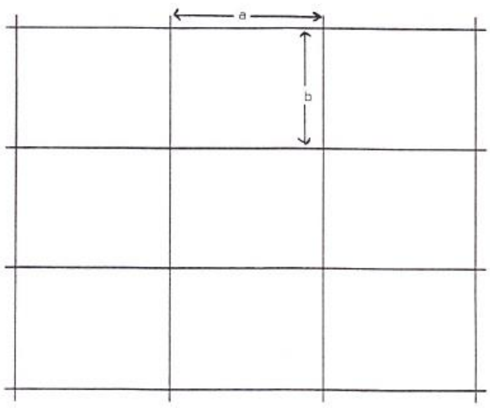
Dated November 11 2020

Playing cards with Buffon

Given wooden floorboards of width b the probability of a needle of length $l \le b$ crossing a join when dropped is $p = \dfrac{2l}{\pi b}$. This can be tested experimentally, preferably using matchsticks rather than needles, though!



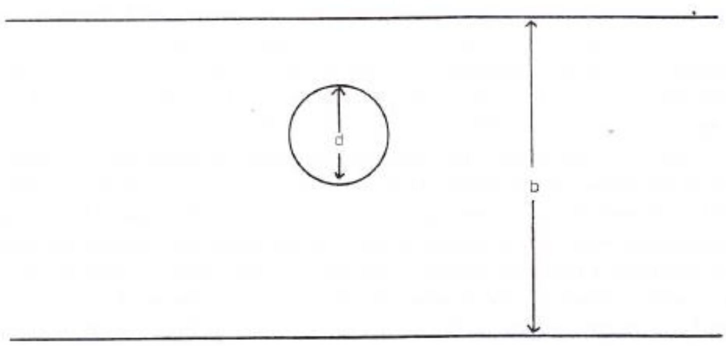
If the needle is now dropped onto rectangular wooden blocks/tiles with dimensions $a \times b$, where $l \le a, b$, what is the corresponding probability?



If the needle is bent into the shape of a circle of diameter d the corresponding probability for floorboards is

$$p = \frac{d}{b} = \frac{l/\pi}{b} = \frac{l}{\pi b}$$

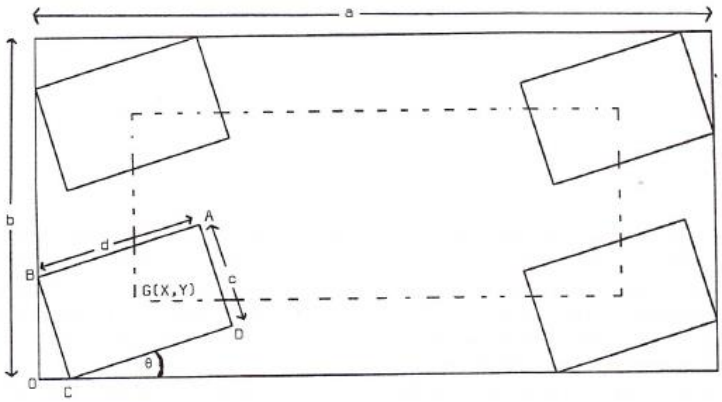
i.e. the probability is **halved** when the needle is bent into the shape of a circle. This can be tested using coins instead of needles bent into the shape of a circle!



What is the corresponding result for blocks/tiles?

What are the corresponding probabilities for polygons – formed by bending a needle (or wire) – when dropped onto floorboards or blocks/tiles?

Results for rectangular shaped objects can be tested by dropping playing cards.

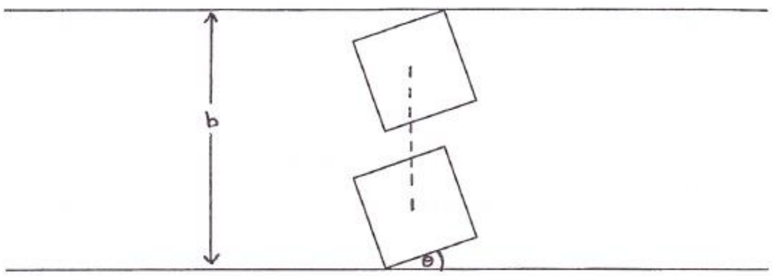


If we view a needle of length l as a rectangle of width zero, then we have

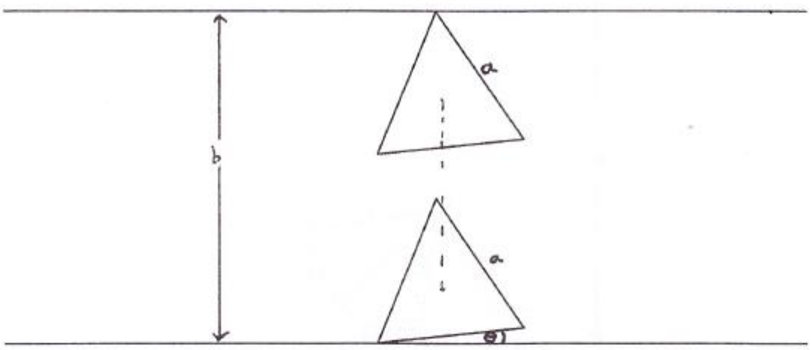
$$p = \frac{2l}{\pi b} = \frac{1}{\pi} \frac{\text{perimeter of shape}}{\text{width of floorboards}}$$

which is the same as for the circle.

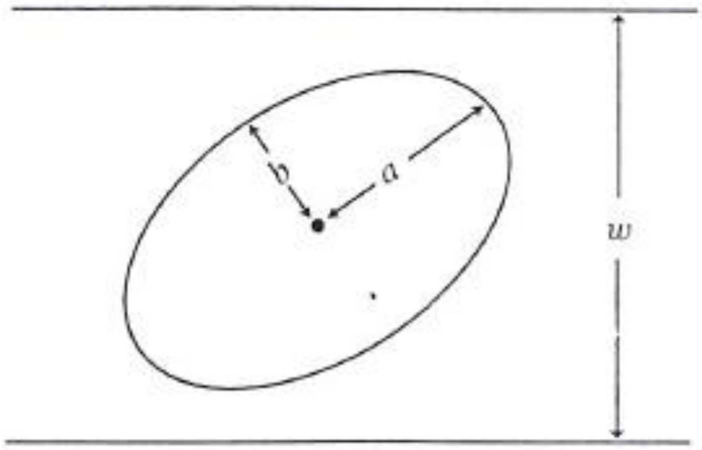
Is this still the case that when a needle/wire is bent into the shape of a rectangle or square, whether on floorboards or blocks/tiles?



What about other regular polygons, e.g. an equilateral triangle?



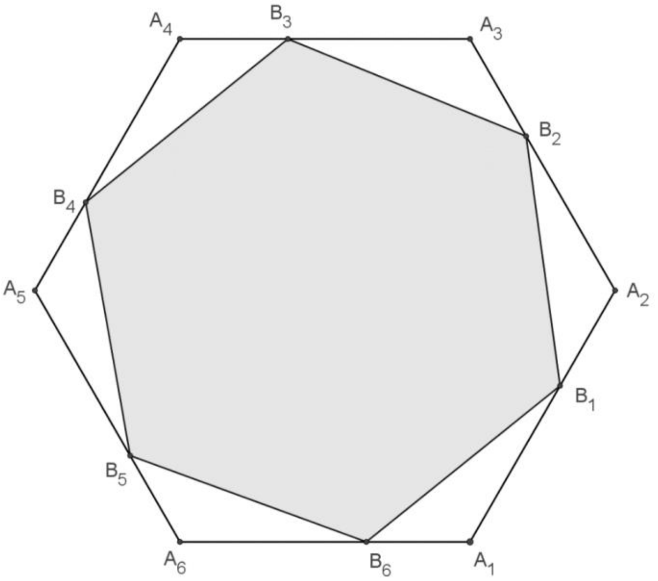
Ellipses?



What are the corresponding probabilities if the restrictions above on the dimensions of the needle, coin, polygon are removed?

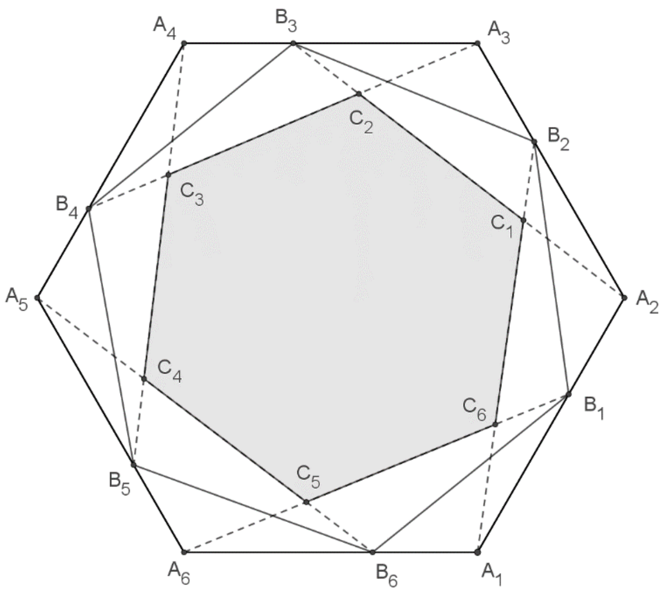
Polygon divisions

In a regular polygon n – sided polygon $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_n$ the points $\mathrm{B}_1, \mathrm{B}_2, \ldots, \mathrm{B}_n$ divide the sides $\mathrm{A}_1\mathrm{A}_2, \mathrm{A}_2\mathrm{A}_3, \ldots, \mathrm{A}_n\mathrm{A}_1$ in the ratio $r:1$.



If the area of the polygon $\mathrm{B}_1, \mathrm{B}_2, \ldots, \mathrm{B}_n$ has unit area, what values of r for each value of n is the area of $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_n$ an integer?

Further, if the points $\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_n$ denote the points of intersection of $\mathrm{A}_1\mathrm{B}_2$ and $\mathrm{A}_2\mathrm{B}_3$, $\mathrm{A}_2\mathrm{B}_3$ and $\mathrm{A}_3\mathrm{B}_4$, …, and of $\mathrm{A}_n\mathrm{B}_1$ and $\mathrm{A}_1\mathrm{B}_2$



and the area of the polygon $\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_n$ has unit area, what values of r for each value of n is the area of $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_n$ an integer?

Potholes

This is particularly relevant to Core Maths.

The BBC News item: http://www.bbc.co.uk/news/business-44065655 (May 11 2018) reads:

> *'A spokesperson for the Department for Transport said it was providing councils in England with more than £6bn to help improve the condition of roads." This funding includes a record £296m through the Pothole Action Fund - enough to fix around 6 million potholes."*

The total length of roads in England (as of 2014) can be found from: https://www.gov.uk/government/statistics/road-lengths-in-great-britain-2014.

There is much of interest to Core Maths in the BBC report, and the links in the Government webpage by way of information in the 2014 report and data in the accompanying spreadsheets.

On the topic of potholes, first note in the Government report:

> *'Major roads in Great Britain are split into trunk roads which are centrally managed, and principal roads which are managed by local authorities (including Transport for London).*
>
> *Trunk motorways and 'A' roads in England are managed by Highways England, in Scotland by Transport Scotland and in Wales by the Welsh Government.'*

So, taking this into account, and the information in the report and/or tables, estimate the length of roads you think the £296m 'Pothole Action Fund' is meant to cover.

Is the figure of 6 million potholes likely to represent all potholes on these roads, based on your experience of how frequently these occur on your recent journeys, or is there a mismatch in the data?

What assumptions have you made?

Assuming the 2014 data is correct, and even allowing for a modest increase in the total length of roads, if we take ALL roads (motorways, and trunk, A, B, C and U roads), across GB, amounting to ~250,000 miles, this would mean the fund targeted at 6 million potholes will be enough to fix these at a rate of 24 potholes per mile across the whole of GB. Might there be enough to 'slack' to cover pavements too?

Power towers

Given the sequence

$$f_1(x) = x^x,\ f_2(x) = x^{f_1(x)} = x^{x^x},\ f_3(x) = x^{f_2(x)} = x^{x^{x^x}},\ldots$$

for $n = 1, 2, \ldots$ what are:

1. $f_n(1)$
2. $f_n(0) = \lim\limits_{x\to 0^+} f_n(x)$
3. $f_n'(1)$
4. $f_n'(0) = \lim\limits_{x\to 0^+} f_n'(x)$
5. corresponding values to 3. and 4. for *any* derivative of $f_n(x)$?

Given the sequence

$$g_1(x) = x^{\frac{1}{x}},\ g_2(x) = x^{g_1(x)} = x^{x^{\frac{1}{x}}},\ g_3(x) = x^{g_2(x)} = x^{x^{x^{\frac{1}{x}}}},\ldots$$

for $n = 1, 2, \ldots$ what are:

1. $g_n(1)$, $g_n(2)$, $g_n(4)$, $g_n(\tfrac{1}{2})$, $g_n(\tfrac{1}{3})$
2. $\lim\limits_{n\to\infty} g_n(1), \lim\limits_{n\to\infty} g_n(2), \lim\limits_{n\to\infty} g_n(4), \lim\limits_{n\to\infty} g_n(\tfrac{1}{2}), \lim\limits_{n\to\infty} g_n(\tfrac{1}{3})$
3. $\lim\limits_{n\to\infty} g_n(x)$?

A problem for Su Doku grids

The table shows a partially completed 9×9 Su Doku grid

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

and the task is to complete this so that:

every row, every column and every one of the 3×3 sub-grids shown contains the digits 1 to 9.

All puzzles set usually have only one solution.

Here we ask how many solutions there are from other given starting conditions where less information is given than that required for a unique solution.

For simplicity we begin with a smaller version, a 4×4 grid as shown

1			
			2
3			
			4

where this time the task is to complete the grid so that:

every row, every column and every one of the 2×2 sub-grids shown contains the digits 1 to 4.

The questions we pose concern the number of solutions to puzzles which initially contain four digits, one each of 1, 2, 3 and 4, in specified locations, and how the number of solutions changes as fewer digits are given initially.

Q1. How many solutions are there starting with:

1			
	2		
		3	
			4

A1. There are two solutions which are mirror images of each other:

1	3	4	2
4	**2**	1	3
2	4	**3**	1
3	1	2	**4**

1	4	2	3
3	**2**	4	1
4	1	**3**	2
2	3	1	**4**

Q2. What happens if the 4 is removed, then the 3, and finally the 2?

A2. The number of solutions increases from the 2 above to 6, so 3 times as many, and then to 24, so 4 times as many again. How many do you think there are with just **1** in the top left hand corner?

Q3. How many solutions are there starting with:

1			**2**
3			**4**

A3. This time there are 7 solutions altogether (which is a little surprising), 2 of which are:

1	4	3	**2**
2	3	4	1
4	1	2	3
3	2	1	**4**

1	4	3	**2**
2	3	4	1
4	2	1	3
3	1	2	**4**

where the only difference is the bottom central block

2	1
1	2

which switches to

1	2
2	1

The remaining 5 solutions have on the top row.

1	3	4	**2**

Q4. What happens if the 4 and then the 3 are removed?

A4. This time the number of solutions increases by only 2, to 9, and then to 24 solutions as in Q2.

The same number of solutions found in Q3 & Q4 are also found for the following cases:

	1	2	
	3	4	

	1	2	
	3	4	

	1	2	
	3	4	

and similar ones.

Q5. What happens if we start with:

1	2	3	4

A5. This time there are 12 solutions.

