

The Mathematics of Scattering by Unbounded Rough Surfaces

Simon Chandler-Wilde

University of Reading, UK

www.reading.ac.uk/~sms03snc

Joint work with: **Bo Zhang, Coventry/Chinese Academy of Sciences**

Roland Potthast, Eric Heinemeyer, Göttingen

Peter Monk, Delaware

Kai Haseloh, Hannover

PhD Students Chris Ross, Tilo Arens, Anja Meier

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Preliminaries

Wave equation, wave speed c :

$$\Delta U = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}$$

Seek **time harmonic** solution $U(x, t) = u(x)e^{-i\omega t}$. Then u satisfies the **Helmholtz equation**

$$\Delta u + k^2 u = 0$$

with $k = \omega/c = 2\pi \times \text{frequency}/c > 0$.

Simplest solution is the **plane wave**

$$u(x) = e^{ikx_1} \Rightarrow U(x, t) = e^{i(kx - \omega t)}$$

with wavelength $\lambda = 2\pi/k$.

The Scattering Problem

$$\Delta u + k^2 u = g$$

$$D \subset \mathbb{R}^n$$

● Support of g

$$u = 0$$

∂D

x_n

$\tilde{x} = (x_1, \dots, x_{n-1})$

The Scattering Problem

$$\Delta u + k^2 u = g$$

$D \subset \mathbb{R}^n$

Support of g

$$u = 0$$

$$x_n = f_+$$

∂D

x_n

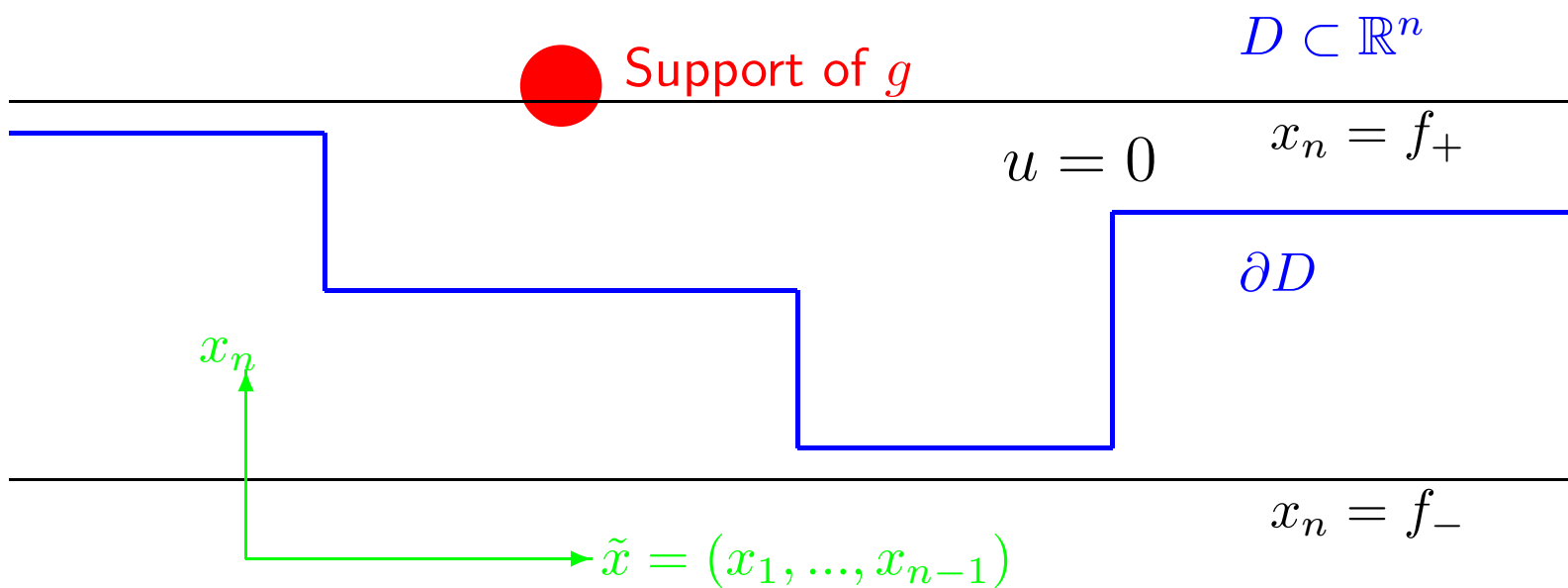
$$\tilde{x} = (x_1, \dots, x_{n-1})$$

$$x_n = f_-$$

The Scattering Problem

SGA: $x = (\tilde{x}, x_n) \in D \Rightarrow (\tilde{x}, s) \in D, \forall s > x_n$

$$\Delta u + k^2 u = g$$



Motivation

This problem models time harmonic acoustic scattering by an unbounded sound soft rough surface (surface on which the perturbation in pressure is zero). More generally, it models (in the sense of having some of the same difficulties) other rough surface scattering problems, e.g.

- Scattering of sonar waves from towed seismic transducers by the sea surface (Schlumberger)
- Outdoor noise propagation over ground surfaces (TRL)
- Scattering of ground penetrating radar by buried surfaces (Rutherford Appleton Laboratory)
- Scattering of radar waves by the sea surface (BAE Systems)
- Scattering of light by diffractive optics structures (e.g. the hologram on your passport) (UK Home Office)

Challenges

- Correct mathematical formulation
- Existence, uniqueness, well-posedness
- Stability and convergence of approximation by problem in finite domain
- Effective numerical schemes, e.g. approximation with few degrees of freedom, effective preconditioners, iterative solvers, fast matrix-vector products
