Boundary element methods for acoustics

Problem Sheet Part One

1. Solve the integral equation

$$y(s) = 1 + \int_0^1 s^2 t^2 y(t) dt$$

by the method which was used in Section 2.1 of the notes to obtain the solution (2.10) to the integral equation (2.5). Check that your solution is correct by substituting back in the equation. (The correct solution is given at the end of the problem sheet.)

2. Let $\nabla_{\mathbf{x}}$ be the usual gradient operator with respect to the components x, y, and z of $\mathbf{x} = (x, y, z)$, i.e.

$$\nabla_{\mathbf{x}} := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

Let $\mathbf{x}_0 = (x_0, y_0, z_0)$ be a fixed position vector, and let R denote the distance between $\mathbf{x}$ and $\mathbf{x}_0$, i.e.

$$R := |\mathbf{x} - \mathbf{x}_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}.$$

Show that

$$\nabla_{\mathbf{x}} R = \left(\frac{\partial R}{\partial x}, \frac{\partial R}{\partial y}, \frac{\partial R}{\partial z} \right) = \frac{\mathbf{x} - \mathbf{x}_0}{R}.$$

Hence, where G is the 3D fundamental solution as defined in Section 2.2 of the notes, show that

$$\nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi} \frac{d}{dR} \left(\frac{\mathrm{e}^{\mathrm{i}kR}}{R} \right) \nabla_{\mathbf{x}} R = -\frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}kR}(\mathrm{i}kR - 1)}{R^3} (\mathbf{x} - \mathbf{x}_0). \quad (1)$$

Note that this calculation confirms that formula (2.21) in the notes is correct. Deduce from formula (2.21) and the basic definition of the normal derivative (1.6) that

$$\frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} = -\frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}kR}(\mathrm{i}kR - 1)}{R^3} (\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}(\mathbf{y}),$$

where now R denotes $R := |\mathbf{x} - \mathbf{y}|$.

Using the fact that the derivative of the Hankel function $H_0^{(1)}$ is the Hankel function $-H_1^{(1)}$, obtain the corresponding explicit expression in the 2D case, that

$$\nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}_0) = -\frac{\mathrm{i}}{4} \frac{d}{dR} H_0^{(1)}(kR)\, \nabla_{\mathbf{x}} R = \frac{\mathrm{i}k}{4} \frac{H_1^{(1)}(kR)}{R} (\mathbf{x} - \mathbf{x}_0), \qquad (2)$$

justifying equation (2.39).

3. Equation (2.29) in the notes has been derived on the assumption that $\mathbf{x} \in D$. The corresponding equation (2.30) holds when $\mathbf{x} \in \partial D$. The **boundary integral equation** method is to first solve (2.30) to determine u on the boundary after which (2.29) is an explicit formula for calculating u throughout D.

In the case that $\mathbf{x}$ is **outside** D, the corresponding formula that holds is that

$$G(\mathbf{x}_0, \mathbf{x}) = \int_{\partial D} \left[ik\beta(\mathbf{y}) G(\mathbf{y}, \mathbf{x}) + \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \right] u(\mathbf{y})\, ds(\mathbf{y}). \qquad (3)$$

Write out a justification of (3), copying and modifying the justification for equation (2.29) in the notes.

Equation (3) is very useful for checking the validity of BEM codes; if the integral on the right hand side (approximated numerically) is not close to $G(\mathbf{x}_0, \mathbf{x})$ when the boundary elements are small, then there must be a problem with the boundary element scheme or its implementation.

4. An attraction of the boundary integral equation method is the explicit analytical formulae for the acoustic field in the domain that it gives (e.g. (2.29) or (2.32) in the notes). One attraction of this is that it enables one to construct further formulae for derivatives of the solution. In particular, one can write down a formula for ∇u, the gradient of the pressure u. This is of considerable interest as the particle velocity is proportional to the gradient of the pressure, precisely

$$\mathbf{v} = -\frac{\mathrm{i}}{\omega\rho} \nabla u.$$

Moreover, once the velocity and pressure are known then the **acoustic intensity vector**, $\mathbf{I}_{av}$, averaged over one period of the time harmonic wave, can be calculated by the formula

$$\mathbf{I}_{av} = \frac{\Im(\bar{u}\nabla u)}{2k\rho c},$$

where $\bar{u}$ denotes the complex conjugate of u, $\Im$ denotes the imaginary part, and ρ and c are air density and sound speed, respectively.

