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Instability of MHD-modified interfacial gravity waves revisited

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Abstract

We reveal the basic mechanism of instability of the two-layer conductive fluid system carrying a normal current and exposed to a uniform external magnetic field. This process is a reflection of a MHD-modified interfacial gravity wave from the boundary. Due to special boundary conditions, the reflection coefficient turns out to be greater than 1 for some directions of the wave propagation. We consider two cases: reflection of a monochromatic plane wave from the plane boundary and reflection of rotating waves in a circular geometry. We believe that the proposed mechanism gives a new understanding of the instability formation in the system 'liquid metal–electrolyte' type. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The problem of instability of MHD-modified interfacial gravity waves is one of great interest from both theoretical and practical points of view. Although it has been the subject of numerous studies (see [1–4] and references therein), the basic physical mechanism underlying this instability has not been clearly understood yet. Understanding of this mechanism, however, is vital for developing practical methods of suppressing the aforementioned instability as it is the major factor limiting operation of the reduction cells in the industrial production of aluminium.

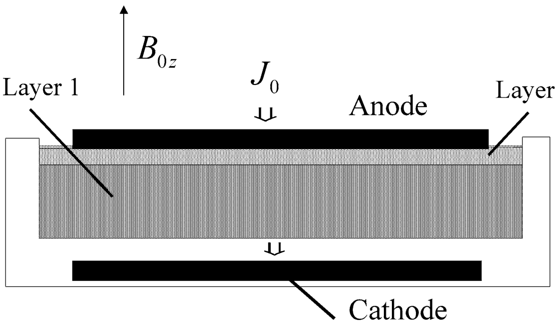

Fig. 1. Two-layer system.

A schematic model of the fluid system considered in this Letter is shown in Fig. 1. Two electrically conducting fluids with substantially different conductivities (liquid metal and electrolyte) are placed in a nonconductive shallow bath (or cell). A large vertical current ($\sim 10^5$ A in the aluminium production) flows down-

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wards from the anode through the fluids and is collected by the cathode on the bottom.

The current is supplied by the external circuit, which has a very complicated geometry. The currents in the circuit induce magnetic field that affects the flow in the bath. It is known that the MHD effects can destabilize gravity waves on the liquid metal–electrolyte interface thus resulting to short circuit and destroying the setup. The instability turned out to develop if the current exceeds some critical value. This effect reduces substantially the efficiency of the cell operation since the main part of the electrical energy is consumed within the poorly conductive electrolyte layer. Moreover, the thickness of the layer cannot be reduced in the case of potentially unstable interface.

In previous investigations, researchers concentrated either on the study of travelling interfacial waves propagating in nonuniform external magnetic field surrounding the cell or on numerical modelling of the global modes for a finite cell geometry. It has been found that the nonuniformity of the external magnetic field can be the cause of instability in travelling waves [2]. In the case of finite geometry instability can develop even in the uniform magnetic field. The stability analysis has been performed numerically in [1,3] using basis of ordinary gravity modes as eigenfunctions. However, pure gravitational modes generally speaking do not satisfy the special (modified by Lorentz force) boundary conditions at the sidewalls of the cell. Nevertheless, the correct boundary conditions can be put into the 'weak' integral formulation of the problem [1]. Although such a formulation is acceptable from the numerical point of view, it obscures the cause of instability.

The crucial role of the boundary conditions in developing the instability thus has been overshadowed. One needs to note, though, that in general there was a clear indication that the boundary inspired the instability [4], since in uniform magnetic field the travelling MHD-modified gravity waves themselves are stable. In the current investigation we will look specifically at the processes taking place in the vicinity of the boundary and describe an underlying mechanism of instability in aluminium reduction cells.