As for Q1 the following each have 2 solutions:

1			
	2		
			3
		4	

1			
2			
		3	
			4

(and similar ones), whereas the following have 4 solutions

1			2
	3	4	

1	2		
		3	4

1			2
	3	4	

	1		
		2	
	3		
		4	

(and similar ones), whereas these have only 1 solution:

1			
		2	
		3	
4			

1			
			2
3			
			4

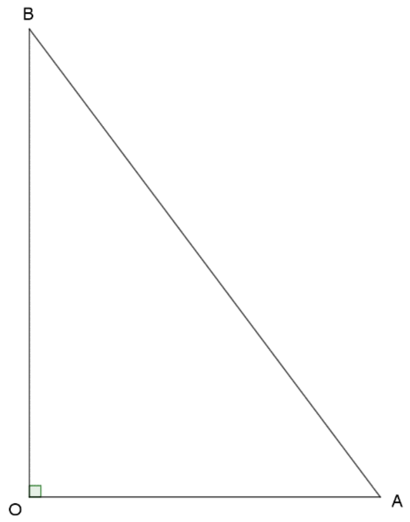
As before we could investigate how many solutions are obtained when one or more of the digits are removed in these.

An obvious variation to consider is the number of solutions when starting with, say, all four $\mathbf{1}$'s in specified locations, and then removing these one at a time.

And now to the $\mathbf{9} \times \mathbf{9}$ case, for which there are many more variations to consider, and beyond…

Pythagoras and cosine rules for tetrahedra

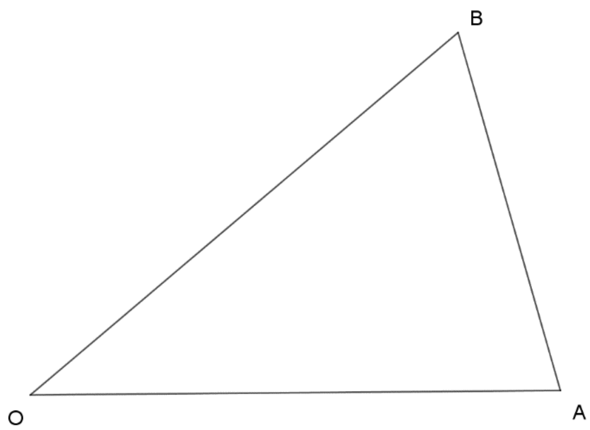
In a right-angled triangle OAB where $\mathrm{AOB} = 90^\circ$



Pythagoras' theorem states that:

$$\mathrm{AB}^2 = \mathrm{OA}^2 + \mathrm{OB}^2$$

The generalisation to any triangle:

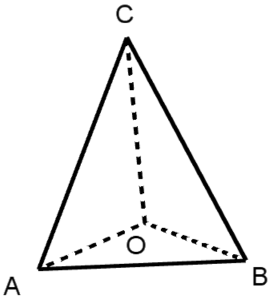


is the cosine rule

$$\mathrm{AB}^2 = \mathrm{OA}^2 + \mathrm{OB}^2 - 2\mathrm{OA}.\mathrm{OB}\cos(\mathrm{AOB}) \qquad .$$

There are analogies for tetrahedra which can be proved using scalar and vector products.

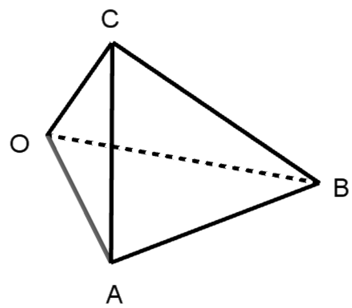
For the right-angled triangle tetrahedron OABC where $\mathrm{AOB}=\mathrm{BOC}=\mathrm{AOC} = 90^\circ$



we have:

$$(\text{area }\Delta\text{ABC})^2 = (\text{area }\Delta\text{OAB})^2 + (\text{area }\Delta\text{OBC})^2 + (\text{area }\Delta\text{OAC})^2 \qquad .$$

The generalisation to any tetrahedron:



is

$$(\text{area }\Delta\text{ABC})^2 = (\text{area }\Delta\text{OAB})^2 + (\text{area }\Delta\text{OBC})^2 + (\text{area }\Delta\text{OAC})^2$$
$$+\tfrac{1}{2}\text{OA.OB.OC}\left(\begin{array}{l}\text{OA}\cos(\text{BOC}) + \text{OB}\cos(\text{AOC}) + \text{OC}\cos(\text{AOB})\\ -\text{OA}\cos(\text{AOB})\cos(\text{AOC}) - \text{OB}\cos(\text{AOB})\cos(\text{BOC}) - \text{OC}\cos(\text{AOC})\cos(\text{BOC})\end{array}\right)$$

Pythagoras for series

What is the value of:

$$\left(2^0\cos^0\theta\cos 1\theta+2^1\cos^1\theta\cos 2\theta+2^2\cos^2\theta\cos 3\theta+2^3\cos^4\theta\cos 4\theta+\cdots\right)^2$$
$$+\left(2^0\sin^0\theta\sin 1\theta+2^1\sin^1\theta\sin 2\theta+2^2\sin^2\theta\sin 3\theta+2^3\sin^4\theta\sin 4\theta+\cdots\right)^2$$

i.e.

$$\left(\cos\theta+2\cos\theta\cos 2\theta+2^2\cos^2\theta\cos 3\theta+\cdots\right)^2+\left(\sin\theta+2\sin\theta\sin 2\theta+2^2\sin^2\theta\sin 3\theta+\cdots\right)^2 \qquad ?$$

If we only retained the first term in each bracket we would have $\cos^2\theta+\sin^2\theta$, which we know is 1 by Pythagoras's theorem in a triangle with hypotenuse of length 1, with one of the angles being $\theta<90^\circ$, i.e.

$$\cos^2\theta+\sin^2\theta\equiv 1 \qquad \text{for all } \theta \qquad .$$

But what about the more general expression above?

This is also 1, i.e.

$$\left(\cos\theta+2\cos\theta\cos 2\theta+2^2\cos^2\theta\cos 3\theta+\cdots\right)^2+\left(\sin\theta+2\sin\theta\sin 2\theta+2^2\sin^2\theta\sin 3\theta+\cdots\right)^2=1$$

but not for all θ, **only for** $\frac{1}{3}(3k+1)\pi<\theta<\frac{1}{3}(3k+2)\pi$, $k=1,2,\ldots$. This does include a range of acute angles (with $k=1$), though, i.e. $\frac{1}{3}\pi<\theta<\frac{1}{2}\pi$, but sadly not quite so universal as Pythagoras!

Pythagorean triples from infinite series

What is special about the right-angled triangle with one of the angles being θ where

$$1\times\tfrac{1}{2}\cos\theta+2\times\tfrac{1}{2^2}\cos 2\theta+3\times\tfrac{1}{2^3}\cos 3\theta+\cdots=0 \qquad ?$$

What about

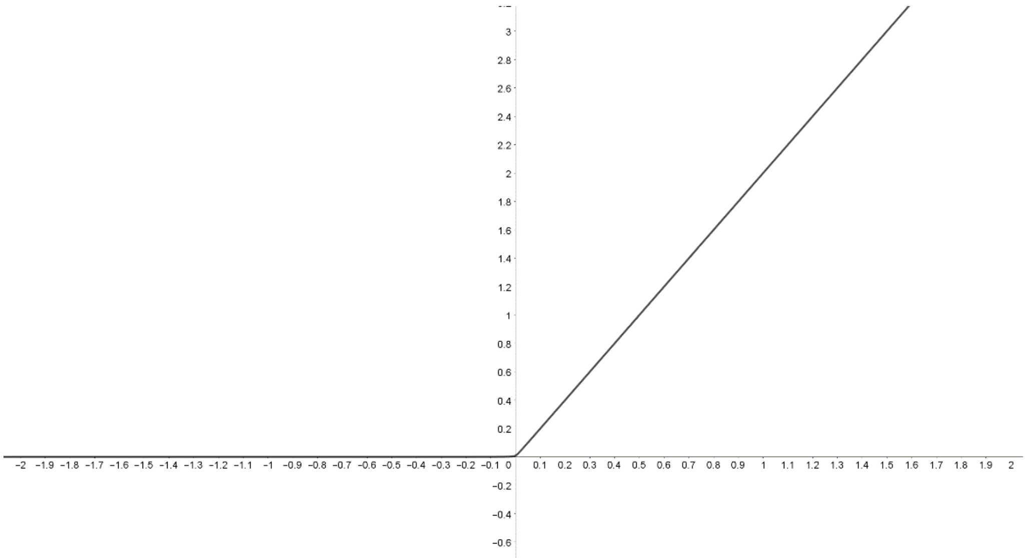
$$1\times\tfrac{1}{3}\cos\theta+2\times\tfrac{1}{3^2}\cos 2\theta+3\times\tfrac{1}{3^3}\cos 3\theta+\cdots=0$$

or

$$1\times\tfrac{1}{4}\cos\theta+2\times\tfrac{1}{4^2}\cos 2\theta+3\times\tfrac{1}{4^3}\cos 3\theta+\cdots=0 \qquad ?$$

Quadratics and turning the corner

What is the equation of the graph shown in the figure?



At first glance it looks like $f(x) = 2x$ for $x \geq 0$ and $f(x) \equiv 0$ for $x < 0$.

Consider the quadratic equation (for t)

$$y^2 - 2xy - p = 0$$

where x, p are real parameters with $p > 0$. The roots of this equation are given by

$$y = x \pm (x^2 + p)^{\frac{1}{2}}.$$

The function we consider here is the one obtained from the positive root, i.e.

$$r(x) = x + (x^2 + p)^{\frac{1}{2}}$$

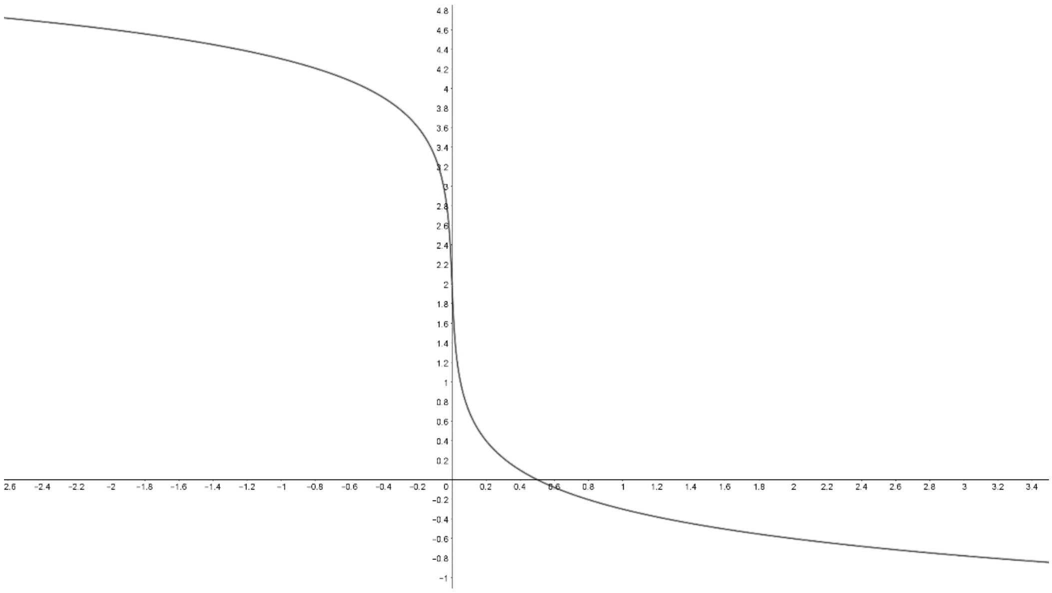
as a function of x, with $p > 0$ a parameter.

The most interesting case is when $p \ll 1$. The figure is actually the graph of $r(x)$ in the case $p = 10^{-4}$, which gives the appearance of having a 'corner' in it (like the graph of $|x|$ has).

If we look closely, though, unlike $|x|$ the function $r(x)$ has a *continuous* derivative, but this changes rapidly near the origin with large *curvature*. For smaller values of p the 'corner' is more pronounced, and less so for larger values of p.

For $p \ll 1$ determine the behaviour of $r(x), r'(x)$ and $r''(x)$, each for $x > 0, x = 0$ and $x < 0$. You will find that $r''(0) = p^{-\frac{1}{2}} \gg 1$ for $p \ll 1$, indicating large curvature near the origin.

One way of magnifying the behaviour of this 'corner' function is to take logs (say to base 10) and the figure shows the graph of $z = -\log_{10} r(x)$ for $p = 10^{-4}$. (The minus sign ensures that the graph is predominantly above the x-axis.) The log graph changes rapidly near $x = 0$, and this is more pronounced the smaller the value of p.



This qualitative behaviour occurs in the field of chemistry where an alkali is added to an acid, e.g sodium hydroxide to hydrochloric acid, and the resulting acidity/alkalinity can be determined by solving a quadratic equation similar to the one above. The quantity x is related to the volume of added alkali and the function $-\log_{10} r(x)$ corresponds to the pH of the solution which is a measure of the acidity. The rapid change in pH occurs when the volume of alkali added is just sufficient to neutralise the acid. Typically the parameter $p = 10^{-14}$ for such problems. We examine this link shortly.

First we consider another way of interpreting the quadratic equation above which proves useful in the application in chemistry.

Rearranging the quadratic equation we have

$$x = \frac{y^2 - p}{2y} \quad .$$

If we now plot x against y, but with the graph rotated so that y is along the horizontal and x is along the vertical, then the first graph with the 'apparent corner' will be repeated, and if we substitute in $y = 10^{-z}$, i.e.

$$x = \frac{(10^{-z})^2 - p}{2(10^{-z})} \qquad \text{or} \qquad 2x(10^{-z}) = (10^{-z})^2 - p$$

and plot x against z, but with the graph rotated so that z is along the horizontal and x is along the vertical, then the second graph above is repeated.

This idea can be useful when drawing graphs of roots of an equation, and this is particularly so in chemistry as we now see.

In chemistry is the pH of a solution is defined by

$$\mathrm{pH} = -\log_{10}[\mathrm{H}^+]$$

where $[\mathrm{H}^+]$ is the concentration of hydrogen ions H^+, and represents the degree of acidity of the solution. For example, for water $[\mathrm{H}^+] = 10^{-7}$ giving a pH of 7. The standard approach is to conduct an experiment, called a titration, whereby the base is added gradually to the acid. An alternative approach is to determine the pH theoretically.

In the case of adding a strong base, say sodium hydroxide (NaOH), of concentration $0\cdot 2$ (in suitable units) is added to a volume 50 (again in suitable units) of a solution of a strong acid, say hydrochloric acid (HCl), of concentration $0\cdot 2$, then $[\mathrm{H}^+]$ is the (positive) solution of the quadratic equation

$$[\mathrm{H}^+]^2 - \left(\frac{10-0\cdot 2V}{50+V}\right)[\mathrm{H}^+] - K_W = 0$$

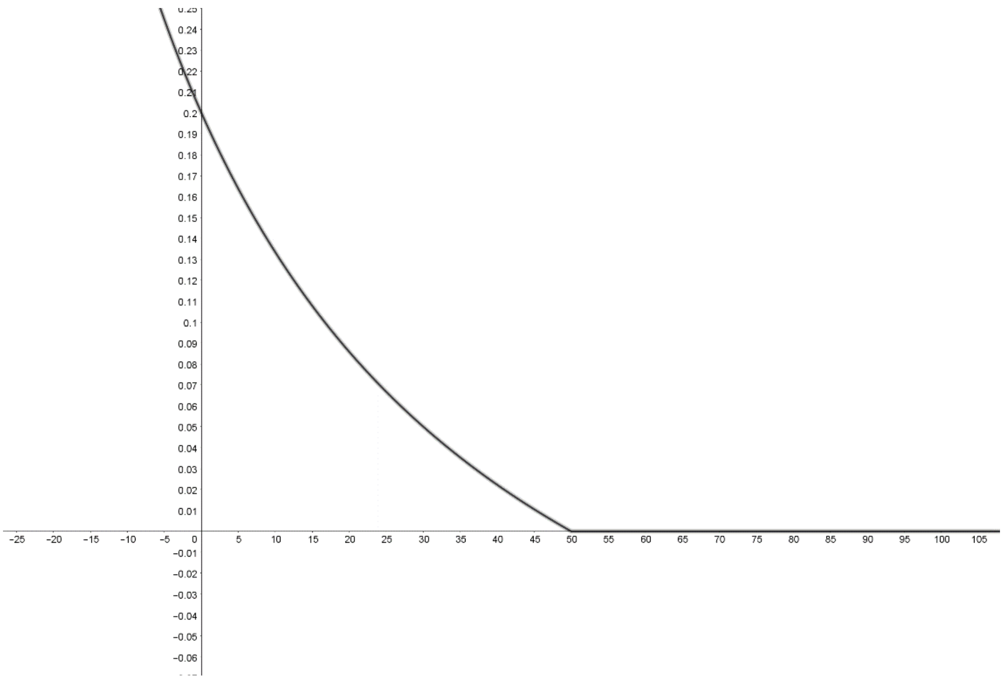
where the V is the volume of base added and K_W is the ionic product of water whose value at temperature 25 deg C is 10^{-14}, which is of the form similar to the above quadratic equation.