In the 2D case, when G is given in terms of the Hankel function as in Section 2.1 of the notes, differentiate equation (2.32), i.e. the equation

$$u(\mathbf{x}) = G(\mathbf{x}_0, \mathbf{x}) + \int_{\partial D} G(\mathbf{y}, \mathbf{x}) \frac{\partial u}{\partial n}(\mathbf{y})\, ds(\mathbf{y}), \quad \mathbf{x} \in D, \tag{4}$$

to obtain the explicit expression for ∇u that

$$\nabla u(\mathbf{x}) = \frac{\mathrm{i}k}{4} \frac{H_1^{(1)}(k|\mathbf{x}-\mathbf{x}_0|)}{|\mathbf{x}-\mathbf{x}_0|} (\mathbf{x}-\mathbf{x}_0) + \frac{\mathrm{i}k}{4} \int_{\partial D} \frac{H_1^{(1)}(k|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|} (\mathbf{x}-\mathbf{y}) \frac{\partial u}{\partial n}(\mathbf{y})\, ds(\mathbf{y}), \tag{5}$$

for $\mathbf{x} \in D$. (To get this expression you will need to assume that it is fine to take the derivative under the integral sign (which it is in this instance) and to use the results from question 2.)

5. In this question we will apply the simple boundary element scheme of Chapter 3 to the integral equation (2.33) and the representation formula (2.32).

First, split ∂D into N boundary elements γ_j, $j = 1, ..., N$, approximate $\partial u/\partial n$ by the constant v_j on the jth element, and deduce from (2.32) that

$$u(\mathbf{x}) \approx G(\mathbf{x}_0, \mathbf{x}) + \sum_{j=1}^{N} v_j \int_{\gamma_j} G(\mathbf{y}, \mathbf{x})\, ds(\mathbf{y}), \quad \mathbf{x} \in D. \tag{6}$$

and from (2.33) that

$$0 \approx G(\mathbf{x}_0, \mathbf{x}) + \sum_{j=1}^{N} v_j \int_{\gamma_j} G(\mathbf{y}, \mathbf{x})\, ds(\mathbf{y}), \quad \mathbf{x} \in \partial D. \tag{7}$$

Let $\mathbf{v}$ denote the vector of unknowns, precisely let $\mathbf{v}$ be the length N column vector with jth entry v_j. As in Chapter 3, let $\mathbf{x}_i$ be some point in the ith element (for instance its centroid). Show that choosing the constants

v_j, $j = 1, ..., N$, so that equation (10) is satisfied exactly at the points $\mathbf{x}_i$, $i = 1, ..., N$, implies that $\mathbf{v}$ satisfies the linear system, in matrix form,

$$A\mathbf{v} = -\mathbf{b}, \tag{8}$$

where $\mathbf{b}$ denotes the column vector whose ith entry is $G(\mathbf{x}_0, \mathbf{x}_i)$, and the $N \times N$ matrix A has ijth entry

$$a_{ij} := \int_{\gamma_j} G(\mathbf{y}, \mathbf{x}_i)\, ds(\mathbf{y}). \tag{9}$$