Our analysis will be based upon a system of linear equations describing interfacial waves in the shallow water approximation. The equations along with the boundary conditions are obtained from the standard 3D MHD equations by means of expansions in small dimensionless parameters naturally determined by specific physical parameters of the problem. The resulting system of equations describing MHD-modified linear gravity waves has a quite classical form (coupled wave and Poisson equations). The appropriate boundary conditions, however, appear to be nontrivial due to the action of the Lorentz force. This makes the problem of finding the global modes in an arbitrary geometry a challenge for analytical study.

To study basic processes taking place near the boundary we consider first the fundamental problem of reflection of a monochromatic wave from a plane wall. The necessity to satisfy specific boundary conditions results in amplification of the waves with incident angles lying in the interval $(0, \pi/2)$ while the waves with the angles $(-\pi/2, 0)$ are damped. The situation reverses with the change of the direction of the external magnetic field.

A similar result is obtained for a circular geometry, where reflection from the boundary amplifies waves rotating in one direction and attenuates the waves rotating in the opposite direction. Although the circular global modes of the problem (in contrast to modes in rectangular domain) can be easily found in a closed form [3], we believe that even in this case the reflection problem gives an insight into the essential processes governing the onset of the 'rotating instabilities' in the industrial aluminium reduction cells.

2. Basic equations and shallow water approximation

This section summarises procedure of derivation of the shallow water wave equations and the boundary conditions for the two-layer conductive fluid system carrying the normal current and exposed to external magnetic field. Our derivation is essentially based on the results of Bojarevics and Romerio [1] but seems to be more straightforward. We also distinguish major factors inspiring the instability.

We start with the general 3D MHD equations for the two-layer system (see Fig. 1):

$$\rho_i \left[\frac{\partial \mathbf{u}_i}{\partial t} + (\mathbf{u}_i \nabla \mathbf{u}_i) \right] + \nabla (P_i + \rho_i g z) = \mathbf{F}_i, \tag{1}$$

$$\nabla \cdot \mathbf{u}_i = 0, \tag{2}$$

$$\mathbf{F}_i = \mathbf{J}_i \times \mathbf{B}, \tag{3}$$

$$\nabla \cdot \mathbf{J}_i = 0. \tag{4}$$

Here $i = 1, 2$ is the layer number, ρ_i is the density of the ith layer, $\mathbf{u}_i(x, y, z, t)$ is the fluid velocity, $P_i(x, y, z)$ is the hydrodynamic pressure, $\mathbf{J}_i$ is the current density, and $\mathbf{B}$ is magnetic field.

In the equilibrium state,

$$\mathbf{J}_0 = (0, 0, -J_0), \qquad \nabla \times [\mathbf{J}_0 \times \mathbf{B}_0] = 0. \tag{5}$$

The last relationship implies $B_{0z}(x, y)$ to be arbitrary (given by the external circuit).

The boundary conditions for (1), (2) are

$$(\mathbf{u}_i \cdot \mathbf{n})_{\text{bath}} = 0, \tag{6}$$

where $\mathbf{n}$ is the normal unit vector to the bath surface (lateral walls and bottom). As the system 'liquid metal–electrolyte' is our main physical concern we will assume the following ranking of conductivities:

$$\sigma_{\text{side walls}} \ll \sigma_2 \ll \sigma_{\text{bottom}} \ll \sigma_1, \tag{7}$$

which is characteristic for the aluminium reduction cells (see [1]). Then, the boundary conditions for the current are

$$(\mathbf{J}_{1,2} \cdot \mathbf{n})_{\text{side walls}} = 0, \qquad (\mathbf{J}_1 \cdot \mathbf{n})_{\text{bottom}} = -J_0,$$
$$(\mathbf{J}_1 \cdot \mathbf{n} - \mathbf{J}_2 \cdot \mathbf{n})_{\text{interface}} = 0. \tag{8}$$

We will describe the interface deviation from the equilibrium position at $z = 0$ by the equation

$$z = h(x, y). \tag{9}$$

The shallow water approximation is based on perturbation expansions in two small parameters

$$\delta = \frac{\max h}{H_1} \ll 1, \qquad \epsilon = \frac{H_1}{L} \ll 1. \tag{10}$$

Here L is the typical horizontal dimension of the problem (characteristic wavelength); we also assume that H_1 and H_2 are of the same order.