The positive root of the equation is

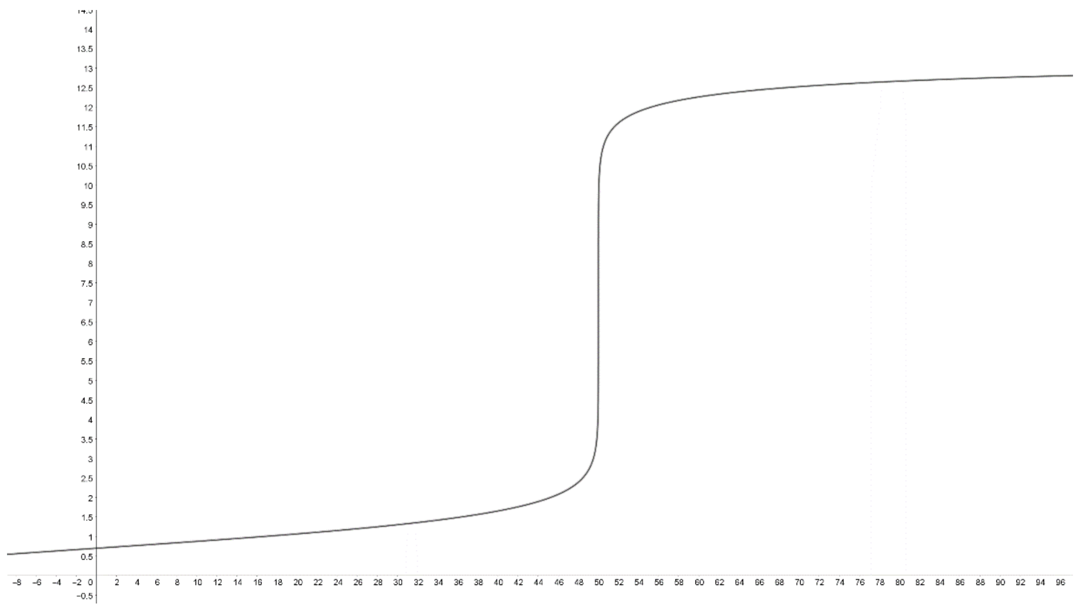
$$[\mathrm{H}^+] = \frac{1}{2}\left(\frac{10-0\cdot 2V}{50+V} + \sqrt{\left(\frac{10-0\cdot 2V}{50+V}\right)^2 + 4\times 10^{-14}}\right)$$

giving $[\mathrm{H}^+]$ in terms of the volume V of NaOH added.

A plot of $[\mathrm{H}^+]$ against V is shown:



and a plot of $\mathrm{pH} = -\log_{10}[\mathrm{H}^+]$ against V is shown:



The 'corner' shown is located at the equivalence point given by $10-0\cdot 2V=0$, i.e. $V=50$ where acid and base are present in equal amounts, and this is where the rapid transition occurs as shown. These features are precisely those discussed for the initial quadratic equation. We note from the expression for $[\mathrm{H}^+]$ that

$$[\mathrm{H}^+]\approx\frac{10-0\cdot 2V}{50+V} \qquad \text{for} \qquad V<50$$

$$[\mathrm{H}^+]=10^{-7} \qquad \text{for} \qquad V=50$$

and

$$\begin{aligned}[\mathrm{H}^+]&=\frac{1}{2}\left(\sqrt{\left(\frac{10-0\cdot 2V}{50+V}\right)^2+4\times10^{-14}}-\frac{0\cdot 2V-10}{50+V}\right)\\ &=\frac{1}{2}\left(\frac{4\times10^{-14}}{\sqrt{\left(\frac{10-0\cdot 2V}{50+V}\right)^2+4\times10^{-14}}+\frac{0\cdot 2V-10}{50+V}}\right) \qquad \text{for } V>50\\ &\approx\frac{10^{-14}}{(0\cdot 2V-10)(50+V)}\end{aligned}$$

The corresponding three parts of the graph for $[\mathrm{H}^+]$ and pH can be seen in the figures above.

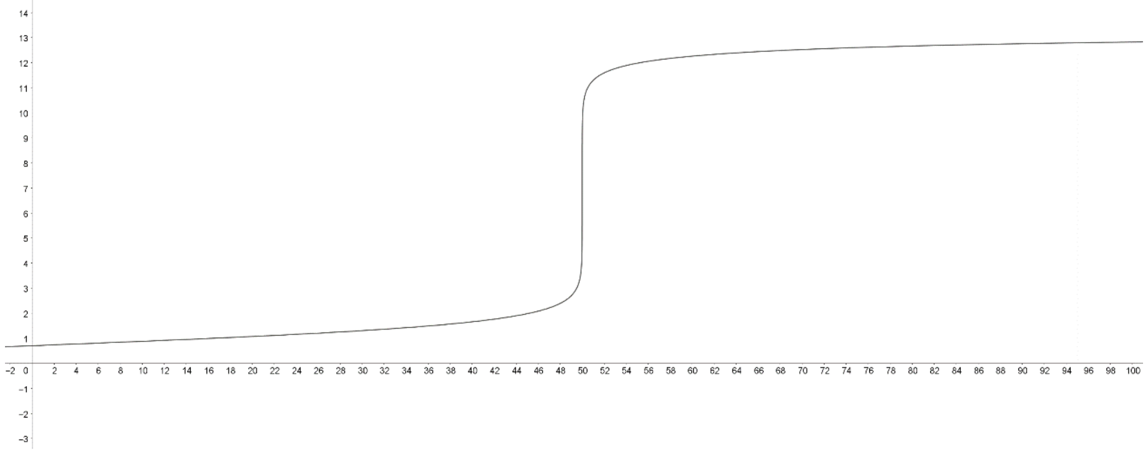
Another way of interpreting the quadratic equation is to rearrange as:

$$V=\frac{50K_W+10[\mathrm{H}^+]-50[\mathrm{H}^+]^2}{[\mathrm{H}^+]^2+0\cdot 2[\mathrm{H}^+]-K_W}$$

and if we substitute $[\mathrm{H}^+]=10^{-\mathrm{pH}}$ we have

$$V = \frac{50K_W + 10\times 10^{\text{-pH}} - 50\times(10^{\text{-pH}})^2}{(10^{\text{-pH}})^2 + 0\cdot 2\times 10^{\text{-pH}} - K_W}$$

Plotting V against pH using this expression, but with V along the horizontal and pH along the vertical gives exactly the same graph as above but without the need to solve the quadratic equation.



We can also read off the pH for a given volume V, again without the need to solve the quadratic equation. This technique proves very useful for titrations where we have a *weak* acid and strong base, a strong acid and a *weak* base, and finally where we have a *weak* acid and a *weak* base.

For example adding a strong base, e.g. sodium hydroxide (NaOH), of concentration $0\cdot 2$ to a volume 50 of weak acid, e.g. ethanoic acid (CH_3COOH), of concentration $0\cdot 2$, then then $[\mathrm{H}^+]$ is the (positive) solution of the cubic equation

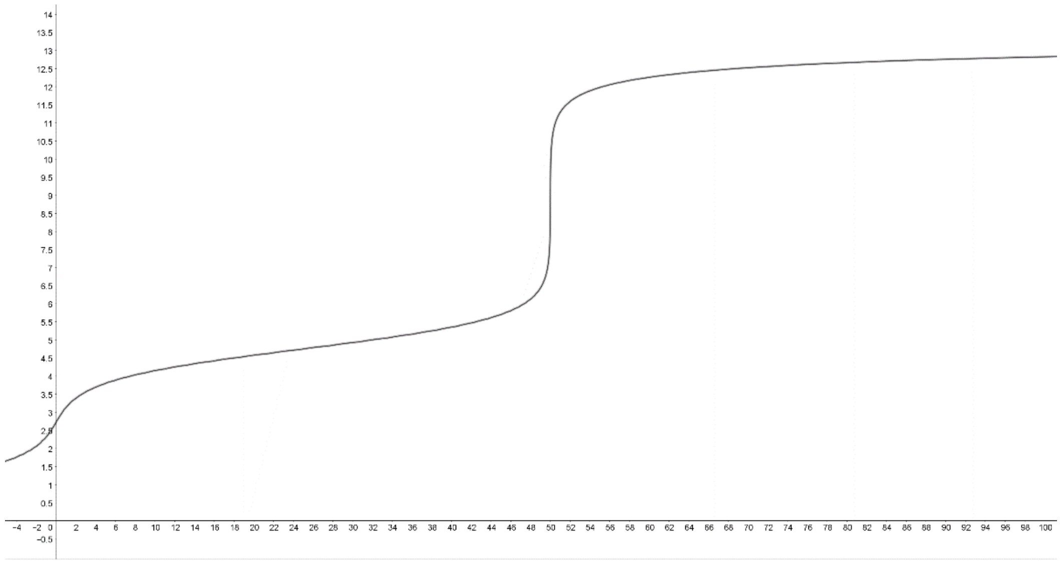
$$(V+50)(K_a+[\mathrm{H}^+])[\mathrm{H}^+]^2 + 0\cdot 2V(K_a+[\mathrm{H}^+])[\mathrm{H}^+] - K_W(50+V)(K_a+[\mathrm{H}^+]) - 0\cdot 2\times 50K_a[\mathrm{H}^+] = 0$$

where K_W is the ionic product of water, as before, whose value at temperature 25 deg C is 10^{-14}, and K_a is the equilibrium constant for the weak acid, whose value for ethanoic acid at 25 deg C is 5×10^{-8}. If we want to see how the $\mathrm{pH} = -\log_{10}[\mathrm{H}^+]$ varies with volume of base added, V, we would need to solve the cubic equation numerically for many values of V, or we could use the formula for the roots of a cubic.

Alternatively we could use the approach above for the quadratic and rearrange the expression to write V in terms of $[\mathrm{H}^+]$ and substite $[\mathrm{H}^+] = 10^{\text{-pH}}$:

$$V = \frac{50\Big((K_a+[\mathrm{H}^+])(K_W - [\mathrm{H}^+]^2) + 0\cdot 2K_a[\mathrm{H}^+]\Big)}{(K_a+[\mathrm{H}^+])\Big([\mathrm{H}^+]^2 + 0\cdot 2[\mathrm{H}^+] - K_W\Big)}$$

and then plot V against pH using this expression, but with V along the horizontal and pH along the vertical:



Similar remarks apply for weak base/strong acid titrations, and for weak acid/weak base titrations the corresponding equation is a quartic but the same approach as above will work in this case too and avoids solving the resulting quartic equation, either exactly or numerically for many different volumes V .

Revealing numerical solutions of a differential equation

Ask students to solve the problem:

$$\frac{dy}{dx} = +(1-y^2)^{\frac{1}{2}} \; 0 < x \le \pi \text{ , } y(0) = 0$$

and you may find they present the solution $y(x) = \sin(x)$ through integration and application of the initial condition.

Now ask them to use Euler's method:

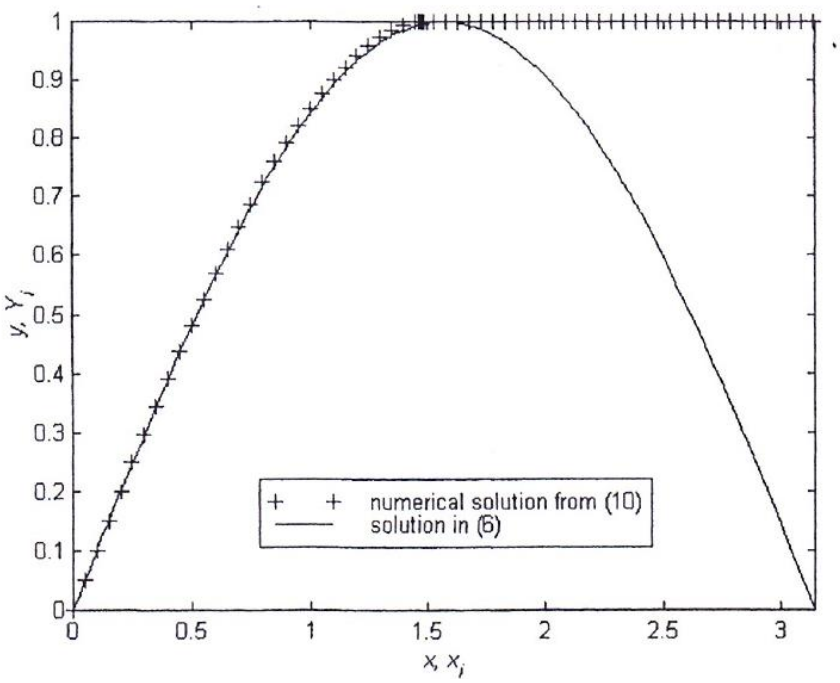
$$\frac{y_{n+1} - y_n}{h} = +(1-y_n^2)^{\frac{1}{2}} \text{ , } y_0 = 0$$

(where they will need to periodically reduce the size of h to prevent the argument under the square root becoming negative) and explain what they get.

They will discover that Euler's method is predicting the solution *correctly* as they have failed to spot in their solution $y(x) = \sin(x)$ that the left hand side of the differential equation is $\cos(x)$, whereas the right hand side is:

$$\begin{aligned} +(1-\sin^2 x)^{\frac{1}{2}} = +(\cos^2 x)^{\frac{1}{2}} &= |\cos(x)| \\ &= \cos(x) \text{ for } 0 \le x \le \tfrac{1}{2}\pi \\ &= -\cos(x) \text{ for } \tfrac{1}{2}\pi < x \le \pi \end{aligned}$$

i.e. their solution of $\sin(x)$ is only valid for $0 \le x \le \frac{1}{2}\pi$, and that the remaining part of the solution for $\frac{1}{2}\pi < x \le \pi$ is $y(x) = 1$, as correctly predicted by Euler's method!



Roots and reactions

The Haber process for the industrial manufacture of ammonia, nitrogen and hydrogen react in the presence of a catalyst as follows:

$$N_2 + 3H_2 \rightleftarrows 2NH_3 \quad .$$

The two-way arrow indicates that the reaction is reversible, and this is because the rate of reaction for a given temperature is proportional to the concentration of the reactants. So when the nitrogen and hydrogen are first mixed ammonia will form at a fast rate, but this will slow as the concentrations fall.

Conversely the rate of decomposition of ammonia grows as the concentration of ammonia increases, and after some time a dynamic equilibrium will be reached.

The amount of nitrogen, hydrogen and ammonia at equilibrium can be determined by solving an algebraic equation. For example, if there are initially a_0 moles of nitrogen, b_0 moles of hydrogen and u_0 moles of ammonia then the equilibrium amounts are $a_0 - x$, $b_0 - 3x$ and $u_0 + 2x$ moles, respectively, where x is a solution of

$$(u_0 + 2x)^2 - K(a_0 - x)(b_0 - 3x)^3 = 0$$

and where the equilibrium constant $K > 0$ is known.