- The inverse problem

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Talk Today

- Review of previous work
- Exact solution when $D = U_h$ is a half-space
- Mathematical formulation, including radiation condition
- Equivalent variational formulation
- Solvability when $\kappa := k(f_+ - f_-)$ is small via Lax-Milgram
- Solvability via inf-sup condition when Secondary Geometrical Assumption holds
- Some open questions

Scattering by Unbounded Surfaces is a Subtle Business

- The standard Sommerfeld radiation condition, which implies (in 2D) that $u(x) = O(|x|^{-1/2})$ as $|x| \rightarrow \infty$ may not be satisfied.
- The surface may support surface waves, i.e. solutions of $\Delta u + k^2 u = 0$ which decay exponentially with distance from the surface (Gotlib 2000, $n = 2$ examples where $D + (L, 0) = D$, for some $L > 0$, and SGA not satisfied).
- Surface waves of this type may be excited by waves incident on the surface. As a result, in $n = 2$, near the boundary ∂D it may hold that $|u(x)| \approx |x|^{-\alpha}$ as $|x| \rightarrow \infty$ with $\alpha = 0, 1/2$ or $3/2$, depending on the geometry/boundary condition on ∂D
- Formulation may depend on the precise geometry
- Behaviour at infinity may be surprising; e.g. in $n = 2$ for plane wave incidence the solution of $\Delta u + k^2 u = 0$ in D , $u = 0$ on ∂D , may not be bounded (C-W & Potthast 2002)

More preliminaries

A solution to

$$\Delta u + k^2 u = -\delta_y$$

is (in 3D) the standard **fundamental solution**

$$u(x) = \Phi(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x \neq y.$$

The Scattering Problem: Special Case

$$\Delta u + k^2 u = -\delta_y$$

$$D = U_h \subset \mathbb{R}^n$$

• y

$$u = 0$$

$$\partial D = \Gamma_h$$

• y'

A solution is

$$u(x) = G(x, y) := \Phi(x, y) - \Phi(x, y'),$$

with G the **Green's function** for U_h . Note that

$$|u(x)| \leq C_y \frac{x_n - h}{1 + |x - y|^{(n+1)/2}}.$$

The Scattering Problem: Special Case

$$\Delta u + k^2 u = g$$

$$D = U_h \subset \mathbb{R}^n$$



Support of g

$$u = 0$$

$$\partial D = \Gamma_h$$

A solution is, for compactly supported $g \in L^2(D)$,

$$u(x) = - \int_D G(x, y) g(y) dy, \quad x \in D,$$

with G the **Green's function** for U_h . Note that

$$|u(x)| \leq C \frac{x_n - h}{(1 + |x|)^{(n+1)/2}},$$

so that $u \in L^2(S_H)$ for every $H > h$, where $S_H := D \setminus \bar{U}_H$.