We introduce the standard decompositions

$$\mathbf{u}_i = \delta \mathbf{v}_i(x, y, t) + O\big(\delta^2, \epsilon\delta, \epsilon^2\big),$$
$$h = \delta \eta(x, y, t) + O\big(\delta^2, \epsilon\delta, \epsilon^2\big),$$
$$\mathbf{F}_i = \delta \mathbf{f}_i(x, y, t) + O\big(\delta^2, \epsilon\delta, \epsilon^2\big). \tag{11}$$

As a consequence, projection of the equation of motion (1) onto z-direction gives the expression for the pressure to the leading order in ϵ,

$$P_i = P_0 + \rho_i g(h - z), \tag{12}$$

where P_0 is the interfacial pressure.

Then, substituting (11), (12) into Eqs. (1), (2), and considering the limit as $\delta \to 0$, $\epsilon \to 0$, leads to the forced linear wave equation for the interface deviation:

$$\frac{\partial^2 \eta}{\partial t^2} - c^2 \nabla_{||}^2 \eta = \alpha \nabla_{||} \cdot \{\mathbf{f}_1 - \mathbf{f}_2\},$$
$$\nabla_{||} \equiv \left(\frac{\partial}{\partial x}; \frac{\partial}{\partial y}\right). \tag{13}$$

Here

$$\alpha = \frac{1}{\rho_1/H_1 + \rho_2/H_2},$$
$$c^2 = \Delta\rho g \alpha, \quad \Delta\rho = \rho_1 - \rho_2.$$

The boundary conditions (6) transform into

$$\big\{g \Delta\rho \nabla_{||} h - (\mathbf{f}_1 - \mathbf{f}_2)\big\} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \tag{14}$$

where Γ is the boundary in the x, y-plane.

Now we shift our attention to the electromagnetic part of the problem. The current redistribution due to the wave motion of the interface is governed by Eq. (4) with boundary conditions (8). We shall assume that $\sigma_2/\sigma_1 \ll \epsilon$, which implies that the current $\mathbf{J}_2$ in poorly conducting upper layer is vertical to the leading order in ϵ. Then one gets the problem for $\mathbf{J}_1$ as follows:

$$\nabla \cdot \mathbf{J}_1 = 0, \tag{15}$$

$$J_{1z}(z = h) = -J_0\left(1 + \frac{h}{H_2}\right),$$
$$J_{1z}(z = -H_1) = -J_0, \tag{16}$$

and

$$\text{on } \Gamma: \quad \mathbf{J}_1 \cdot \mathbf{n} = 0. \tag{17}$$

We shall seek $\mathbf{J}_1$ in the form of the decomposition

$$\mathbf{J}_1 = \mathbf{J}_0 + \delta(\mathbf{j}_{||} + \mathbf{j}_z) + O\big(\delta^2\big). \tag{18}$$

In the shallow water approximation, we have to use

$$\mathbf{j}_{||} = \mathbf{j}_{||}(x, y, t) + O(\epsilon),$$
$$\mathbf{j}_z = \mathbf{j}_z(x, y, \bar{z}, \epsilon, t), \quad \bar{z} = z/\epsilon. \tag{19}$$

One can see then that Eq. (15) implies the following decomposition for $\mathbf{j}_z$:

$$j_z = c_1(x, y, t) + \epsilon c_2(x, y, t)\bar{z} + O\big(\epsilon^2\big), \tag{20}$$

where the coefficients $c_{1,2}$ are determined from conditions (16). Then, Eq. (15) eventually takes the form, as $\epsilon \to 0$, $\delta \to 0$,

$$\nabla_{||} \cdot \mathbf{j}_{||} = \frac{J_0 \eta}{H_1 H_2},$$
$$\text{on } \Gamma: \quad \mathbf{j}_{||} \cdot \mathbf{n} = 0. \tag{21}$$

Eq. (21) implies that $|\mathbf{j}_{||}| = O(J_0/\epsilon)$, which has the following important consequences:

(i) The current perturbation in the liquid metal layer is horizontal (with the accuracy $O(\epsilon)$).