To make sense physically the amounts at equilibrium must be positive, so that x must lie in $\left(-u_0/2, \min(a_0, b_0/3)\right)$. Moreover, there should be only one position of equilibrium. Thus

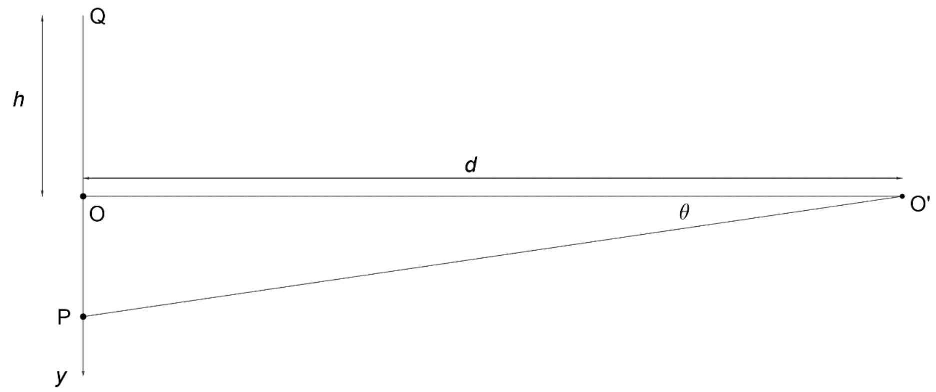
$$Q(x) \equiv (u_0 + 2x)^2 - K(a_0 - x)(b_0 - 3x)^3$$

should have only one solution in $\left(-u_0/2, \min(a_0, b_0/3\right)$.

Prove, using calculus, that this is the case.

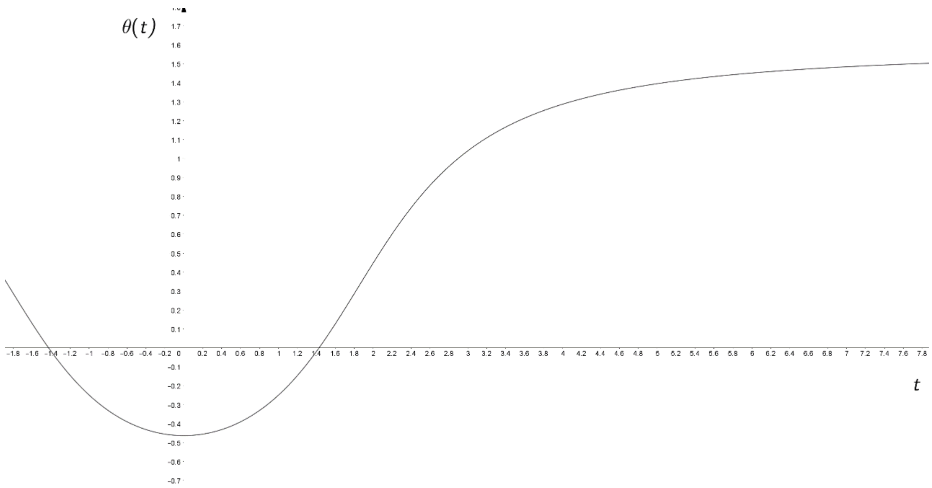
Skyers

A ball is dropped from rest when $t = 0$ from a point Q at a height h above a horizontal plane through O and O' where OO'= d , as shown:

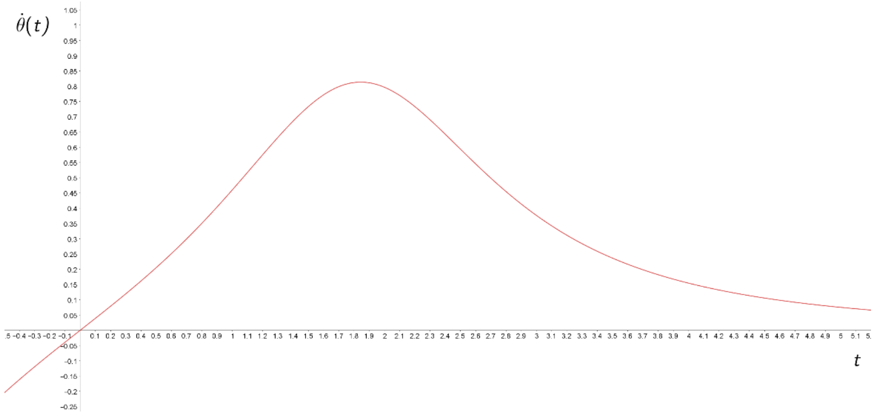


If the ball is at point P when the time elapsed is t, and is viewed by an observer at point O', how does the 'observation angle' θ vary as a function of t, and what observations do you make from this in terms of the rate at which the observer's head rotates while watching the ball fall (which will be accelerating due to gravity).

With $h = 10\,\mathrm{m}$, $d = 20\,\mathrm{m}$, the acceleration due to gravity $g = 9{\cdot}8\,\mathrm{ms}^{-2}$, θ varies as a function of t as shown:

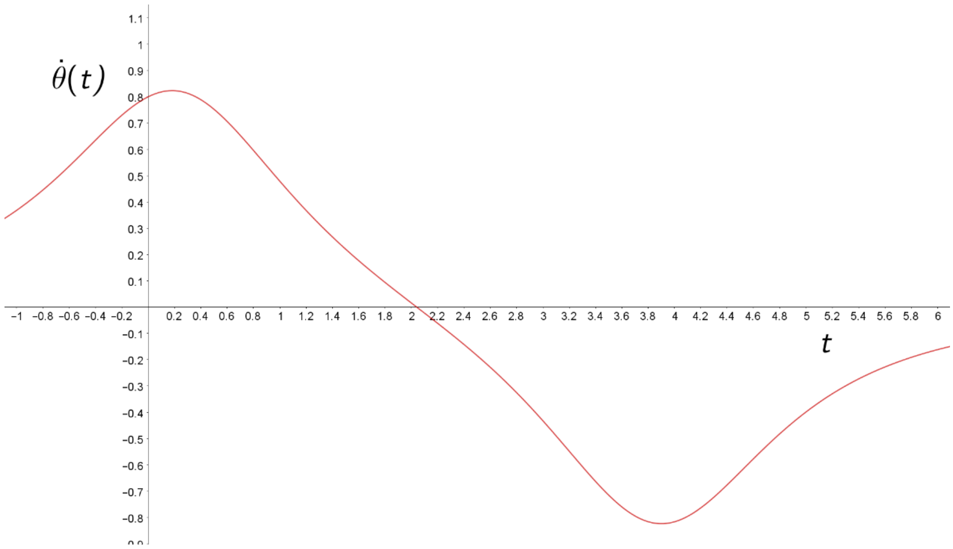


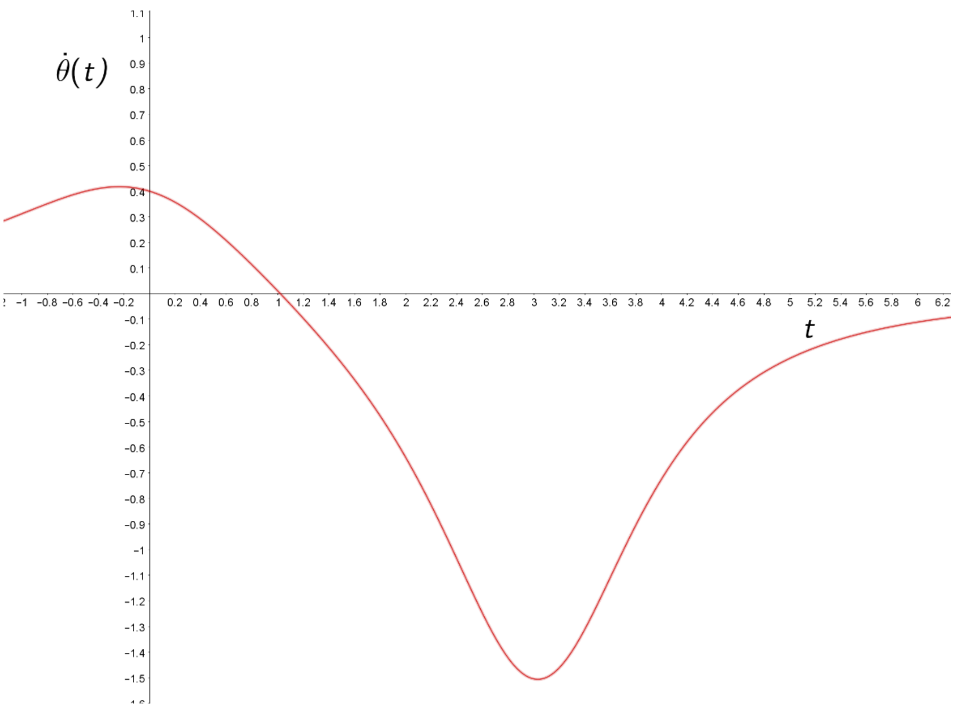
from which we see that the rate of 'rotation' $\dot{\theta}$ reaches a maximum before, and that is clear from the graph of $\dot{\theta}$ shown:



Determine where the ball has fallen to when this maximum is attained. (When $\frac{h}{d}$ is the same as that in the figure above the ball will be at the point P shown in that figure.) What happens if the ball is initially projected downwards with a speed u ?

A natural extension is to consider a ball projected vertically *upwards* from a point *below* the horizontal plane through OO'. Graphs of $\dot{\theta}$ for two different scenarios are shown below, where in the first case $|\dot{\theta}|$ attains a maximum (below the horizontal) on the way up and on the way down, whereas in the second case $|\dot{\theta}|$ attains just one maximum value (again on the way down and below the horizontal).

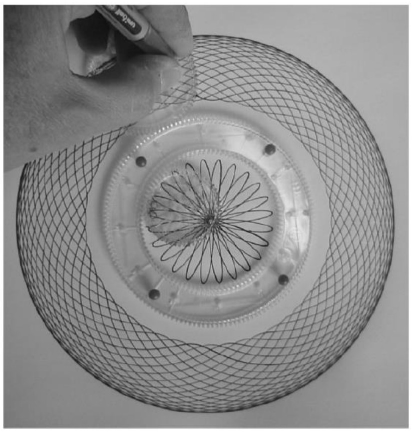




Show that these two cases are distinguished according to whether or not (h, d) lies outside the circle centre $\left(u^2/g, 0\right)$, radius u^2/g. In both cases there are some interesting features which one could investigate, for example the oblique points of inflexion in the graphs of $\dot{\theta}$, i.e. where $\dddot{\theta} = 0$, $\ddot{\theta} < 0$

Spidergraph

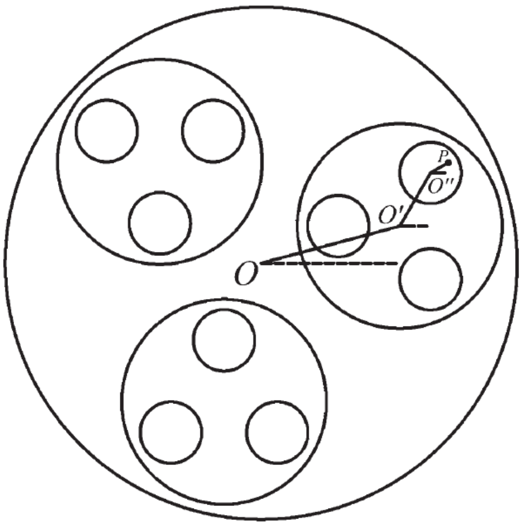
Some of you may be familiar with Spirograph:



where the locus traced out are curves such as *epitrochoids*, *hypotrochoids*, with special cases of *epicycloids* and *hypocycloids*, as well as *cycloids* when the rolling disk moves along a straight line as opposed to inside or outside a circle.

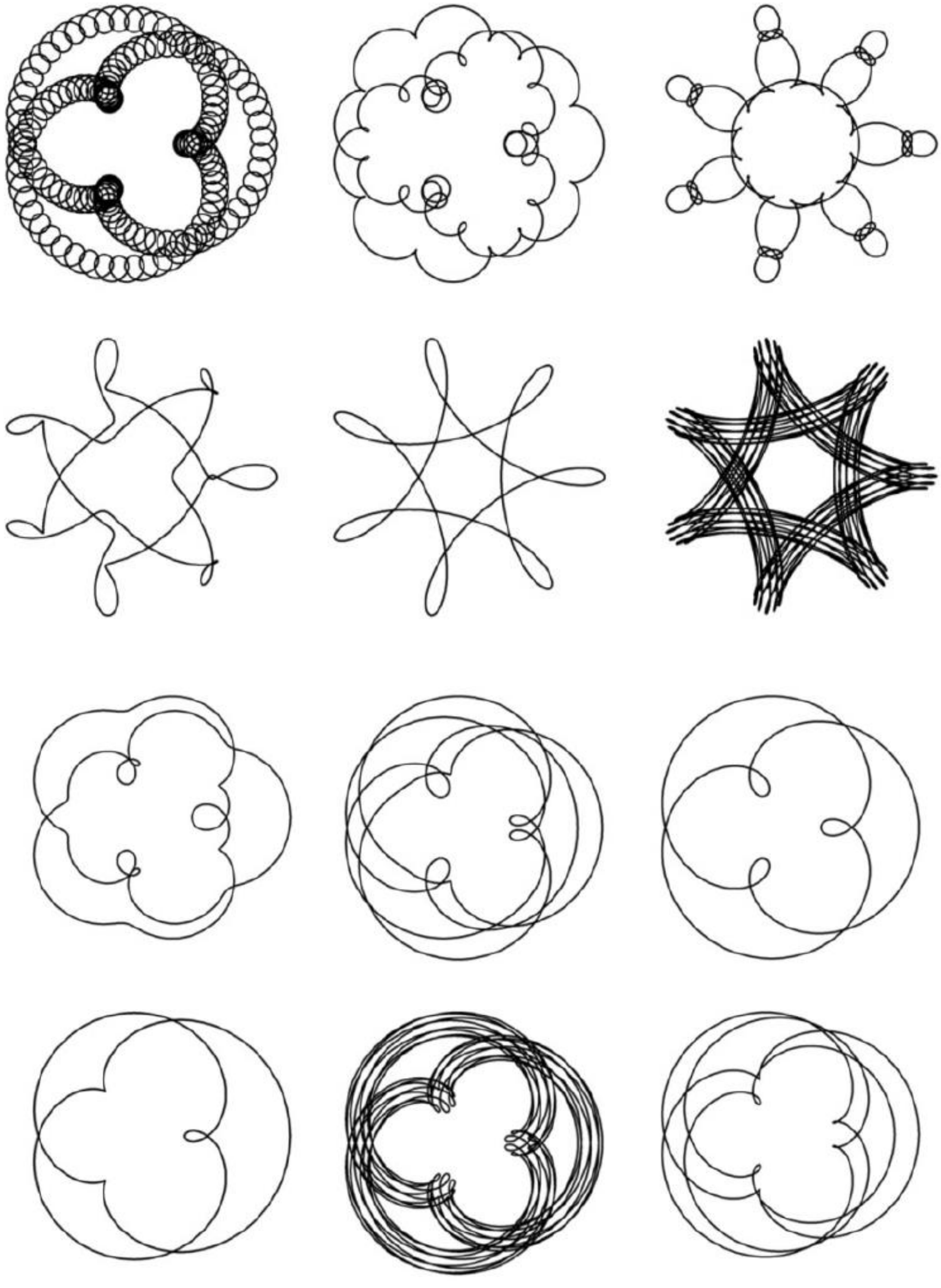
But what are the loci traced out by the riders on Disneyland's Mad Hatter's Teacup ride, shown, or Legoland's spinning spider:

