ON THE INTERACTIONS BETWEEN INVARIANT TUBES IN VOLUME PRESERVING MAPS

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Abstract
We consider a class of nearly integrable volume preserving mappings for which invariant tubes appear according to a simple resonance condition. By controlling the nonlinearities that appear within both the limiting integrable map and its perturbations, we are able to discuss a novel phenomenon where distinct tubes of orthogonal orientation pass through one another and interfere as they do so.

1. Introduction
In this paper we shall consider the interactions between invariant structures within volume preserving (Liouvillian) mappings. By considering a class of such mappings that contain general nonlinear terms (within both the limiting integrable map and its perturbation) we can give a description of the genesis of invariant tubes (observed in numerical examples) through a generalised resonance condition for nearly integrable systems, and we can control and examine how such tubes may interfere with one another. This last gives rise to a novel phenomenon where two tubes, with orthogonal axial directions, must pass “through” one another as we control a suitable small parameter within the nonlinear terms.

The class of Liouville mappings we consider are nearly integrable systems having a single action variable (z in $\mathbb{R}$), and two angular variables (x and y, calculated modulo 1). In the integrable limit the motion takes place on surfaces given by $z$ constant. At each iteration the angular variables are rotated by a vector $\omega = (F(z), G(z))$. Here $F$ and $G$ are any smooth non-linear functions, and we shall vary these so as to control the existence and orientation of invariant tubes in the corresponding nearly integrable maps.

By considering a small perturbation of the integrable map with terms of amplitude $\epsilon$ we shall show that the nature of the breakdown of the perturbed planar invariants depends upon the existence of a solution to a certain linear functional equation. This scheme is similar to that in [1] where a different class of maps was considered. As there, the functional equation has no solution when a resonance condition is satisfied. In our simplest case the condition takes the form

$$(1, \pm 1). \omega = \text{integer}.$$ 

Hence, as in [1] we might expect a version of the KAM theorem will hold and that for small $\epsilon$, the slightly deformed invariant surfaces will persist for non resonant action values, $z$. However at values of the action in the resonant case the breakdown

happens at first order in $\epsilon$, and a family of invariant tubes is observed. These have an (axial) orientation along a line of the form $x + y = \text{constant}$ or $x - y = \text{constant}$ according to the sign of the $\pm$ in the resonance condition.

By considering the $\omega$-plane, our resonance condition, given above, defines a simple “diagonal” grid over which we superimpose the “angular increment” curve, $\omega = (F(z), G(z))$, parameterised by the action variable $z$ (for the nearby integrable mapping). Wherever this curve intersects the grid we observe some invariant tubes for $\epsilon$ near zero. Next we may arrange the functions $F$ and $G$ so as to control and perturb the angular increment curve within a neighbourhood of some crossing node of the grid (which corresponds to both types of resonance), we can create the conditions where two (families of) tubes having orthogonal orientations must move, from below and above some value $z^*$, respectively, to lying above and below $z^*$ respectively. In the non-generic case where the curve runs exactly through the grid crossing node (at $z = z^*$) the families of tubes are seen to interfere with one another and the motions become much more chaotic. We shall illustrate this novel phenomenon with some numerical examples.

We are also able to arrange the functions $F$ and $G$ so that two distinct (families of) tubes having parallel orientations must move together and coalesce at some value of $z^*$ (where the angular increment curve is tangent to some line within the grid) before vanishing. In this case we observe the motion in the cross section to resemble the reconnection phenomena seen for two dimensional problems such as the standard map.

2. Asymptotic analysis for small $\epsilon$

Consider the mapping

$$
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  x + F(z + \epsilon H(x, y)) \\
  y + G(z + \epsilon H(x, y)) \\
  z + \epsilon H(x, y)
\end{pmatrix},
$$

where $x$ and $y$ are angular configuration variables, and are computed “mod 1”, and $z$ is an “action” variable in $\mathbb{R}$. Here $F$ and $G$ are smooth scalar mappings, $\epsilon$ is a constant and $H$ is a doubly periodic function given by

$$
H(x, y) = \frac{\sin(2\pi x) \sin(2\pi y)}{2\pi}.
$$

For $\epsilon = 0$ this represents a simple integrable shear mapping (volume preserving) with $x$ and $y$ rotated by $F(z)$ and $G(z)$ respectively.

For $\epsilon \neq 0$ the nearly integrable map (2.1) is still volume preserving since

$$
\det
\begin{pmatrix}
  1 + F'\epsilon H_x & F'\epsilon H_y & F'' \\
  G'\epsilon H_x & 1 + G'\epsilon H_y & G'' \\
  \epsilon H_x & \epsilon H_y & 1
\end{pmatrix}
\equiv 1.
$$

We may plot orbits for this map. For example, if we set $\epsilon = 0.5$, and

$$
F(u) = u^2, \\
G(u) = u
$$

then immediately we see that some orbits move across curved surfaces (close to the shear planes when $\epsilon = 0$) whereas at least one plotted orbit moves around an invariant tube, close to $z = 0.6$. 