(ii) One can neglect the induced motion of the liquid metal ($|\mathbf{v}_1 \times \mathbf{B}| \ll |\mathbf{j}_{||}|$) and, therefore,

$$\mathbf{j}_{||} = -\sigma_1 \nabla_{||} \varphi, \tag{22}$$

where φ is the electric potential.

(iii) The shallow water decomposition for the Lorentz force is

$$\mathbf{f}_1 = \mathbf{j}_{||} \times \mathbf{B}_{0z} + O(1). \tag{23}$$

(iv) The Lorentz force acting on electrolyte is much less than that acting on liquid metal:

$$|\mathbf{f}_2|/|\mathbf{f}_1| = O(\epsilon). \tag{24}$$

It is also clear that above decompositions are valid only if $\delta \ll \epsilon$.

Substituting (22)–(24) into (13), (21) we eventually arrive at the desired system describing MHD-modified interfacial gravity waves in the shallow water approximation,

$$\frac{\partial^2 \eta}{\partial t^2} - c^2 \nabla_{||}^2 \eta = c^2 \nabla_{||} \phi \cdot \left[\nabla_{||} \times b(x, y) \mathbf{e}_z\right], \tag{25}$$
$$\nabla_{||}^2 \phi = -\beta \eta, \tag{26}$$

where

$$\phi = \frac{\sigma_1 B_0}{\Delta \rho g} \varphi, \qquad \beta = \frac{J_0 B_0}{H_1 H_2 \Delta \rho g}, \tag{27}$$

and

$$\mathbf{B}_{0z}(x, y) = B_0 b(x, y) \mathbf{e}_z$$

is a given function. The boundary conditions for (25), (26) take the form (see (14), (22)–(24)):

$$\text{on } \Gamma: \quad \frac{\partial \phi}{\partial n} = 0, \quad \frac{\partial \eta}{\partial n} = -b \frac{\partial \phi}{\partial \tau}. \tag{28}$$

Here $\partial/\partial n$ stands for the normal derivative at the boundary and $\partial/\partial \tau$ denotes the derivative in the direction tangential to the boundary. In conclusion, we note that our basic physical assumption $\sigma_2/\sigma_1 \ll \epsilon \ll 1$ is realistic and corresponds to the actual parameter values of industrial aluminium reduction cells where $\sigma_{\text{electrolyte}}/\sigma_{\text{alum}} \sim (H_{\text{alum}}/L)^2 \sim 10^{-4}$ [1].

Now one can distinguish two factors that could lead to an unbounded growth of the solution to (25)–(28):

(i) spatial nonuniformity of the external magnetic field $B_{0z}(x, y)$ entering the right-hand part of the wave equation (25).

(ii) the second boundary conditions (28), which can act as a source of instability even in the uniform magnetic field.

It is important to emphasize that although both factors act simultaneously, the physical mechanisms inspiring instability in each case are completely different. The first type of instability occurs even in travelling waves [2] in an infinite space and is quite understandable from both physical and mathematical point of view. In the second case, the existence of boundaries becomes a crucial factor. We believe that the second type of instability, although obviously studied before, has not been understood so far. From our point of view this type of instability is much more intriguing as the mentioned special boundary conditions change dramatically the structure of solutions compared to the standard case of gravity waves in a closed domain.