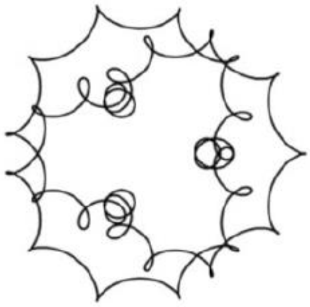
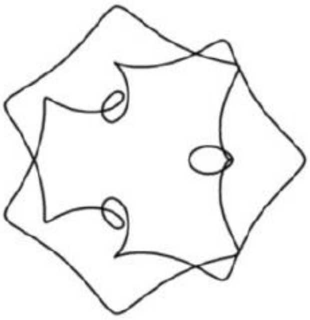
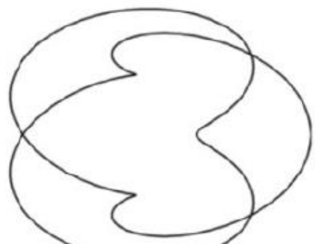
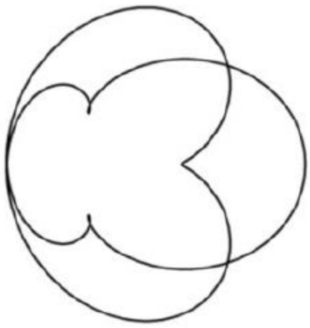
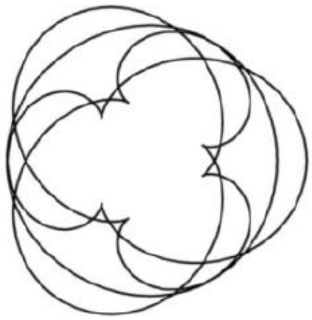
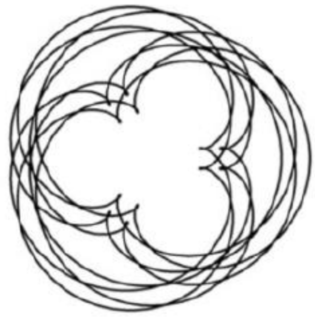
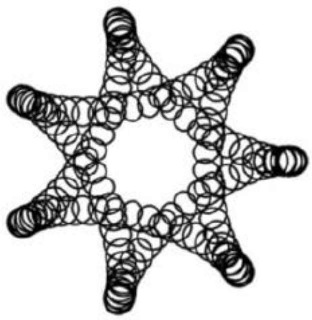
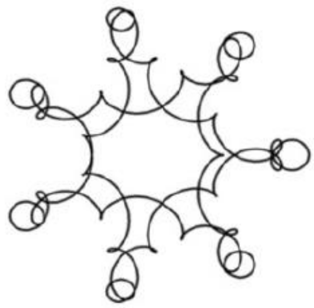
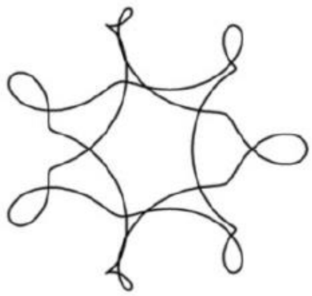
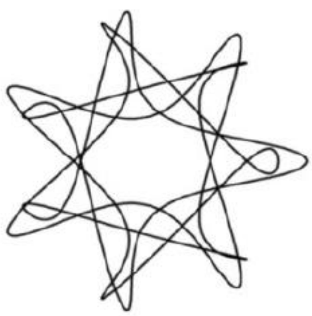
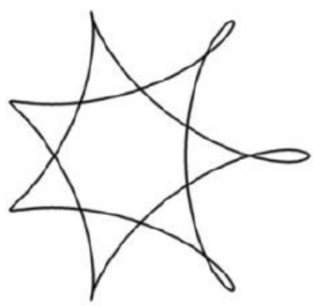
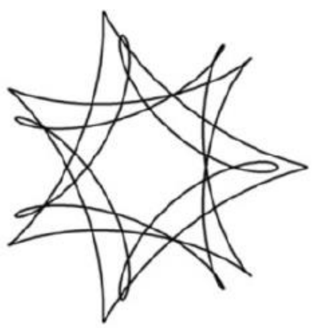
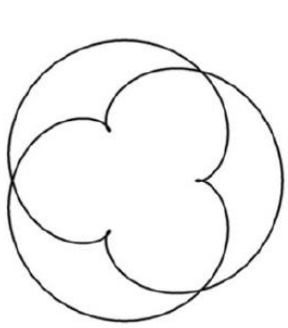
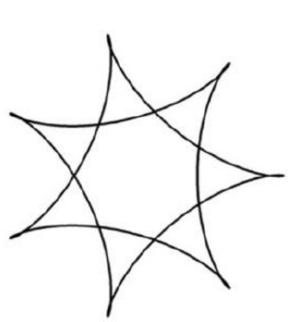
Here is a schematic of the ride

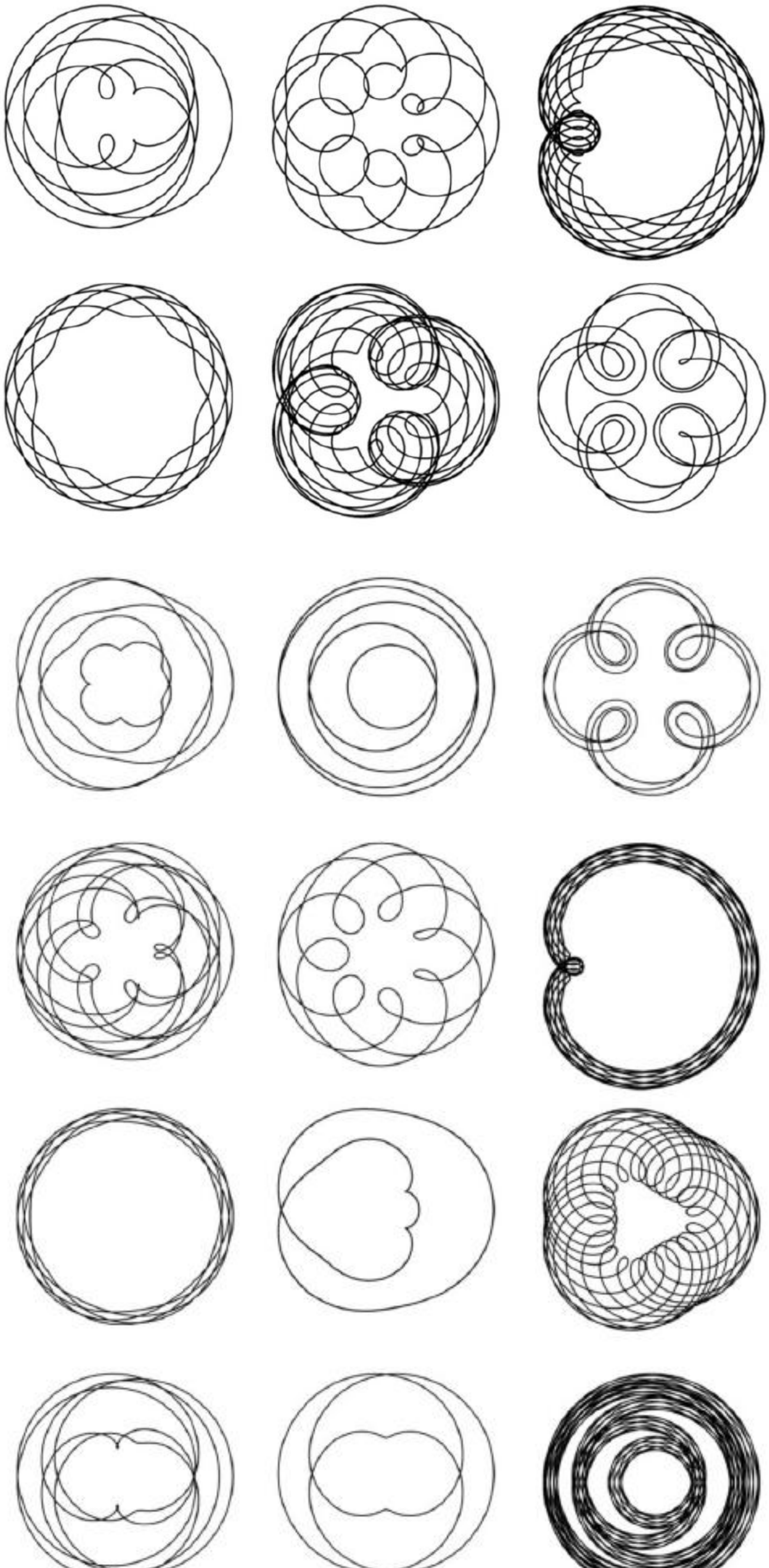


Each of the large and medium size disks rotate (increasing in speed during the ride), while the speed of the small disks varies according to whoever has taken charge of the wheel to turn the teacup (or similar).

Write down expressions for the parametric equations for the locus of the riders. Depending on the particular ride (and the rider), which will dictate the various radii of the three disks, and the various speeds of all three disks, and which direction they are spinning in, you will see many different patterns for the loci, as follows:

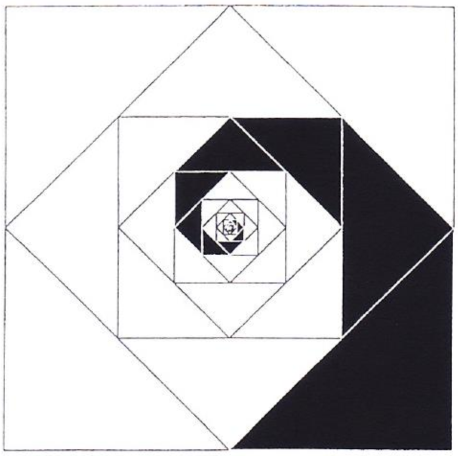
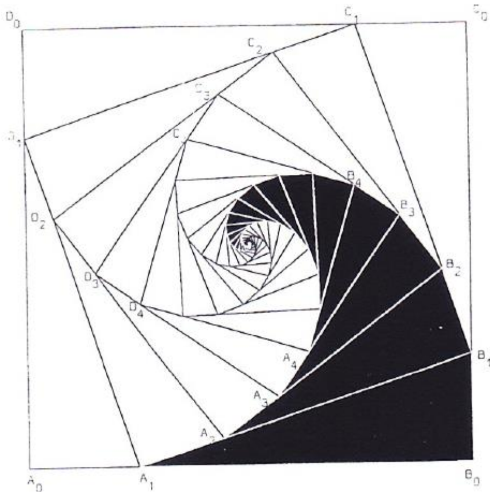


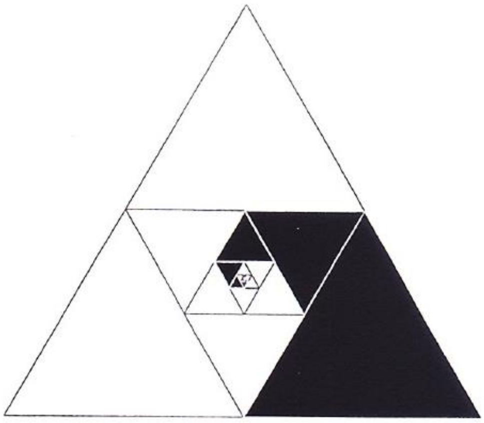
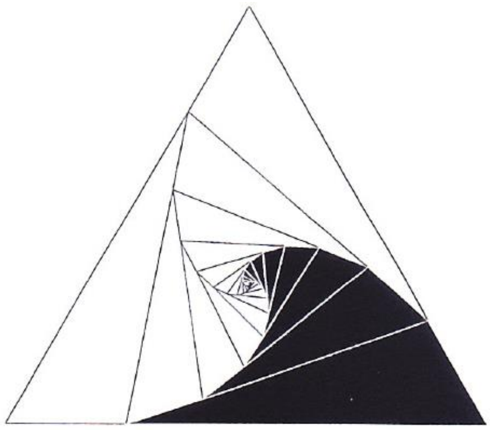


Spirals galore

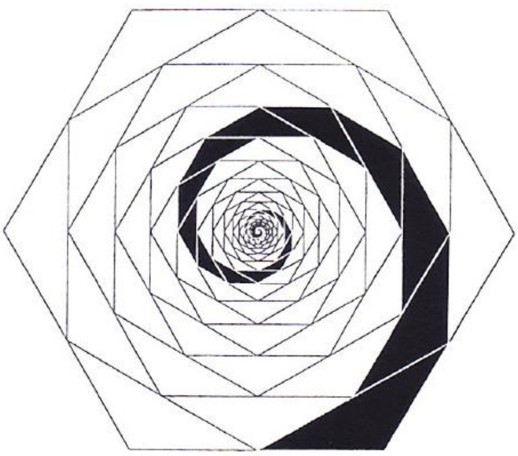
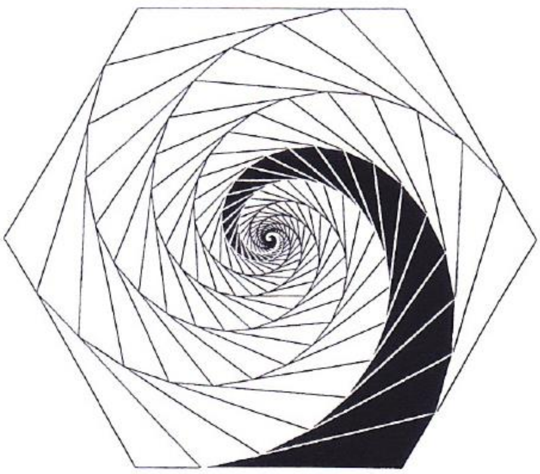
What is the area of the spiral shown on the left, and what about the general case on the right?

What about a triangle, and the corresponding general case?

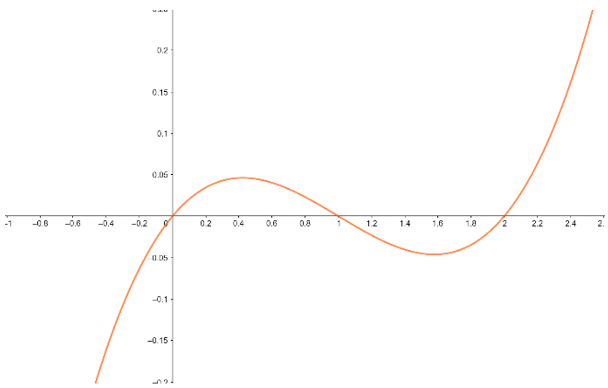
A hexagon?

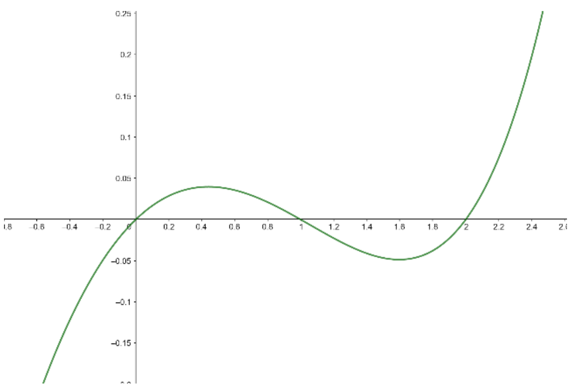



Any regular polygon?

Spot the difference

Can you spot the difference between these two curves?





The one on the left is $f(x)=0{\cdot}115x(x-1)(x-2)$ while the other one is $g(x)=2^x-\frac{1}{2}x(x+1)$.

The function $g(x)$ and its generalisation to $h(x)=a^x-\frac{1}{2}x(x+1)$, $\quad a>1$, have some properties in common with cubic functions like $f(x)$.

Here are some properties of $g(x)$ and $h(x)$.