Previous Work

Previous Work: Science/Engineering

Huge effort in science, engineering, applied maths literature on rough surface scattering aimed at:

- Approximate calculation methods e.g. small perturbation methods $D \approx U_h$, small slope approximations, Kirchhoff approximations
- Statistical analysis, where ∂D is random
- Effective numerical calculation schemes

E.g. books: Ogilvy 1991, Voronovich 1998. Review papers: DeSanto 2002, Warnick & Chew 2001, Saillard & Sentenac 2001.

Previous Work: Mathematical Analysis

Very considerable mathematical analysis for the case when D is periodic or bi-periodic. E.g., for the case $n = 2$, $D + (L, 0) = D$, monograph by Wilcox 1983, **Elschner & Yamamoto 2002** (variational formulation for D Lipschitz). Problem reduces to one in a single period.

Large body of work on scattering by unbounded surfaces, studied via a priori estimates in weighted Sobolev spaces, limiting absorption arguments, e.g. Eidus & Vinnik 1974, Vogelsang 1975, Minskii 1983. The arguments require that D is **starlike** and/or that D approaches a half-space at infinity.

Previous Work: Mathematical Analysis

For the 2D case, with $D = \{x = (x_1, x_2) : x_2 > f(x_1)\}$ and f bounded and continuous and sufficiently smooth ($C^{1,1}$), the rough surface scattering problem has been studied via **integral equation** methods.

Where $h < f_-$ and G is the Green's function for U_h , look for a solution as $u = u^i + u^s$, with

$$u^s(x) = \int_{\partial D} \left(\frac{\partial G(x, y)}{\partial n(y)} + i\eta G(x, y) \right) \phi(y) ds(y), \quad x \in D,$$

leading to a second kind boundary integral equation for ϕ , in operator form

$$(I + K)\phi = -2u^i|_{\partial D}.$$

It has been shown that $I + K$ is invertible as an operator on $L^p(\partial D)$, $1 \leq p \leq \infty$, and as an operator on weighted spaces of continuous functions (Zhang & C-W 2003, Arens, C-W & Haseloh 2002, 2003).

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In the corresponding 3D case it is shown in (C-W, Heinemeyer, Potthast 2005) that $I + K$ is invertible as an operator on $L^2(\partial D)$ for mildly rough surfaces.

As a consequence, our problem has a solution for $n = 2$ and $g \in L^2(D)$ compactly supported, and that solution satisfies that $u \in L^2(S_H)$, for every $H > f_+$, in fact

$$u \in H^1(S_H) := \{u \in L^2(S_H) : \nabla u \in L^2(S_H)\}$$

a Hilbert space with the inner product

$$(u, v)_H := \int_{S_H} (u\bar{v} + k^2 \nabla u \cdot \nabla \bar{v}) dx.$$

and norm

$$\|u\|_H := |(u, u)|^{1/2} = \left\{ \int_{S_H} (|u|^2 + k^2 |\nabla u|^2) dx \right\}^{1/2}.$$

In fact, since $u = 0$ on ∂D and D is Lipschitz (in fact pretty smooth), the 2D solution computed by integral equation methods satisfies $u \in V_H$, where V_H is the subspace of $H^1(S_H)$ that is the closure of

$$\{v|_{S_H} : v \in C_0^\infty(D)\}$$

in the norm of $H^1(S_H)$. For Lipschitz D , equivalently

$$V_H = \{v \in H^1(S_H) : \text{trace of } v \text{ is zero on } \partial D\}.$$

Moreover, the 2D solution computed by integral equation methods has, for $H > f_+$, the explicit representation for $x = (x_1, x_2) \in U_H$ that

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(i[(x_2 - H)\sqrt{k^2 - \xi^2} + x_1\xi]) \hat{F}_H(\xi) d\xi,$$

where $F_H(x_1) = u((x_1, H))$, $\hat{F}_H \in L^2(\mathbb{R})$ is the Fourier transform of $F_H \in L^2(\mathbb{R})$, and $\sqrt{k^2 - \xi^2} = i\sqrt{\xi^2 - k^2}$ when $|\xi| > k$.

