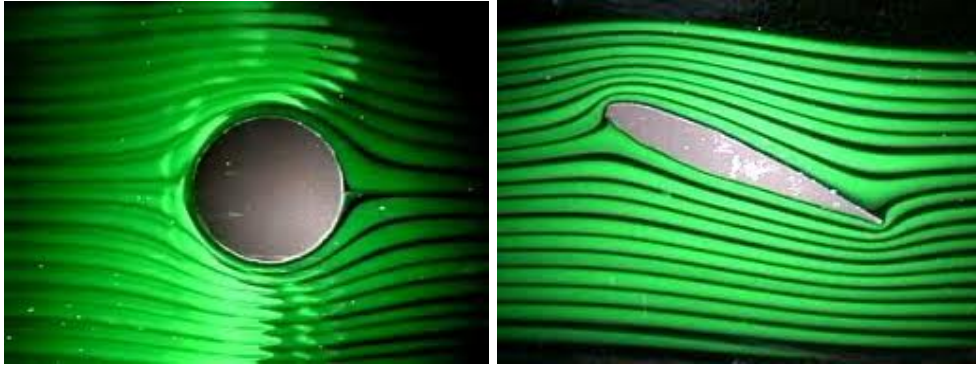


MA3FM/MA4FM

A Mathematical Introduction to Fluid
Mechanics and Its Environmental Applications

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These Lecture Notes are taken from the course given by Dr. A.Lukyanov in the Autumn term of 2016, at the Department of Mathematics, the University of Reading.



Potential flows past a cylinder and a wing.

Introduction

Fluid Mechanics is an integral part of a more general applied mathematical discipline - Continuum Mechanics, which deals with media motion modelled as continuum. In this course, we will first briefly consider basic notions and principles of Continuum Mechanics, such as tensors and operations with them, deformations, rate of deformations and stress tensors, conservation laws, which are common for both Fluid Mechanics and Theory of Elasticity. Then we will concentrate on the fluid mechanical description and consider in detail the main formulation of Fluid Mechanics - the Navier-Stokes equations.

The system of Navier-Stokes equations has been the basis for theoretical consideration of fluid motion during two centuries and arguably is the most intricate system of partial differential equations so far. Not surprisingly, the Navier - Stokes existence and uniqueness problem is one of the seven "**The Millennium Prize Problems**". Its first solution has been claimed recently by Prof. Otelbayev, but it appeared that the proof had **a flaw**. In the course, we will analyze both model "academic" inviscid potential flows and more realistic examples featured in applications such as river flows and origin of the aircraft lift. We will consider important applications of Fluid Mechanics to environmental flows, such as flows in porous media, concluding with the shallow water model of tsunami waves.

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