1. $g(x)$ has three real roots at $x=0,1,2$.
2. $g(n)\geq 0$ for all non-negative integers n.
3. $g(x)<0$ for $x<0$ and for $1<x<2$, and is non-negative otherwise.
4. $g(x)$ has a local maximum in $(0,1)$ and a local minimum in $(1,2)$.
5. $g(x)$ has a point of inflexion at $x\approx 1{\cdot}06$ ($f(x)$ has its inflexion at $x=1$).
6. $h(x)$ always has a root at $x=0$.
7. $h(x)$ has only one real root ($x=0$) and a horizontal point of inflexion at $x\approx 0{\cdot}875$ when $a\approx 2{\cdot}07$.
8. $h(x)$ has a positive, double real root at $x\approx 1{\cdot}44$ when $a\approx 2{\cdot}02$.
9. $h(x)$ has a double real root at $x=0$ when $a=\sqrt{e}$.

Can you prove these?

Squaring up to factorials

The series: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, valid for all x, tells us that

$$e^{-1} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots = \frac{1}{\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots} = \frac{1}{e} \quad (\text{where } e = 2\cdot 71828\cdots)\ .$$

What about:

$$\begin{aligned}
&\frac{1}{(0!)^2} + \frac{1}{(1!)^2} + \frac{1}{(2!)^2} + \cdots \\
&\frac{1}{(0!)^2} - \frac{1}{(1!)^2} + \frac{1}{(2!)^2} - \cdots \\
&\frac{1}{(0!)^n} + \frac{1}{(1!)^n} + \frac{1}{(2!)^n} + \cdots,\ n = 3, 4, \ldots \\
&\frac{1}{(0!)^0} + \frac{1}{(1!)^1} + \frac{1}{(2!)^2} + \cdots
\end{aligned}
\qquad ?$$

A tale of two series and a Dickens of an integral

You know that:

$$\frac{d}{dx}x^x = (1+\ln x)x^x \qquad .$$

Have you ever wondered about:

$$\int x^x dx \qquad \text{or} \qquad \int_0^1 x^x dx \qquad ?$$

You also know that:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots \qquad \text{is unbounded}$$

whereas:

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\cdots = \ln 2 \qquad .$$

Have you ever wondered about:

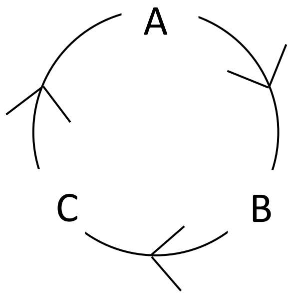
$$\frac{1}{1^1}+\frac{1}{2^2}+\frac{1}{3^3}+\cdots \qquad \text{or} \qquad \frac{1}{1^1}-\frac{1}{2^2}+\frac{1}{3^3}-\cdots \qquad ?$$

What about others such as:

$$\frac{1}{1^1}-\frac{1}{3^2}+\frac{1}{5^3}+\cdots \qquad \text{or} \qquad \frac{1}{2^0}+\frac{1}{4^1}+\frac{1}{6^2}+\cdots \qquad ?$$

The other Bernoulli trials

The three Bernoulli brothers, Daniel, Johann II and Nicolaus II were playing a game with three six-sided dice, labelled: A (Daniel), B (Johann II) and C (Nicolaus II), given to them by their father, Johann. The dice had the numbers 1 to 18 written on them so that the sum of the digits on each die is the same, i.e. 57. The game consists of rolling either two or three of the dice at a time, and a win occurs for the die showing the highest number on the upper face. The dice are said to be *non-transitive* if when A rolls against B then A wins on average, and when B rolls against C then B wins on average, and, importantly, when C rolls against A then C also wins on average. In other words there is a cyclic pattern as illustrated in the figure, where A > B means A beats B on average, etc.



The first set of dice that Johann produced are shown in Table 1.

Die A				Die B				Die C		
1	6	8		3	5	7		2	4	9
11	15	16		10	14	18		12	13	17

Table 1

These are examples of non-transitive dice. The likelihoods of A beating B, B beating C and C beating A are same, and after a careful check (by comparing each of three possible pairings of the three dice) the winning die would win 19 and lose 17 times out of a possible 36 outcomes, and so the odds of A beating B etc. are $19:17$. Table 2 shows the 36 possible outcomes for the pair A and B and which die wins for each outcome.

		Die B						
		3	5	7	10	14	18	sum = 57
Die A	1	B	B	B	B	B	B	
	6	**A**	**A**	B	B	B	B	
	8	**A**	**A**	**A**	B	B	B	
	11	**A**	**A**	**A**	**A**	B	B	
	15	**A**	**A**	**A**	**A**	**A**	B	
	16	**A**	**A**	**A**	**A**	**A**	B	
sum =	57							

Table 2

Exactly the same result occurs for the pair B and C and the pair C and A, with the difference between wins for A and wins for B in the trial between A and B being $19-17=2$, and similarly for B and C, and C and A. This information is shown in Table 3.

Die A				Die B				Die C			Wins A - B	Wins B - C	Wins C - A
1	6	8		3	5	7		2	4	9	2	2	2
11	15	16		10	14	18		12	13	17			

Table 3

They next played a game where all three dice were to be rolled and the winner each time would be the one whose die showed the *highest* score.

Out of 216 possible outcomes on average they would each win the same number of times, i.e. 72 each, and in this case the odds of winning are all equal. This information is summarised in Table 4.

Die A				Die B				Die C			Wins A - B	Wins B - C	Wins C - A	Wins A	Wins B	Wins C
1	6	8		3	5	7		2	4	9	2	2	2	72	72	72
11	15	16		10	14	18		12	13	17						

Table 4

Is this always the case? What other possibilities are there? Table 5 shows another scenario.

Die A				Die B				Die C			Wins A - B	Wins B - C	Wins C - A	Wins A	Wins B	Wins C
1	6	9		4	5	7		2	3	11	4	4	4	72	72	72
10	13	18		8	16	17		12	14	15						

Table 5

Table 6 shows a range of other scenarios with some interesting variations.

Die A		Die B		Die C	Wins A - B	Wins B - C	Wins C - A	Wins A	Wins B	Wins C
1 2 4 15 17 18		3 5 10 12 13 14		6 7 8 9 11 16	2	2	2	102	57	57
1 6 9 10 13 18		4 5 7 8 16 17		2 3 11 12 14 15	4	4	4	72	72	72
1 2 5 14 17 18		3 4 9 12 13 16		6 7 8 10 11 15	2	2	2	97	63	56
1 2 5 15 16 18		3 4 11 12 13 14		6 7 8 9 10 17	4	4	4	96	60	60
1 2 7 14 15 18		5 6 10 11 12 13		3 4 8 9 16 17	4	4	4	88	56	72
1 2 9 14 15 16		6 7 8 11 12 13		3 4 5 10 17 18	6	6	6	81	54	81
1 5 7 10 16 18		2 3 6 14 15 17		4 8 9 11 12 13	2	2	2	80	80	56
1 5 8 13 14 16		3 4 9 11 12 18		2 6 7 10 15 17	2	2	2	73	71	72
1 6 8 11 15 16		3 5 7 10 14 18		2 4 9 12 13 17	2	2	2	72	72	72
1 6 11 12 13 14		5 7 8 9 10 18		2 3 4 15 16 17	6	6	6	63	63	90
1 7 8 10 14 17		2 5 6 13 15 16		3 4 9 11 12 18	2	2	2	71	74	71
1 7 8 10 14 17		3 5 6 12 13 18		2 4 9 11 15 16	2	2	2	71	73	72
1 7 8 11 12 18		3 5 6 10 16 17		2 4 9 13 14 15	4	4	4	72	74	70
1 7 8 11 12 18		4 5 6 9 16 17		2 3 10 13 14 15	4	4	4	72	72	72
1 6 11 12 13 14		5 7 8 9 10 18		2 3 4 15 16 17	6	6	6	63	63	90
1 2 11 12 15 16		5 6 9 10 13 14		3 4 7 8 17 18	4	4	4	80	56	80
1 2 9 10 17 18		5 6 7 8 15 16		3 4 11 12 13 14	4	4	4	88	64	64

Table 6

Variations on this problem include the simpler problem of tossing three coins with the numbers 1 to 6 written on the faces so that the sum on each coin is the same.

Table 7 shows the only possible case in which we see that the coins are *equally likely* to win against each other in a two-coin tossing competition, whereas in a three-coin tossing competition with these values A wins 4 times out of the $2\times2\times2=8$ possible outcomes, with B and C winning 2 out of 8 each. Because 8 is not divisible by 3 it could not have been possible that in this case each would win the same number of times.

Coin A		Coin B		Coin C	Wins A - Wins B	Wins B - Wins C	Wins C - Wins A	Wins A	Wins B	Wins C
1 6		2 5		3 4	0	0	0	4	2	2

Table 7

A simple way of checking the number of wins in the three-coin game, and indeed for any game with all three (or more) coins or dice (or spinners) is to write down all the possibilities as shown in Table 8 for the above example and tally up the wins (which are highlighted).

Coin A	Coin B	Coin C
1	2	3
1	2	4
1	5	3
1	5	4
6	2	3
6	2	4
6	5	3
6	5	4

Table 8

For three spinners made of equilateral triangular card, and the numbers 1 to 9 written on them, and a total for each spinner of 15, two results are found as shown in Table 9.

Spinner A		Spinner B		Spinner C	Wins A - B	Wins B - C	Wins C - A	Wins A	Wins B	Wins C
1 5 9		3 4 8		2 6 7	1	1	1	11	8	8
1 6 8		3 5 7		2 4 9	1	1	1	10	7	10

Table 9

Only in the first case, with 11 wins out a possible $3\times3\times3=27$ outcomes, is there an outright winner in the three-spinner game. Each spinner wins 5 and loses 4 times out of the 9 possible outcomes when it competes against its neighbour. Even though 27 is divisible by 3, there is no solution where each spinner wins 9 times in the three-spinner game.

Turning now to three tetrahedral dice, with the numbers 1 to 12 written on them with the sum of faces being 26, it is again not possible to achieve equally likely outcomes in the three-dice game because the number of possible outcomes $4\times4\times4=64$ is not divisible by 3. Three examples are shown in Table 10, and the last example is one in which the outcomes are as near equal as is possible, with 21, 21 and 22 wins for A, B and C, respectively, with C narrowly winning. The dice are again non-transitive with A winning 9 times and losing 7 times in 16 possible outcomes when competing against B, and similarly for the other two pairings.

Die A		Die B		Die C	Wins A - B	Wins B - C	Wins C - A	Wins A	Wins B	Wins C
1 3 10 12		2 7 8 9		4 5 6 11	2	2	2	28	18	18
1 4 10 11		2 7 8 9		3 5 6 12	2	2	2	25	18	21
1 7 8 10		3 5 6 12		2 4 9 11	2	2	2	21	21	22

Table 10

Three examples for pentagonal spinners are shown in Table 11. Here we see that the odds of A beating B, B beating C, and C beating A, vary from $13:12$, $14:11$ and $3:2$ in rows 1, 2 and 3, respectively. Because $5 \times 5 \times 5 = 125$ is not divisible by 3 we know for certain that there cannot be a case where there are equally likely outcomes of winning in the three-spinner game.

Die A		Die B		Die C	Wins A - B	Wins B - C	Wins C - A	Wins A	Wins B	Wins C
1 3 11 12 13		2 7 8 9 14		4 5 6 10 15	1	1	1	48	38	39
1 4 6 14 15		2 3 10 12 13		5 7 8 9 11	3	3	3	52	42	31
1 3 11 12 13		6 7 8 9 10		2 4 5 14 15	5	5	5	45	30	50

Table 10

Clearly there are many more variations to consider. Apart from other regular polygonal spinners, other obvious cases include the possibilities arising from polyhedral dice in the shape of the three remaining Platonic solids: the 8-sided octahedron, the 12-sided dodecahedron and the 20-sided icosahedron. What we know for certain is that because $12^3 = 1728$ *is* divisible by 3, while $8^3 = 512$ and $20^3 = 8000$ are not, it is only in the dodecahedral case where we could have the non-transitive property with each die winning against its neighbour *and* equally likely outcomes of winning in the three dice game.

What about using *four* dice. Can these all be non-transitive, and with equally likely outcomes when all four roll? Because $4^4 = 256$ is divisible by 4 it is theoretically possible to have equally likely outcomes in the four tetrahedral dice game when all four are rolled, but can you find a solution in which the dice are also non-transitive? In this case even with four non-transitive dice there are solutions in which the odds of A beating B are larger than those of B beating C, etc, which is not the case above in the three-dice game.

Similarly with $6^4 = 1296$ divisible by 4 it is theoretically possible to have equally likely outcomes in the four cubic dice game when all four are rolled, but can you find a solution in this case in which the dice are also non-transitive? With $3^4 = 81$ not divisible by 4 it is not possible to have equally likely outcomes in the four triangular spinner game when all four are spun, and in any case it is not possible to have four non-transitive triangular spinners with the numbers 1 to 12 written on them with each spinner having the same sum, or indeed four spinners with the same sum because the sum of the numbers is 78 which is not divisible by 4.

With coins, spinners and polyhedral dice in the shape of the five Platonic solids, including the regular six-sided dice, and any number of them, the possibilities are endless.

Two obvious facts that can easily be established are as follows.

- For k lots of n-sided spinners/dice with consecutive integers starting at m up to $m+nk-1$, the sum of the numbers is $\frac{1}{2}nk(2m+nk-1)$, and this is divisible by k (giving the sum on each spinner as $\frac{1}{2}n(2m+nk-1)$) only if n is even or nk is odd. Clearly in the case where $n=3$ and $k=4$, i.e. four triangular spinners, we cannot achieve the same sum on each spinner with consecutive integers regardless of which integer we start with. With $n=6$ and $k=4$, however, i.e. four cubic dice, then n is even and the sum on each dice is $3(2m+23)$. If we have the same number of spinners/dice as the number of sides on them, i.e. $n=k$, then we would need either n to be even or $nk=n^2$ to be odd, and one or other of these is always true, for example with $n=k=6$, i.e. 6 cubic dice.

- For k lots of n-sided spinners/dice the number of outcomes in the k spinner/die game is n^k, and for it to be theoretically possible to have equally likely outcomes then n^k must be divisible by k. With $n=6$ and $k=4$, i.e. four cubic dice, then $n^k/k=6^4/4=324$, and it is theoretically possible to have equally likely outcomes in this case as we saw earlier. If we again have the same number of spinners/dice as the number of sides on them, i.e. $n=k$, then we would need n^n to be divisible by n, and since $n^n/n=n^{n-1}$, which is certainly true for integers $n \geq 2$, and hence it is always theoretically possible to have equally likely outcomes in this case, for example with $n=k=6$, i.e. 6 cubic dice.

As a final challenge can you find an example of 6 non-transitive cubic dice with the numbers 1 to 36 written on the faces, with the sum on each die of 111, where in the 6-dice game each wins on average $6^6/6=6^5=7776$ times out of the $6^6=46656$ possible outcomes?

The revenge of the hare over the tortoise

This alternative to the original fable apportions a win to whoever of the hare and the tortoise travels the furthest based on a set of prescribed rules (regardless how long it would take) rather than whoever crosses a line first. The options are as follows:

Option 1: Move forward in turn the following distances: $\binom{4}{i} = \frac{4!}{(4-i)!i!}$, $i = 0,1,2,3,4$, i.e. the distances $1,4,6,4,1$.

Option 2: Move forward in turn the following distances: $\frac{1}{2^i}\binom{3+i}{i} = \frac{(3+i)!}{2^i\, 3!i!}$, $i = 0,1,2,\ldots$, i.e. the distances $1,2,\frac{5}{2},\frac{5}{2},\frac{35}{16},\frac{7}{4},\ldots$.

The hare offered the tortoise the choice of moves, pointing out that Option 1 comprised a finite number of moves, whereas Option 2 required infinitely moves. Ignoring the clear fact that infinitely many moves cannot be made in a finite time, the winner would be the one who would cover the furthest distance when one considered the *limit* of the number of moves in Option 2.