To set separate this particular mechanism acting in a finite geometry we put the magnetic field to be spatially uniform ($b \equiv 1$), which immediately leads to the most simple form of the governing system,

$$\frac{\partial^2 \eta}{\partial t^2} - c^2 \Delta \eta = 0,$$
$$\Delta \phi = -\beta \eta, \tag{29}$$

while the boundary conditions take the form

$$\text{on } \Gamma: \quad \frac{\partial \phi}{\partial n} = 0, \quad \frac{\partial \eta}{\partial n} = -\frac{\partial \phi}{\partial \tau}. \tag{30}$$

Here we have dropped the subscript '||' for the differential operators assuming $\eta = \eta(x, y)$, $\phi = \phi(x, y)$.

We note that the problems for η and ϕ are essentially coupled only at the boundary.

3. Reflection of the plane wave from the wall

Although system (29), (30) might seem to be simple enough it is quite nontrivial from an analytical point of view owing to the coupled boundary conditions for η and ϕ. The main difficulty is that the standard gravity modes (the trigonometric Fourier basis for the rectangular domain) do not satisfy the second boundary condition (30) and cannot be applied (at least directly) to the construction of the solution to our boundary value problem. The 'weak' formulation of the problem in [1], however, allows to include this boundary condition into the integral setting, which is convenient for numerical studies but somewhat overshadows the simple physics underlying the processes in the two-layer system under consideration. The obvious drawback of this formulation from the analytical point of view is that the solution is represented in the form of the infinite series, each term of which does not satisfy the boundary conditions.

On the other hand, the formal analytical solution of the problem with the aid of the implicitly defined Green function given in [4] seems to provide no more information than the system itself and does not reveal the basic mechanism of the instability.

To get a better understanding of the properties of the solution to the boundary value problem (29), (30) we shift our attention from finding the global modes in various geometries to the fundamental problem of the reflection of the plane wave from the wall.

Let a monochromatic wave with the frequency ω be incident on an infinite plain boundary layer located at $x = 0$ (Fig. 2). Far from the boundary, the wave is just a monochromatic plain wave $\eta \sim \exp(ik_x x + ik_y y - i\omega t)$, $\omega^2 = c^2(k_x^2 + k_y^2)$ moving towards the boundary. Now we shall be concerned with what happens after the wave is reflected from the boundary.

Let us find a general solution to (29) using Fourier transform over t and y:

$$\eta = \hat{\eta}(x)\exp(ik_y y - i\omega t),$$
$$\phi = \hat{\phi}(x)\exp(ik_y y - i\omega t). \tag{31}$$

Then, we end up with a system of two ODEs with respect to x,

$$\frac{d^2\hat{\eta}}{dx^2} + \hat{\eta}\big(\omega^2/c^2 - k_y^2\big) = 0,$$
$$\frac{d^2\hat{\phi}}{dx^2} - k_y^2\hat{\phi} = -\beta\hat{\eta}, \tag{32}$$

with the boundary conditions at $x = 0$:

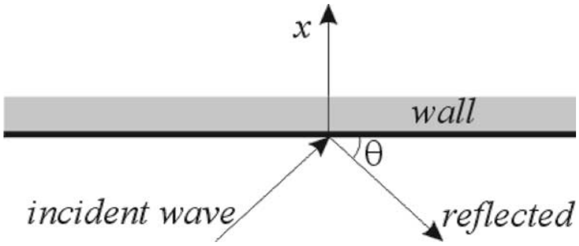

Fig. 2. Plane reflection geometry.

$$\frac{d\hat{\phi}}{dx} = 0, \qquad \frac{d\hat{\eta}}{dx} = -ik_y\hat{\phi}. \tag{33}$$

A general solution to (32), bounded as $x \to -\infty$, is given by the expressions

$$\hat{\eta} = C_1\exp(ik_x x) + C_2\exp(-ik_x x),$$
$$\hat{\phi} = A\exp\big(|k_y|x\big) + \Omega\big\{C_1\exp(ik_x x) + C_2\exp(-ik_x x)\big\}. \tag{34}$$

Here $\Omega = \beta c^2/\omega^2$.