A crude, but not completely useless, approximation for the integrals in (6) and (9), is to make the approximation that

$$G(\mathbf{y}, \mathbf{x}) \approx G(\mathbf{x}_j, \mathbf{x})$$

for $\mathbf{y} \in \gamma_j$, so that

$$\int_{\gamma_j} G(\mathbf{y}, \mathbf{x})\, ds(\mathbf{y}) \approx G(\mathbf{x}_j, \mathbf{x}) \int_{\gamma_j} ds(\mathbf{y}) = G(\mathbf{x}_j, \mathbf{x})\, A_j,$$

where

$$A_j := \int_{\gamma_j} ds(\mathbf{y})$$

is the area of the element γ_j in 3D, the arc-length of γ_j in 2D. Making this approximation in (6) gives the completely explicit formula for $u(\mathbf{x})$ that

$$u(\mathbf{x}) \approx G(\mathbf{x}_0, \mathbf{x}) + \sum_{j=1}^{N} v_j A_j G(\mathbf{x}_j, \mathbf{x}), \quad \mathbf{x} \in D. \tag{10}$$

The same approximation can be made in (9), except when $i = j$ since $G(\mathbf{x}_j, \mathbf{x}_j)$ is undefined (is infinite). A crude, but simple, explicit, and not completely useless approximation is to take

$$a_{ij} \approx \tilde{a}_{ij} := \begin{cases} A_j G(\mathbf{x}_j, \mathbf{x}_i), & i \neq j, \\ 0, & i = j, \end{cases} \tag{11}$$

and so solve

$$\tilde{A}\mathbf{v} = -\mathbf{b}, \tag{12}$$

instead of (8), where $\tilde{A}$ is the matrix with ijth entry $\tilde{a}_{ij}$.

[Solution to question 1 is $y(s) = 1 + \frac{5}{12}s^2$.]

Boundary element methods for acoustics

Problem Sheet Part Two: Matlab Problems

Start Matlab up and create a directory bem_matlab. Go to the web site

www.reading.ac.uk/~sms03snc/smart_numerics.html

and download to this directory the six files listed. The files are 5 Matlab functions: G.m, bem_solve.m, circ.m, pressure.m, pressure_exact.m, plus one short script file bem_complete.m which calls four of these functions to solve a particular interior problem for the Helmholtz equation. The interior problem solved is the one described at the beginning of section 2.3.2 of the notes. The BEM used is the simplest possible, as described in question 5 of part one of the problem sheet. The various functions implement this as follows:

1. bem_solve.m sets up and solves the matrix equation (12) on the problem sheet (type help bem_solve in Matlab for the in-built documentation, and similarly for the other functions);
2. pressure.m then computes the pressure using equation (10);
3. circ.m sets up a boundary element mesh on a circle;
4. pressure_exact.m computes the exact solution in the case when the domain is a circle centred on the origin and the point source is at the centre of the circle, this exact solution being

$$u(\mathbf{x}) = -\frac{\mathrm{i}}{4} H_0^{(1)}(kr) + cJ_0(kr), \tag{13}$$

where $r = |\mathbf{x}|$ and

$$c = \frac{\frac{\mathrm{i}}{4} H_0^{(1)}(kR)}{J_0(kR)},$$

where R is the radius of the circle.
5. G.m computes the 2D free-field Green's function.

The script file bem_complete.m uses these functions to solve by BEM a simple problem of a source at the centre of a circle of radius R, comparing the BEM solution with the exact solution, and seeing how the relative error depends on the length, h, of the boundary elements.

1. Read through the Matlab files and compare with the notes and, in particular, with question 5 on part one of the problem sheet. Then run `bem_complete.m` yourself. Note how the error decreases roughly in proportion with h, so that the BEM appears to be **first order convergent**, so that very small elements compared to the wavelength are needed for very small relative errors.

2. The test of the BEM method and implementation that `bem_complete` provides is not very satisfactory because of the radial symmetry of the problem solved. Because of this symmetry $\partial u/\partial n$ is constant on ∂D, so that the approximation that $\partial u/\partial n$ is constant on each element is exact in this case!