The tortoise immediately chose Option 2 because he thought that by taking infinitely many moves, even though they started by increasing in size, but then decreasing, eventually he would cover more ground than in Option 1. He also thought that the moves in Option 2 were very much in keeping with his *style*. The hare was delighted with this. He was confident that the tortoise would never come close to covering the distance he would travel by the moves in Option 1. Option 1 was also more in keeping with the hare's style – a few large moves, increasing at first and then decreasing (as he tired), but altogether a finite number.

Did the tortoise make the correct choice?

After 5 moves the hare had covered a distance $1+4+6+4+1=16$.

After 4 moves the tortoise had travelled a distance of $1+2+\frac{5}{2}+\frac{5}{2}=8$, so already one-half of that travelled by the hare.

The total distance each covers is:

$$\binom{4}{0}+\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4} \qquad \underline{\text{Option 1}}$$

$$\binom{3}{0}+\frac{1}{2}\binom{4}{1}+\frac{1}{4}\binom{5}{2}+\frac{1}{8}\binom{6}{3}+\cdots \qquad \underline{\text{Option 2}}$$

The sum in Option 1 is 16.

What is the infinite sum in Option 2; is it larger or smaller than 16?

Generalisations of this problem are:

$$\binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5} \qquad \underline{\text{Option 1}}$$

(which gives a distance of 32), and

$$\binom{4}{0}+\frac{1}{2}\binom{5}{1}+\frac{1}{4}\binom{6}{2}+\frac{1}{8}\binom{7}{3}+\cdots \qquad \underline{\text{Option 2}}$$

so is sum is larger or smaller than 32?

Here is a solution to determine the sum, S_n, of the infinite geometric series with binomial coefficients

$$S_n = \sum_{i=0}^{\infty} \frac{1}{a^i}\binom{n-1+i}{i} = \binom{n-1}{0} + \frac{1}{a}\binom{n}{1} + \frac{1}{a^2}\binom{n+1}{2} + \frac{1}{a^3}\binom{n+2}{3} + \cdots, n = 1, 2, \ldots, a > 1 \quad (1)$$

where the original problem is with $a = 2$ and $n = 4$.

Consider first the expression $\left(1 - \frac{1}{a}\right)^{-n}$ for $n = 1, 2, \ldots, a > 1$ using the infinite (convergent) binomial expansion

$$\begin{aligned}
\left(1-\frac{1}{a}\right)^{-n} &= 1 + \frac{(-n)\times\left(-\frac{1}{a}\right)}{1!} + \frac{(-n)\times(-n-1)\times\left(-\frac{1}{a}\right)^2}{2!} + \frac{(-n)\times(-n-1)\times(-n-2)\times\left(-\frac{1}{a}\right)^3}{3!} + \cdots \\
&= 1 + \frac{n}{1!}\times\frac{1}{a} + \frac{(n+1)n}{2!}\times\left(\frac{1}{a}\right)^2 + \frac{(n+2)(n+1)n}{3!}\times\left(\frac{1}{a}\right)^3 \cdots \\
&= \binom{n-1}{0} + \frac{1}{a}\binom{n}{1} + \frac{1}{a^2}\binom{n+1}{2} + \frac{1}{a^3}\binom{n+2}{3} + \cdots \qquad (2) \\
&= \sum_{i=0}^{\infty}\frac{1}{a^i}\binom{n-1+i}{i} \\
&= S_n \quad , \quad n = 1, 2, \ldots
\end{aligned}$$

i.e. the sum in (1). (Note that (2) also converges for $a < -1$.) Hence the sum in (1) is $\left(1-\frac{1}{a}\right)^{-n}$, representing the (general) distance travelled using Option 2. With $a = 2$ and $n = 4$ this becomes $\left(1-\frac{1}{2}\right)^{-4} = \left(\frac{1}{2}\right)^{-4} = 2^4 = 16$, which is precisely the distance travelled for the sum in Option 1 in the original challenge., i.e. it would appear that the tortoise would get closer to the place where the hare finished, and eventually draw level in the limit! So the tortoise might as well take the offer of the truce and save himself the arduous journey of approaching the hare! But what about the general case?

Returning expression for the sum in (2) we have, using the finite binomial expansion, the standard result

$$\begin{aligned}
\left(1-\frac{1}{a}\right)^{-n} &= \left(\frac{a-1}{a}\right)^{-n} = \left(\frac{a}{a-1}\right)^{n} = \left(1+\frac{1}{a-1}\right)^{n} \\
&= \sum_{i=0}^{n}\frac{1}{(a-1)^i}\binom{n}{i} \qquad (3) \\
&= \binom{n}{0} + \frac{1}{a-1}\binom{n}{1} + \frac{1}{(a-1)^2}\binom{n}{2} + \cdots + \frac{1}{(a-1)^n}\binom{n}{n} \quad , \quad n = 1, 2, \ldots
\end{aligned}$$

a finite geometric series with binomial coefficients and the value of S_n from (2). (Note (3) is also valid for $n = 0$). With $a = 2$ and $n = 4$ the finite series of moves implicit in (3) is precisely those of the original ones for Option 1 in the general case, i.e. $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3}$, confirming what we have already established – Option 1 and Option 2 give identical distances in this case.

However, we also see in general from (2) and (3) that

$$\sum_{i=0}^{\infty}\frac{1}{a^i}\binom{n-1+i}{i} = \left(\frac{a}{a-1}\right)^n = \sum_{i=0}^{n}\frac{1}{(a-1)^i}\binom{n}{i} \quad , \quad n = 1, 2, \ldots, \quad a > 1 \quad . \qquad (4)$$

i.e.

$$\begin{aligned}
&\binom{n-1}{0}+\frac{1}{a}\binom{n}{1}+\frac{1}{a^2}\binom{n+1}{2}+\frac{1}{a^3}\binom{n+2}{3}+\cdots \\
&=\left(\frac{a}{a-1}\right)^n \qquad (5)\\
&=\binom{n}{0}+\frac{1}{a-1}\binom{n}{1}+\frac{1}{(a-1)^2}\binom{n}{2}+\cdots+\frac{1}{(a-1)^n}\binom{n}{n} \quad , \quad n=1,2,\ldots, a>1
\end{aligned}$$

and with $a=2$ (5) gives

$$\binom{n-1}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{4}\binom{n+1}{2}+\frac{1}{8}\binom{n+2}{3}+\cdots=2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n} \quad , \quad n=1,2,\ldots \qquad (6)$$

showing that in the general case for the original problem, Option 1 with a finite number of moves (representing the right hand side of (6)) and Option 2 with an infinite number of moves (representing the left hand side of (6)) cover exactly the same distances. We also see that that by replacing n by $n+1$ on the left hand side of (6) the distance covered will be $2^{n+1}=2\times 2^n$, i.e. if we start the moves with n increased by 1 then the distance covered in both Options 1 and 2 will be doubled.

The result in (5) also showed the hare and the tortoise what would happen in a range of other Options 1 and 2 by varying the individual moves, i.e.

Option 1: Move forward in turn the following distances: $\frac{1}{(a-1)^i}\binom{n}{i}, i=0,1,\ldots,n$.

Option 2: Move forward in turn the following distances: $\frac{1}{a^i}\binom{n-1+i}{i}, i=0,1,2,\ldots$.

giving in each case a distance travelled of $\left(\frac{a}{a-1}\right)^n$.

For example, with $a=3,4,5,\frac{3}{2}$ in (5) gives, in turn:

$$\binom{n-1}{0}+\frac{1}{3}\binom{n}{1}+\frac{1}{9}\binom{n+1}{2}+\frac{1}{27}\binom{n+2}{3}+\cdots=\left(\tfrac{3}{2}\right)^n=\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{4}\binom{n}{2}+\cdots+\frac{1}{2^n}\binom{n}{n}$$

$$\binom{n-1}{0}+\frac{1}{4}\binom{n}{1}+\frac{1}{16}\binom{n+1}{2}+\frac{1}{64}\binom{n+2}{3}+\cdots=\left(\tfrac{4}{3}\right)^n=\binom{n}{0}+\frac{1}{3}\binom{n}{1}+\frac{1}{9}\binom{n}{2}+\cdots+\frac{1}{3^n}\binom{n}{n}$$

$$\binom{n-1}{0}+\frac{1}{5}\binom{n}{1}+\frac{1}{25}\binom{n+1}{2}+\frac{1}{125}\binom{n+2}{3}+\cdots=\left(\tfrac{5}{4}\right)^n=\binom{n}{0}+\frac{1}{4}\binom{n}{1}+\frac{1}{16}\binom{n}{2}+\cdots+\frac{1}{4^n}\binom{n}{n}$$

$$\binom{n-1}{0}+\frac{2}{3}\binom{n}{1}+\frac{4}{9}\binom{n+1}{2}+\frac{8}{27}\binom{n+2}{3}+\cdots=3^n=\binom{n}{0}+2\binom{n}{1}+4\binom{n}{2}+\cdots+2^n\binom{n}{n}$$

(all for $n=1,2,\ldots$), where in each case Option 1 with a finite number of moves (representing the right hand side), and Option 2 with an infinite number of moves (representing the left hand side) cover exactly the same distances.

As a variation what about varying their moves so that for every other one they would step backwards?

For the alternating moves the series would now be alternating ones whose sums could be determined by replacing 'a' by '$-a$' in (4) and simplifying, i.e.

$$\sum_{i=0}^{\infty}\frac{(-1)^i}{a^i}\binom{n-1+i}{i}=\left(\frac{a}{a+1}\right)^n=\sum_{i=0}^{n}\frac{(-1)^i}{(a+1)^i}\binom{n}{i} \quad , \quad n=1,2,\ldots, \quad a>1 \tag{7}$$

or by doing the same in (5), i.e.

$$\begin{aligned}&\binom{n-1}{0}-\frac{1}{a}\binom{n}{1}+\frac{1}{a^2}\binom{n+1}{2}-\frac{1}{a^3}\binom{n+2}{3}+\cdots\\&=\left(\frac{a}{a+1}\right)^n\\&=\binom{n}{0}-\frac{1}{a+1}\binom{n}{1}+\frac{1}{(a+1)^2}\binom{n}{2}-\cdots+\frac{(-1)^n}{(a+1)^n}\binom{n}{n} \quad , \quad n=1,2,\ldots, a>1\end{aligned} \quad . \tag{8}$$

Note the change in the denominator of the geometric coefficients from $a-1$ in (5) to $a+1$ in (8).

For example, with $a=2,3,4,\frac{3}{2}$ in (8) gives, in turn (all for $n=1,2,\ldots$)

$$\binom{n-1}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{4}\binom{n+1}{2}-\frac{1}{8}\binom{n+2}{3}+\cdots=\left(\tfrac{2}{3}\right)^n=\binom{n}{0}-\frac{1}{3}\binom{n}{1}+\frac{1}{9}\binom{n}{2}+\cdots+\frac{(-1)^n}{3^n}\binom{n}{n}$$

$$\binom{n-1}{0}-\frac{1}{3}\binom{n}{1}+\frac{1}{9}\binom{n+1}{2}-\frac{1}{27}\binom{n+2}{3}+\cdots=\left(\tfrac{3}{4}\right)^n=\binom{n}{0}-\frac{1}{4}\binom{n}{1}+\frac{1}{16}\binom{n}{2}+\cdots+\frac{(-1)^n}{4^n}\binom{n}{n}$$

$$\binom{n-1}{0}-\frac{1}{4}\binom{n}{1}+\frac{1}{16}\binom{n+1}{2}-\frac{1}{64}\binom{n+2}{3}+\cdots=\left(\tfrac{4}{5}\right)^n=\binom{n}{0}-\frac{1}{5}\binom{n}{1}+\frac{1}{25}\binom{n}{2}+\cdots+\frac{(-1)^n}{5^n}\binom{n}{n}$$

$$\binom{n-1}{0}-\frac{2}{3}\binom{n}{1}+\frac{4}{9}\binom{n+1}{2}-\frac{8}{27}\binom{n+2}{3}+\cdots=\left(\tfrac{3}{5}\right)^n=\binom{n}{0}-\frac{2}{5}\binom{n}{1}+\frac{4}{25}\binom{n}{2}+\cdots+\frac{(-1)^n 2^n}{5^n}\binom{n}{n}$$

where in each case Option 1 with a finite number of alternating forwards and backwards moves (representing the right hand side) and Option 2 with an infinite number of alternating forwards and backwards moves (representing the left hand side) cover exactly the same distances.

Finally we see that for the positive series (5) in the case that the geometric coefficients on the left hand side are powers of the unit fractions $\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots$ then the corresponding geometric coefficients on the right hand side are the 'shifted' (in the denominator) unit fractions $\frac{1}{1},\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots$.

Similarly for alternating series (8) in the case that the geometric coefficients on the left hand side are powers of the unit fractions $\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots$ then the corresponding geometric coefficients on the right hand side are the unit fractions ('shifted' in the denominator the other way): $\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots$.

We now consider alternative moves to the Options above which comprised only a finite number of moves in each case.

These new moves involved using the binomial coefficients to determine the distance to move each time.

For example, taking the 5th row in Pascal's triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array} \tag{9}
$$

one would move forwards, in turn, $1,4,6,4,1$ units, representing the binomial coefficients

$\binom{4}{i} = \frac{4!}{(4-i)!i!}$, $i = 0,1,2,3,4$. This means ones ends up

$$1+4+6+4+1=16=2^4$$

units from the starting point; whereas moving forwards 1 unit, backwards 4 units, forwards 6 units, backwards 4 units, and finally forwards 1 unit, means ones ends up

$$1-4+6-4+1=0$$

units from the starting point, i.e. in this case one ends up where one started (although the total distance travelled is still 16 units).

Both of these results are particular cases of the general results

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}+\binom{n}{n}=2^n \quad , \quad n=0,1,\ldots \tag{10}$$

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^{n-1}\binom{n}{n-1}+(-1)^n\binom{n}{n}=0 \quad , \quad n=1,2,\ldots \tag{11}$$

where the first result (10) corresponds to moving forwards at every turn. The second one (11) corresponds to moving alternately forwards and backwards, and is interesting because it means that the moves would always ultimately result in ending up back at the starting point regardless of which row in (9) one used to determine the size of the individual jumps.

The two became interested in variations on these two versions where the individual moves were scaled according to a prescribed formula. They were mainly interested in alternating jumps, which is what we mostly consider here, mainly because they didn't want to end up too far from the start! We leave you to experiment with non-alternating moves.

The first variation was where each move was scaled by $\frac{1}{2}$ of the previous jump, which would end up

$$\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{4}\binom{n}{2}-\cdots+(-1)^{n-1}\frac{1}{2^{n-1}}\binom{n}{n-1}+(-1)^n\frac{1}{2^n}\binom{n}{n} \quad , \quad n=0,1,\ldots \tag{12}$$

units from the starting point. (They also considered other moves, e.g. where the fractional scaling $\frac{1}{2}$ was replaced by $\frac{1}{3}$.) They discovered that the moves would end up

$$\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{4}\binom{n}{2}-\cdots+(-1)^{n-1}\frac{1}{2^{n-1}}\binom{n}{n-1}+(-1)^{n}\frac{1}{2^{n}}\binom{n}{n}=\frac{1}{2^{n}}\text{ , } n=0,1,\ldots \qquad (13)$$

units from the starting point. For example, with $n=4$ one would end up

$$\binom{4}{0}-\frac{1}{2}\binom{4}{1}+\frac{1}{4}\binom{4}{2}-\frac{1}{8}\binom{4}{3}+\frac{1}{16}\binom{4}{4}=1-\frac{1}{2}\times 4+\frac{1}{4}\times 6-\frac{1}{8}\times 4+\frac{1}{16}\times 1=\frac{1}{16}=\frac{1}{2^{4}}$$

units from the starting point, as predicted by (13). Note from (13) that the end point is exactly where one would end up by merely moving *forwards* the distance moved in the very last move only, of value $\frac{1}{2^{n}}\binom{n}{n}=\frac{1}{2^{n}}$ units. Other examples include ending up

$$\binom{n}{0}-\frac{1}{3}\binom{n}{1}+\frac{1}{9}\binom{n}{2}-\cdots+(-1)^{n-1}\frac{1}{3^{n-1}}\binom{n}{n-1}+(-1)^{n}\frac{1}{3^{n}}\binom{n}{n}=\frac{2^{n}}{3^{n}}\text{ , } n=0,1,\ldots$$

units from the starting point, where we scale each move of the original by $\frac{1}{3}$ instead of $\frac{1}{2}$, but note that in this case the end point is *not* where one would end up by merely moving forwards the distance moved in the very last move only.