From the first boundary condition (33) it follows that

$$A = -i\Omega\frac{k_x}{k_y}(C_1 - C_2).$$

According to (34) the reflection coefficient is given by $\mu = |C_2|^2/|C_1|^2$. Using the second boundary condition we finally get for μ the expression

$$\mu = \frac{\Omega^2 + (1 + \Omega\tan\theta)^2}{\Omega^2 + (1 - \Omega\tan\theta)^2}, \tag{35}$$

where $\tan\theta = k_y/k_x$, $-\pi/2 < \theta < \pi/2$.

It is seen from (35) that, for positive angles of incidence θ, the reflection coefficient $\mu > 1$, i.e., the wave is inevitably amplified at the boundary. On the other hand, if $\theta < 0$, then $\mu < 1$ and the wave is damped.

The graph of function (35) is shown in Fig. 3 versus the angle of incidence. The antisymmetry in behaviour of the reflection coefficient is due to the existence of the distinguished direction in the problem, the direction of the external magnetic field (sign of β). This asymmetry, as we will show below, has its consequence in appearance of 'rotating' instabilities in closed domains.

It is interesting to note that the reflection coefficient first grows up with Ω, if $\Omega < 1$. Further, when $\Omega > 1$, the reflection coefficient is slowing down (see Fig. 4).

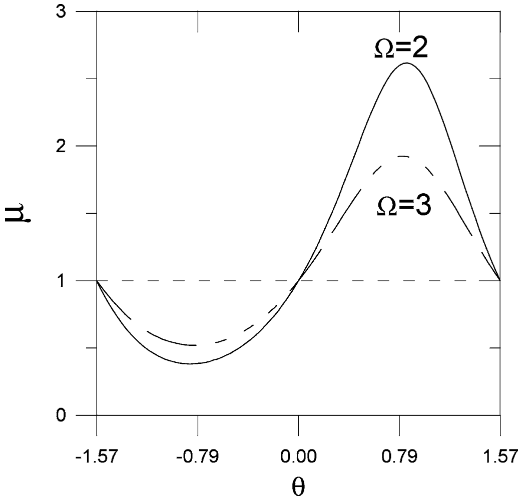

Fig. 3. Reflection coefficient as a function of angle of incidence.

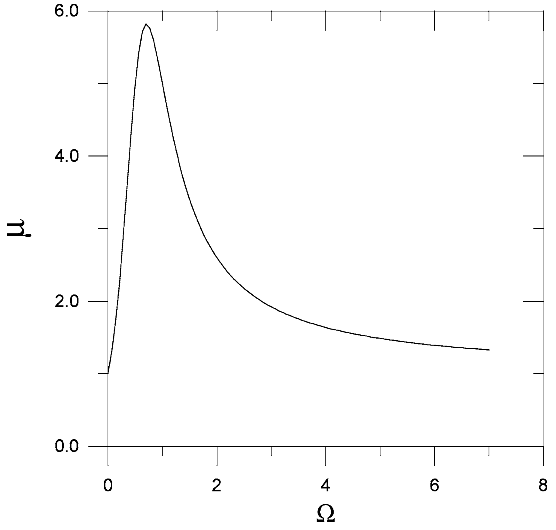

Fig. 4. Reflection coefficient as a function of the parameter $\Omega = \beta c^2/\omega^2$.

To illustrate analytically how the reflection mechanism works in the case of finite geometry we consider the simplest analytic solution of problem (32), (33) for circular domains.