Next we discuss the existence of such tubes even in the limit of small $\epsilon$. This is an example of KAM theory for volume preserving maps and such invariant tubes only occur for some specific values for the action $z$ (representing the unperturbed planes as $\epsilon \to 0$) we follow \[1\].

In order to understand how some invariant planes cannot persist as periodically curved surfaces as $\epsilon$ increases from zero we analyse the system in the limit of small $\epsilon$. Here we shall show that invariant planes, for which $z = z^\ast$, which do not persist even to first order in $\epsilon$ must satisfy a non-linear resonance equation in the form $F(z^\ast) \pm G(z^\ast) = \text{integer}$. Hence we can give a complete list of such values and validate an asymptotic approach with our numerical solutions.

We have the following equations for our map:

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} F(z + \epsilon H) \\ G(z + \epsilon H) \\ \epsilon H \end{bmatrix}.$$  \hspace{1cm} (2.3)

If $\epsilon = 0$ then $z$ is a constant, $z = z^\ast$ which is an invariant plane and

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \omega(z^\ast), \text{ where } \omega(z^\ast) = \begin{bmatrix} F(z^\ast) \\ G(z^\ast) \end{bmatrix} \equiv \begin{bmatrix} f^\ast \\ g^\ast \end{bmatrix}.$$  \hspace{1cm} (2.3)

Next we seek a regular asymptotic expansion for this invariant surface valid for small $\epsilon$. The idea is to perturb away from the invariant plane.

For small $\epsilon$ we write

$$z = z^\ast + \epsilon A(x,y) + O(\epsilon^2),$$  \hspace{1cm} (2.4)
where \( A \) is some smooth uniformly bounded doubly periodic function to be determined. Here implicitly we assume that this perturbed surface exists and is an invariant for the mapping. We are interested in those cases where this assumption is not valid - that is, there \( A \) cannot be determined as required. Then we have:

\[
\begin{align*}
\left( \frac{\dot{x}}{\dot{y}} \right) &= \left( \frac{x}{y} \right) + \omega(z^*) + O(\epsilon) \\
\dot{z} &= z + \epsilon H(x, y) + O(\epsilon^2) \\
\Rightarrow \dot{z} &= z^* + \epsilon A(x, y) + H(x, y) + O(\epsilon^2).
\end{align*}
\]

But we must also have

\[
\dot{z} = z^* + \epsilon A(\hat{x}, \hat{y}) + O(\epsilon^2) \quad \text{since (2.4) holds at both points}
\]

\[
= z^* + \epsilon(A(x + f^*, y + g^*) + O(\epsilon^2).
\]

Therefore to \( O(\epsilon) \) we have

\[
A(x + f^*, y + g^*) = A(x, y) + H(x, y). \tag{2.5}
\]

This last equation has the solution for \( f^* + g^*, f^* - g^* \notin \mathbb{Z} \) (Brian Sleeman, personal communication):

\[
A(x, y) = \frac{1}{16\pi^2} \left( \frac{\sin(2\pi(x + y - \frac{(f^* + g^*)}{2}))}{\sin\pi(f^* + g^*)} - \frac{\sin(2\pi(x - y - \frac{(f^* - g^*)}{2}))}{\sin\pi(f^* - g^*)} \right) \tag{2.6}
\]

which is dependent on \( f^* \) and \( g^* \) (which in turn depend on \( z^* \)). Note here we have used the specific form assumed for \( H \). We shall generalise it at the end of this section.

This solution is singular when either \( f^* + g^* \) or \( f^* - g^* \) is an integer. We refer to such cases as “resonant”.

Of course the asymptotic representation that is employed in (2.4) can (and will) also break down at higher orders of \( \epsilon \) (at other values of \( z^* \)). This is straightforward to see since, if we assume \( z^* \) is nonresonant, and then seek to solve for the next term \( O(\epsilon^2) \) in the series (2.4), we obtain an equations similar to (2.5) with the \( H \) replaced by a term of the form

\[
\nabla A. \frac{d\omega}{dz}(z^*). H.
\]

Clearly this introduces higher order resonances, where this second term of the series is unbounded. Similarly at all higher orders. Hence the singularity condition, \( F(z^*) \pm G(z^*) = \text{integer} \), is sufficient though not necessary for the nonpersistence of the near planar invariants. However it clearly explains the tubes (and their orientations) that we observe numerically for small \( \epsilon \).

For the numerical example considered above, \( f^* = F(z^*) = z^*^2 \) and \( g^* = G(z^*) = z^* \), singular values of \( z^* \) are solutions of the equation \( z^*^2 \pm z^* = \text{integer} \).

