Finite element methods for acoustic scattering

Problem sheet

1. (a) Verify that the function

$$u(x) = \frac{\mathrm{e}^{\mathrm{i}kx}}{k} \int_0^x \sin(ks) f(s)\,\mathrm{d}s + \frac{\sin(kx)}{k} \int_x^1 \mathrm{e}^{\mathrm{i}ks} f(s)\,\mathrm{d}s,$$

satisfies each of

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - k^2 u = f, \quad \text{in } \Omega := (0,1), \tag{1}$$

$$u(0) = 0, \tag{2}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}(1) - \mathrm{i}ku(1) = 0. \tag{3}$$

(b) In the particular case that $f \equiv 1$, show that the solution to (1)–(3) is given by

$$u(x) = \frac{1}{k^2}\left[\mathrm{e}^{\mathrm{i}kx} - 1 - \mathrm{i}\mathrm{e}^{\mathrm{i}k}\sin(kx)\right].$$

2. With the hat functions $\chi_j(x)$ defined for $j = 1, \ldots, N-1$ by

$$\chi_j(x) = \begin{cases} \frac{1}{h}(x - x_{j-1}), & x \in [x_{j-1}, x_j], \\ \frac{1}{h}(x_{j+1} - x), & x \in [x_j, x_{j+1}], \\ 0 & \text{elsewhere,} \end{cases}$$

and for $j = N$ by

$$\chi_N(x) = \begin{cases} \frac{1}{h}(x - x_{N-1}), & x \in [x_{N-1}, 1], \\ 0 & \text{elsewhere,} \end{cases}$$

where $x_j = jh$, $j = 0, \ldots, N$, with $h = 1/N$, show that the formulae

(a)

$$\int_0^1 \chi_j'(x)\chi_m'(x)\,\mathrm{d}x = \begin{cases} 0, & \text{if } |j-m| > 1, \\ -1/h & \text{if } |j-m| = 1, \\ 2/h & \text{if } j = m \neq N, \\ 1/h & \text{if } j = m = N, \end{cases}$$

(b)

$$\int_0^1 \chi_j(x)\chi_m(x)\,\mathrm{d}x = \begin{cases} 0, & \text{if } |j-m|>1, \\ h/6 & \text{if } |j-m|=1, \\ 2h/3 & \text{if } j=m\neq N, \\ h/3 & \text{if } j=m=N, \end{cases}$$

are correct.

3. Consider the one dimensional wave equation

$$\frac{\partial^2 P}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = 0. \tag{4}$$

(a) By seeking a separable solution of the form

$$P(x,t) = u(x)\mathrm{e}^{-\mathrm{i}\omega t},$$

show that the general solution of (4) is given by

$$P(x,t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} + B\mathrm{e}^{-\mathrm{i}(kx+\omega t)}, \tag{5}$$

where A and B are constants.

(b) Show that the value of the function $P_1 := \mathrm{e}^{\mathrm{i}(kx-\omega t)}$ does not change if

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega}{k},$$

and that the value of the function $P_2 := \mathrm{e}^{-\mathrm{i}(kx+\omega t)}$ does not change if

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-\omega}{k}.$$

(c) Deduce which of the terms on the right hand side of (5) is the *outgoing wave*, travelling from left to right, and which is the *incoming wave*, travelling from right to left. Hence show that the nonreflecting boundary condition

$$\frac{\omega}{k}\frac{\partial P}{\partial x}(x_0) + \frac{\partial P}{\partial t}(x_0) = 0$$

eliminates the incoming wave.

(d) Determine a boundary condition that would eliminate the outgoing wave.

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Matlab problem sheet

The matlab code `wave1d.m` produces a numerical solution to the problem (1)–(3) in the particular case that $f \equiv 1$. To run `wave1d.m` in matlab, at the command prompt type

```
>> wave1d
```

and you will then be prompted to enter a value of k, and a value of N, where N is the number of elements used in the Galerkin finite element scheme.

If you know how to program in Matlab, you can look at the code to see how it works, and try editing it to see how that affects the results. However, if you don't know Matlab at all, it is sufficient just to run the code as described above.

WARNING - running the code for a very large value of k or N may take a long time. It is probably best to start with small values, and make them larger gradually, so that you can get a feel for how long the code takes to run.

1. For $k = 1$, try running `wave1d.m` with $N = 1, 2, 4, 8, 16, \ldots$. Look at the plots of the true solution u and the approximate solution U (real parts top left, imaginary parts top right). Is it clear how the finite element approximation U behaves as N increases? What can you say about the relative errors as N increases? When N is large enough, do they satisfy the asymptotic error bound

$$\frac{|u - U|}{|u|} \leq Ch^2, \tag{6}$$

   where $h = 1/N$?

2. Repeating for $k = 10$ and $k = 100$, how do the results compare? What happens to the exact solution as k increases? And as N increases with k fixed, does the relative error still behave like (6)?

3. Now try increasing N and k together. As a first step, fix k to be some (small) value, and increase N until the relative error is below a certain threshold (say $0.01 = 1\%$ relative error). Taking this as your starting point (you may find it useful to draw up a table of the form below);

(a) Increase k and N in such a way that $hk = k/N$ remains fixed (i.e. if you double k then you should also double N). What can you say about the relative errors? Is this a good thing?

(b) Now increase k and N in such a way that $hk^2 = k^2/N$ remains fixed (i.e. if you double k then you should quadruple N). What can you say about the relative errors now? And what can you say about the computing time for the method? Is this a good thing?

4. In practice, a common engineering goal is to keep the error constant as k increases. Can you find a value of α such that choosing $N = k^\alpha$ achieves this?

5. Describe how your findings fit with the error estimate

$$\frac{\|(u-U)'\|}{\|u'\|} \le C_1 hk + C_2 k^3 h^2.$$

k	N	$\max(\lvert u-U\rvert/\lvert u\rvert)$

Table 1: Relative errors, various k and N