4. Reflection mechanism in circular domains

Problem (29), (30) for a circular domain, in contrast to the one in a rectangular domain, admits a complete analytical solution by standard separation of variables:

$$\eta = \hat{\eta}(r)\exp(in\vartheta - i\omega t),$$
$$\phi = \hat{\phi}(r)\exp(in\vartheta - i\omega t). \tag{36}$$

The functions $\hat{\eta}, \hat{\phi}$ are subject to the boundary conditions (30), which in polar coordinates take the form

$$\hat{\phi}'(R) = 0, \qquad \hat{\eta}'(R) = -\frac{in}{R}\hat{\phi}(R), \tag{37}$$

R being the radius of the domain.

The solution bounded at $r = 0$ and satisfying the first boundary condition (37), has the form

$$\hat{\eta} = CJ_n(kr), \tag{38}$$
$$\hat{\phi} = \frac{\beta}{k^2}\left\{\hat{\eta} - \frac{R}{n}\hat{\eta}'(R)\left(\frac{r}{R}\right)^n\right\}, \tag{39}$$

where $k = \omega/c$, $J_n(x)$ is the Bessel function of the first kind, and C is a constant.

Substitution into the second boundary condition yields the dispersion relation (for $n = 1$ this relation in a slightly different form can be found in [3])

$$i\zeta^2 + \beta R^2 = n\zeta\frac{J_n(\zeta)}{J_{n+1}(\zeta)}, \quad \text{where } \zeta = kR. \tag{40}$$

The dispersion relation (40) implies complex roots $\zeta = \xi + i\gamma$ and, therefore, complex frequencies ω for all modes. The graph of functions $\gamma(\beta R^2)$ and $\xi(\beta R^2)$ for $n = 1$ is presented in Fig. 5. One can see that the dispersion relation (40) predicts an absolute instability for waves rotating in one direction and damping for waves rotating in the opposite direction. This 'antisymmetric' property of the amplification clearly resembles the principal feature of the plane reflection considered in the previous section.

To extract effects of reflection governing the instability we consider solution (38)–(40) for $\beta R^2 \gg 1$ (which can be obviously achieved by either increasing the radius of the domain or by increasing the value of the magnetic field). This implies $|\zeta|^2 \sim \beta R^2$ owing to the dispersion relation (40).

The Bessel function has the following asymptotics at large values of its argument [5]:

$$J_n(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi n}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{z^{3/2}}\right),$$
$$|z| \gg 1, \quad |z| \gg n, \quad |\arg z| < \pi. \tag{41}$$

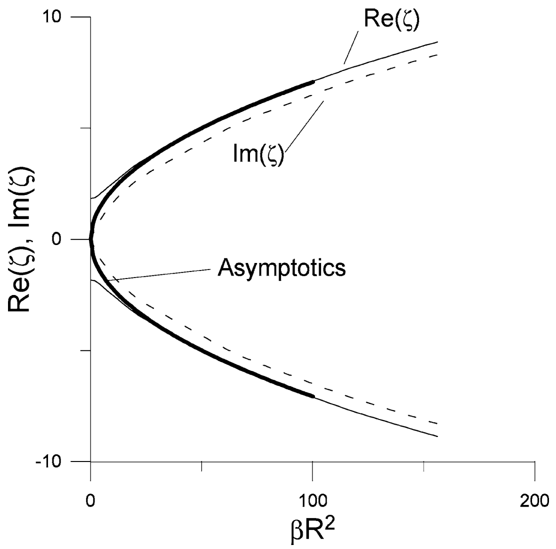

Fig. 5. Roots of the dispersion relation (39) for $n = 1$ as functions of parameter βR^2 in a circular domain (bold line: asymptotics as $\beta R^2 \gg 1$).

Then, in the neighbourhood of the boundary ($R - r/R \ll 1$) one can represent solution (38) in the form

$$\hat{\eta} = C\left\{C_1(r)\frac{\exp\{i(\xi r/R - n\pi/2 - \pi/4)\}}{(\xi r/R)^{1/2}} + C_2(r)\frac{\exp\{-i(\xi r/R - n\pi/2 - \pi/4)\}}{(\xi r/R)^{1/2}}\right\} + \cdots, \tag{42}$$

where

$$C_1(r) = \exp\left(-\gamma\frac{r}{R}\right), \qquad C_2(r) = \exp\left(\gamma\frac{r}{R}\right). \tag{43}$$

Solution (42) represents the ‘circular’ analog of the plane wave solution (34).