Immediately we see that for \( z^*^2 + z^* = 1 \) we have \( z^* \) is equal to 0.618 as a solution which corresponds to the tube seen above in Figure 1.

Another obvious value is 1.618 which solves \( z^*^2 - z^* = 1 \).

The full list of values of \( z^* \) between 0 and 2 that give a singular solution are given in Table 1. Tubes appear at all of these values as depicted in Figure 2 below. Of these, all except 1.61803 occur when \( f^* + g^* \) is an integer. The tubes at \( z^* \approx 1.61803 \)
occurs when \( f^* - g^* = 1 \). This is the reason that the tube at 1.61803 is orthogonal to the others. When \( f^* - g^* \) is an integer, \( f^* \) and \( g^* \) are the same mod 1 so, for \( \epsilon \) small, with each iteration of the map, \( x \) and \( y \) increase by the same amount. Projecting the orbit onto the \((x, y)\) plane shows that the tube is parallel to the line \( x = y \). For the other tubes, the fact that \( f^* + g^* \) is an integer means that the combined increase in \( x \) and \( y \) is 1 so the tubes are parallel to the line \( x + y = 1 \) which is orthogonal to the tube at \( z^* = 1.618 \).

Below are the values of \( z^* \) that give singular solutions to (2.6).

Our example contains one nongeneric point at \((f^*, g^*) = (1, 1)\). Let us examine the resonance condition more carefully. In the \( \omega = (f^*, g^*) \) plane we see that \( \omega = (f^*, g^*) = (F(z^*), G(z^*)) \) describe a locus of points parameterised by \( z^* \). Where this intersects the grid \((1, \pm1)\), \( \omega = \) integer we have the formation of invariant tubes. Let us call tubes where \( f^* + g^* = \) integer “negative” tubes and those tubes
where \( f^* - g^* = \text{integer “positive” tubes.} \) To indicate the correspondence of \( f^* \) and \( g^* \mod 1 \) (positive meaning the same).

Then for an \( F \) and \( G \) we may simply locate the corresponding tubes in the \( \omega \)-plane as in the diagram shown in Figure 3.

![Figure 3. Existence of positive and negative invariant tubes](image)

If \( H \), the biperiodic term, is more general than that above then the number of possible resonances becomes far greater. For example if

\[
H = \sum_{p,q=1}^{\infty} h_{pq} \sin 2\pi px \sin 2\pi qy
\]

then the resonance condition becomes

\[
(p, \pm q).\omega(z) = \text{integer}
\]

for all \( p \) and \( q \) such that \( h_{pq} \) is nonzero. Thus the grid may be much more dense.

3. Interaction between invariant tubes

Immediately two phenomena are of interest

a) Where a single point corresponds to both a negative and a positive tube, see Figure 4 (as in our earlier example at (1,1)). Here we introduce a small parameter \( \delta \) into the definition of \( F \) and \( G \) so to move through this point. The situation must evolve from \( \delta < 0 \), where a (family of) positive tubes occurs at a slightly higher value of \( z \) than a (family of) negative tubes; to \( \delta > 0 \), where the (family of) positive tube

<table>
<thead>
<tr>
<th>( z^* )</th>
<th>( f^* )</th>
<th>( g^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.61803</td>
<td>0.38197</td>
<td>0.61803</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>1.30278</td>
<td>1.69723</td>
<td>1.30278</td>
</tr>
<tr>
<td>1.56155</td>
<td>2.43845</td>
<td>1.56155</td>
</tr>
<tr>
<td>1.61803</td>
<td>2.61803</td>
<td>1.61803</td>
</tr>
<tr>
<td>1.79123</td>
<td>3.20871</td>
<td>1.79129</td>
</tr>
</tbody>
</table>
tubes occurs at a slightly lower value of $z$ than the (family of) negative tubes. An example of this is shown in Figure 5 below. Here

$$F(u) = u^2 + \delta \quad \text{and} \quad G(u) = u.$$ 

As $\delta$ varies from 0.1 to -0.1 the two tubes get closer to each other and interfere with each other when $\delta$ nears 0.
b) Where two tubes of the same polarity annihilate each other as $F$ and or $G$ are perturbed through some small parameter $\delta$, see Figure 6. An example of this can be seen in Figure 7. Here

$$F(u) = \rho_1 \cos u \quad \text{and} \quad G(u) = \rho_2 \sin u$$

where $\rho_2 = 1$ and $\rho_1 = \sqrt{3} + \delta$. $\delta$ starts at 0.01 and as it gets closer to 0 the two negative families of tubes get closer and closer and eventually annihilate one another.

References


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Figure 7. As $\delta$ moves through zero the two families (the green and blue tubes are in the same family and the purple and red tubes are in the other family) of negative tubes annihilate one another. In these graphs $\delta = 10^{-2.5}, 10^{-3}, 10^{-3.5}, 10^{-6}$. 