This intrigued the pair so they now looked at a simple variation on the last one where the distances were not scaled by the powers of $\frac{1}{2}$, namely $\frac{1}{2},\frac{1}{4},\frac{1}{8},\ldots$, but instead by the unit fractions $\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots$; in other words, the moves would result in ending up

$$\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\cdots+(-1)^{n-1}\frac{1}{n}\binom{n}{n-1}+(-1)^{n}\frac{1}{n+1}\binom{n}{n}\quad , \quad n=0,1,\ldots \qquad (14)$$

units from the start.

They first noted that if the powers $\frac{1}{2},\frac{1}{4},\frac{1}{8},\ldots$ were replaced by the unit fractions $\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots$. and only moved forwards (indefinitely) by the unit fraction distances they would end up moving further and further away from the start because

$$1+\frac{1}{2}+\frac{1}{3}+\cdots=\infty$$

the well-known harmonic series, although if they alternated these moves forwards and then backwards, they found that they approached an end point which was

$$1-\frac{1}{2}+\frac{1}{3}-\cdots=\ln(2)\approx 0\cdot 7$$

units from the start.

Returning to (14), they discovered that they ended up where they would have done by merely moving forwards the distance in the very last move only, of value $\frac{1}{n+1}\binom{n}{n}=\frac{1}{n+1}$ units, i.e.

$$\binom{n}{0}-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\cdots+(-1)^{n-1}\frac{1}{n}\binom{n}{n-1}+(-1)^{n}\frac{1}{n+1}\binom{n}{n}=\frac{1}{n+1},\ n=0,1,\ldots \qquad (15)$$

in common with the moves given in (13). For example, with $n=4$ one ends

$$\binom{4}{0}-\frac{1}{2}\binom{4}{1}+\frac{1}{3}\binom{4}{2}-\frac{1}{4}\binom{4}{3}+\frac{1}{5}\binom{4}{4}=1-\frac{1}{2}\times 4+\frac{1}{3}\times 6-\frac{1}{4}\times 4+\frac{1}{5}\times 1=\frac{1}{5}$$

units from the starting point, as predicted by (15).

(For forward moves:

$$\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\cdots+\frac{1}{n}\binom{n}{n-1}+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1},\ n=0,1,\ldots)$$

One last variation comprises translating one place to the left the unit fractions which they scaled the binomial coefficients by, i.e. move forwards $\frac{1}{2}\binom{n}{0}$, backwards $\frac{1}{3}\binom{n}{1}$, forwards $\frac{1}{4}\binom{n}{2}$, and so on, ending up

$$\frac{1}{2}\binom{n}{0}-\frac{1}{3}\binom{n}{1}+\frac{1}{4}\binom{n}{2}-\cdots+(-1)^{n-1}\frac{1}{n+1}\binom{n}{n-1}+(-1)^{n}\frac{1}{n+2}\binom{n}{n}\quad,\quad n=0,1,\ldots$$

units from the starting point. For example, with $n=4$ this results in ending up

$$\frac{1}{2}\binom{4}{0}-\frac{1}{3}\binom{4}{1}+\frac{1}{4}\binom{4}{2}-\frac{1}{5}\binom{4}{3}+\frac{1}{6}\binom{4}{4}=\frac{1}{2}\times 1-\frac{1}{3}\times 4+\frac{1}{4}\times 6-\frac{1}{5}\times 4+\frac{1}{6}\times 1=\frac{1}{30}$$

units from the starting point, and with $n=3$ this results in ending up

$$\frac{1}{2}\binom{3}{0}-\frac{1}{3}\binom{3}{1}+\frac{1}{4}\binom{3}{2}-\frac{1}{5}\binom{3}{3}=\frac{1}{2}\times 1-\frac{1}{3}\times 3+\frac{1}{4}\times 3-\frac{1}{5}\times 1=\frac{1}{20}\qquad .$$

units from the starting point. This suggests the result that

$$\frac{1}{2}\binom{n}{0}-\frac{1}{3}\binom{n}{1}+\frac{1}{4}\binom{n}{2}-\cdots+(-1)^{n-1}\frac{1}{n+1}\binom{n}{n-1}+(-1)^{n}\frac{1}{n+2}\binom{n}{n}=\frac{1}{(n+1)(n+2)},\ n=0,1,\ldots \qquad (16)$$

Can you prove this? What about forward moves?

$$\frac{1}{2}\binom{n}{0}+\frac{1}{3}\binom{n}{1}+\frac{1}{4}\binom{n}{2}+\cdots+\frac{1}{n+1}\binom{n}{n-1}+\frac{1}{n+2}\binom{n}{n},\ n=0,1,\ldots$$

Or more general moves:

$$\frac{1}{k}\binom{n}{0}-\frac{1}{k+1}\binom{n}{1}+\frac{1}{k+2}\binom{n}{2}-\cdots+(-1)^{n-1}\frac{1}{k+n-1}\binom{n}{n-1}+(-1)^{n}\frac{1}{k+n}\binom{n}{n},\ n=0,1,\ldots \quad (17)$$

for $k=1,2,\ldots$ together with the corresponding case of moving only *forwards*?

For completeness, the result for the alternating trial in (17) is

$\dfrac{n!(k-1)!}{(n+k)!}$, $n=0,1,\ldots$, $k=1,2,\ldots$, coinciding with the results in (15) and (16) in the cases $k=1,2$, respectively.

Times of flight

An object is projected vertically upwards with speed U in a uniform gravitational field where g is the constant acceleration due to gravity. The speed of return to the point of projection is denoted by V, and the times 'up' to the highest point, and 'down' from the highest point to the point of projection, are denoted by t_u and t_d, with $T = t_u + t_d$ denoting the total time of flight.

If it is assumed that there is no (air) resistance to motion, show that:

$$U = V \ , t_u = t_d = \frac{U}{g} \ , T = \frac{U+V}{g} \qquad .$$

Extension 1 Assume now that there is a resistance to motion of magnitude $k \times (speed)^2$, where $k > 0$ is a constant.

Show that

$$t_u = \frac{1}{\sqrt{gk}} \tan^{-1}\left(\sqrt{\frac{k}{g}}U\right) \qquad t_d = \frac{1}{\sqrt{gk}} \tanh^{-1}\left(\sqrt{\frac{k}{g}}V\right) = \frac{1}{\sqrt{gk}} \sinh^{-1}\left(\sqrt{\frac{k}{g}}U\right) \qquad .$$

and that $t_d > t_u$.

Extension 2 Assume now that there is a resistance to motion of magnitude $k \times (speed)$, where $k > 0$ is a constant.

Show that

$$U + V = \frac{g}{k} \ln\left(\frac{g+kU}{g-kV}\right) \qquad t_u = \frac{1}{k} \ln\left(1 + \frac{kU}{g}\right) \qquad t_d = -\frac{1}{k} \ln\left(1 - \frac{kV}{g}\right)$$

and that

$$T = \frac{U+V}{g}$$

as for no resistance.

Show that $t_d > t_u$.

Extension 3 Assume now that there is a resistance to motion of magnitude $f(speed)$ for some positive function f (for positive arguments).

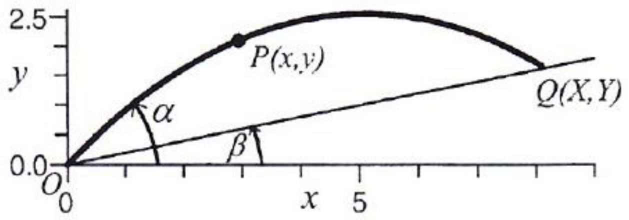
Show that $t_d > t_u$.

Extension 4 Assume now that there is a resistance to motion where mechanical energy is <u>not</u> conserved (which is also the case for Extensions 2,3,4),and show that

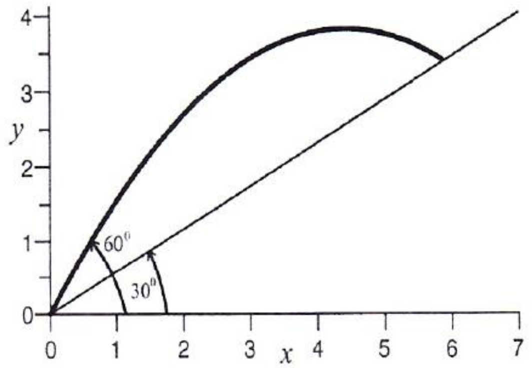
$$V < U \qquad \text{and} \qquad t_d > t_u \qquad .$$

Two problems thrown up by projectiles

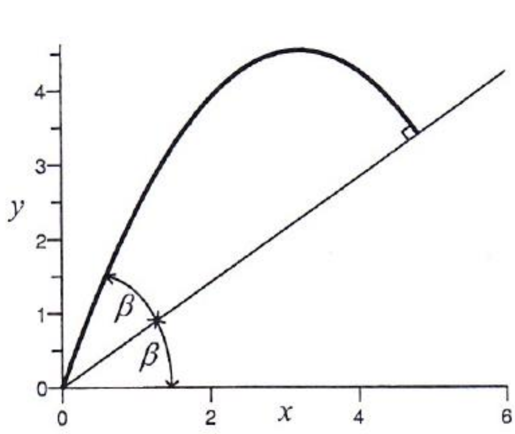
An object is projected at an angle α (to the horizontal) up a plane inclined at angle β (to the horizontal). What should α be so that the object lands as far as possible up the plane from the point of projection, attaining its maximum 'range' up the plane?



A simple calculation shows that this achieved when $\alpha-\beta=\frac{1}{2}(\frac{1}{2}\pi-\beta)$, i.e. when the initial direction of motion *bisects* the angle between the plane and the vertical. The special case where the plane is horizontal with $\beta=0$, and so the required $\alpha=\frac{1}{4}\pi\ \ (45^\circ)$ will be familiar. Another interesting case is when $\beta=\frac{1}{6}\pi\ \ (30^\circ)$ so that the angle for the maximum 'range' is $\alpha=\frac{1}{3}\pi\ \ (60^\circ)$, i.e. where $\alpha=2\beta$.



What about the 'angle of impact' when the object hits the plane? Suppose the object hits the plane at a right-angle. What is the minimum angle of projection, α, for which this occurs as β varies? It turns out that this occurs when $\alpha=2\beta$ (with $\beta=\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$ and $\alpha=\tan^{-1}\left(2\sqrt{2}\right)$).



What are the corresponding results when there is an (air) resistance proportional to $(\text{speed})^n$ for $n>0$, e.g. a linear ($n=1$) and a quadratic ($n=2$) law of resistance?

Is 3 unique?

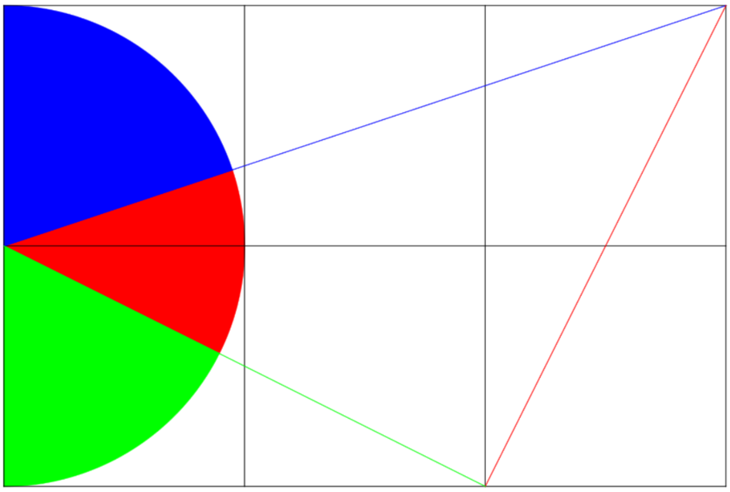
The familiar result

$$\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi \tag{1}$$

Immediately suggests the general problem of finding other natural numbers $n>1$ for which

$$\tan^{-1}1+\tan^{-1}2+\cdots+\tan^{-1}n=k\pi \tag{2}$$

for some $k\in\mathbb{N}$.



Pictorial representation of $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$.

Prove that

$$\tan^{-1}1+\tan^{-1}2+\cdots+\tan^{-1}n=\tan^{-1}\left(\frac{f(n)}{g(n)}\right)+m\pi \tag{3}$$

for some $m\in\mathbb{N}$ satisfying $\dfrac{n-2}{4}<m<\dfrac{n+1}{2}$, where the expressions $f(n)$ and $g(n)$ in (3) are different according to whether n is odd or even:

$$\left.\begin{aligned} f(n)&=S(n,1)-S(n,3)+S(n,5)-\cdots+(-1)^{\frac{n}{2}-1}S(n,n-1)\\ g(n)&=1-S(n,2)+S(n,4)-\cdots+(-1)^{\frac{n}{2}}S(n,n)\end{aligned}\right\}\quad n \text{ is even} \qquad , \tag{4}$$

$$\left.\begin{aligned} f(n)&=S(n,1)-S(n,3)+S(n,5)-\cdots+(-1)^{\frac{n-1}{2}}S(n,n)\\ g(n)&=1-S(n,2)+S(n,4)-\cdots+(-1)^{\frac{n-1}{2}}S(n,n-1)\end{aligned}\right\}\quad n \text{ is odd} \qquad . \tag{5}$$

and where

$$S(n,j)=\sum_{i_1<i_2<\cdots<i_j} i_1i_2\cdots i_j\ ,\ n\in\mathbb{N}\ ,\ j\in\{1,2,\ldots,n\} \tag{6}$$

representing the sum of all possible $\binom{n}{j}$ combinations of j distinct values chosen from the n values.

$1,2,\ldots,n$.

To find general solutions of (2) you will need to find values of n for which $f(n)=0$ and $g(n)\neq 0$ so that the first term on the right hand side of (3) vanishes. To explore this further you need to generate expressions for $f(n)$ and $g(n)$.

Prove that

$$S(n,n)=n! \quad , \quad n=1,\ldots$$

$$S(n,1)=\tfrac{1}{2}n(n+1) \quad , \quad n=1,\ldots$$

$$S(n,2)=\tfrac{1}{24}n(n^2-1)(3n+2) \quad , \quad n=2,\ldots$$

and (to be able to determine $S(n,j)$ for $j=3,4,\ldots,n-1$):

$$S(n+1,j)=S(n,j)+(n+1)S(n,j-1) \ , \ 1\leq j\leq n$$

where $S(n,0)=1$ for all $n\in\mathbb{N}$.

Use technology to generate $S(n,j), f(n), g(n)$ and investigate their properties to explore other possible solutions of (2) (other than (1)), i.e. for some value of n other than 3.

There are some interesting features of $f(n)$ and $g(n)$, and thus $\tan^{-1}1+\tan^{-1}2+\cdots+\tan^{-1}n$, for $n=15,80,395,1904$, and once you have discovered what these are, find the next value of n where these occur – as a hint the value is some $n<10000$. Investigating this may provide you with some further insight into solving the original problem (2).