Asymptotics (42) implies the direct relation of the reflection coefficient with the normalized instability increment γ:

$$\mu = \frac{|C_2(R)|^2}{|C_1(R)|^2} = \exp(4\gamma). \tag{44}$$

The dependence of γ on βR^2 is readily found from the decomposition of the dispersion relation (40) for $|\zeta| \gg 1$. To the leading order in ζ one gets

$$i\zeta^2 = -\beta R^2 + O(\zeta), \tag{45}$$

which immediately yields

$$\xi \approx \gamma \approx \pm\sqrt{\frac{|\beta|R^2}{2}}. \tag{46}$$

From Fig. 5 it is seen that asymptotics (46) is in a good agreement with the exact dispersion curve for $n = 1$ even for comparatively moderate values of βR^2. Another remarkable feature of this asymptotics is also the fact that it depends neither on the number of the mode nor on the number of the root for the chosen mode (both numbers are supposed to be finite). Asymptotics (46) has also its consequence in the finite limiting value of the actual (physical) instability increment $\mathrm{Im}\,\omega = c\gamma/R$ as $R \to \infty$ provided $\beta = O(1)$.

One should note that although Eq. (40) has an infinite number of roots for each n given, for practical needs one can be interested only in the first ones as they provide the greatest instability increment at moderate values of βR^2. The same statement can be made about the mode numbers n themselves. We will show, however, that consideration of the modes with large n leads to establishing a direct correspondence with the case of plane reflection considered in the previous section and therefore is interesting from the theoretical point of view.

To establish this correspondence, we consider asymptotics of solution (38) at $\beta R^2 \gg 1$ under the additional restriction $n \sim \sqrt{\beta R^2}$. It can be shown that in this case $\gamma \ll \xi \sim n$. For large $n \sim z$, a more general asymptotics of the Bessel function should be used (see, for example, [5]):

$$J_n(z) \approx \sqrt{\frac{2}{\pi z}}A(z,n)\sin(z - n\pi/2 - \pi/4) + \cdots, \tag{47}$$

where $A(z,n)$ is a certain function. Taking this into account, substitution of solution (38) into the second boundary condition (37) provided $|\zeta| \approx \xi$ gives a simple expression for the reflection coefficient,

$$\mu = \exp 4\gamma = \frac{\Omega^2 + (1 + \Omega n/\xi)^2}{\Omega^2 + (1 - \Omega n/\xi)^2}, \tag{48}$$

where $\Omega = \beta R^2/\xi^2$. One can see that relationship (48) coincides with the one for the plane reflection case (35) provided the angle of incidence is given by $\tan\theta = n/\xi$. It should be noted, however, that while in the plane case the angle θ is arbitrary, it is a function of βR^2 due to the dispersion relation (40) in the circular case. Also, $\gamma = O(1)$ now, which means that the physical instability increment for the higher modes vanishes as $R \to \infty$.

Thus, the above consideration confirms that the reflection mechanism indeed governs the formation of instabilities of the MHD-modified interfacial gravity waves in closed domains in a uniform magnetic field.

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References

[1] V. Bojarevics, M.V. Romerio, Eur. J. Mech. 13 (1994) 33.
[2] A.D. Sneyd, A. Wang, J. Fluid Mech. 263 (1994) 343.
[3] P.A. Davidson, R.I. Lindsay, J. Fluid Mech. 362 (1998) 273.
[4] Ch. Droste, M. Segatz, D. Vogelsang, Light Metals (1998) 419.
[5] E.T. Whittacker, G.N. Watson, A Course in Modern Analysis, Cambridge University Press, 1927.