To remove the symmetry move the source to $\mathbf{x}_0 = (R/4, 0)$ so that it is off centre. Of course, the exact solution above no longer applies. An exact solution can be computed as an infinite series of Bessel functions, but to test the code we will use instead the idea from question 3 on part one of problem sheet. It can be shown (see question 3 and compare with equations (2.32) and (2.33) in the notes) that, for $\mathbf{x}$ outside D,

$$0 = G(\mathbf{x}_0, \mathbf{x}) + \int_{\partial D} G(\mathbf{y}, \mathbf{x}) \frac{\partial u}{\partial n}(\mathbf{y})\, ds(\mathbf{y}). \qquad (14)$$

Now the function `pressure.m` computes an approximation to the right hand side of this equation, which indeed is the pressure when $\mathbf{x}$ is inside D. It is a good test of the BEM scheme and the coding to see if `pressure.m` predicts the correct value of 0 when given a point $\mathbf{x}$ outside D.

So save a new version of `bem_complete.m`, as `bem_complete2.m`, that:

(i) has $\mathbf{x}_0 = (R/4, 0)$ off-centre;

(ii) has $\mathbf{x} = (2R, 2R)$ outside D;

(iii) no longer computes the exact pressure or the relative error but, instead, tabulates and plots the absolute value of the predicted pressure at $\mathbf{x}$ as a function of h/λ.

[A version of such a program (i.e. a worked solution to this question) is downloadable from the web site above, as are the solutions to question 3.]

3. Return to `bem_complete.m`. Compute, by differentiating the exact solution (13) above, an expression for $\partial u/\partial r$, and an expression for $\partial u/\partial n$ on ∂D. (Note that the derivatives of $H_0^{(1)}$ and J_0 are $-H_1^{(1)}$ and $-J_1$, respectively.) Now carry out the following computational tasks:

(i) Write a function `pressure_exact_dr.m` which has the same inputs as `pressure_exact.m` but calculates, as output, the value of $\partial u/\partial r$ at $\mathbf{x}$ rather than u at $\mathbf{x}$.

(ii) Modify `bem_complete.m` (call the new version `bem_complete3.m`) so that it no longer computes $u(\mathbf{x})$ or the exact pressure at $\mathbf{x}$. Instead, get the program to compute, for each discretisation (i.e. each value of m in the program), the difference between the exact value of $\partial u/\partial n$ and the values in the vector `v` as calculated by `bem_solve.m` (all these values are identical because of the rotational symmetry of the problem). The exact value of $\partial u/\partial n$ can be calculated by calling `pressure_exact_dr.m` (remember that the normal is directed *into* D). You should find that the relative errors in the BEM values of the normal derivative decrease approximately proportionally to h, but are much larger than the relative errors in the predicted pressure in question 1.

(iii) Write a function `pressure_exact_grad.m` which has the same inputs as `pressure_exact.m` but calculates, as output, the value of the vector ∇u at $\mathbf{x}$ rather than u at $\mathbf{x}$. The function can call `pressure_exact_dr.m` which already does most of the work.

(iv) Write a function `G_grad.m` which has the same inputs as `G.m` but calculates, as output, the gradient of G, i.e. $\nabla_{\mathbf{x}} G(\mathbf{x},\mathbf{y}) = \nabla_{\mathbf{x}} G(\mathbf{y},\mathbf{x})$, rather than the value of G itself. Write a function `pressure_grad.m` which has the same inputs as `pressure_grad.m` but calculates, as output, a BEM approximation to $\nabla u(\mathbf{x})$ rather than $u(\mathbf{x})$. (To get an approximation for ∇u take the gradient of the approximation for u, equation (10) on the problem sheet, and use your new function `G_grad.m`.) Create a modified version of `bem_complete.m` (called `bem_complete4.m`) which computes and compares the BEM and exact values of $\nabla u(\mathbf{x})$, rather than the values of $u(\mathbf{x})$.

[The exact expression for ∇u is $\nabla u(\mathbf{x}) = \left[\frac{\mathrm{i}k}{4} H_1^{(1)}(kr) - ckJ_1(kr)\right]\mathbf{r}$ where $r = |\mathbf{x}|$, $\mathbf{r} = \mathbf{x}/r$, and $c = \frac{\mathrm{i}}{4} H_0^{(1)}(kR)/J_0(kR)$.]