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Note that $\exp(i[(x_2 - H)\sqrt{k^2 - \xi^2} + x_1\xi])$ is a solution of the Helmholtz equation that is

- a plane wave travelling upwards for $|\xi| < k$
- a plane wave travelling horizontally for $|\xi| = k$
- an evanescent, exponentially decaying plane wave for $|\xi| > k$.

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See Arens & Hohage 2005 for the interpretation of this representation as the action of a tempered distribution in the case when $F_H \in L^\infty(\mathbb{R})$ (appropriate in the case of plane wave incidence).

Moreover, the 2D solution computed by integral equation methods has, for $H > f_+$, the explicit representation for $x = (x_1, x_2) \in U_H$ that

$$u(x) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}^{n-1}} \exp(i[(x_n - H)\sqrt{k^2 - \xi^2} + \tilde{x} \cdot \xi]) \hat{F}_H(\xi) d\xi,$$

where $F_H = u|_{\Gamma_H} \in L^2(\mathbb{R}^{n-1})$, $\hat{F}_H \in L^2(\mathbb{R}^{n-1})$ is the Fourier transform of $F_H \in L^2(\mathbb{R})$, and $\sqrt{k^2 - \xi^2} = i\sqrt{\xi^2 - k^2}$ when $|\xi| > k$.

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THE BOUNDARY VALUE PROBLEM *Given $g \in L^2(D)$, whose support lies in $\overline{S_H}$, for some $H \geq f_+$, find $u : D \rightarrow \mathbb{C}$ such that $u|_{S_a} \in V_a$ for every $a > f_+$,*

$$\Delta u + k^2 u = g \quad \text{in } D$$

in a distributional sense, and

$$u(x) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}^{n-1}} \exp(i[(x_n - H)\sqrt{k^2 - \xi^2} + \tilde{x} \cdot \xi]) \hat{F}_H(\xi) d\xi,$$

for $x \in U_H$, where $F_H = u|_{\Gamma_H}$.

MAIN THEOREM (C-W & Monk 2005) *If*

$$\kappa := k(H - f_-) < \sqrt{2}$$

or the Secondary Geometric Assumption (SGA) holds, then the BVP has exactly one solution. Moreover

$$k\|u\|_H \leq \frac{\kappa}{\sqrt{2}} \frac{1 + \kappa^2/2}{1 - \kappa^2/2} \|g\|_2,$$

where $\|g\|_2 = \left\{ \int_{S_H} |g|^2 dx \right\}^{1/2}$, in the case that $\kappa < \sqrt{2}$, while

$$k\|u\|_H \leq \frac{\kappa}{\sqrt{2}} (\kappa + 1)^2 \|g\|_2.$$

whenever the SGA holds.

The Variational Formulation

For $v \in V_H$,

$$(g, v) := \int_{S_H} g \bar{v} \, dx = \int_{S_H} (\Delta u + k^2 u) \bar{v} \, dx.$$

Integrating by parts, since $u = 0$ on ∂D , we see that

$$-(g, v) = \int_{S_H} (\nabla u \cdot \nabla \bar{v} - k^2 u \bar{v}) \, dx + \int_{\Gamma_H} \bar{v} \frac{\partial u}{\partial x_n} \, ds.$$

The Variational Formulation cont ...

Now

$$u(x) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}^{n-1}} \exp(i[(x_n - H)\sqrt{k^2 - \xi^2} + \tilde{x} \cdot \xi]) \hat{F}_H(\xi) d\xi,$$

for $x \in U_H$, where $F_H = u|_{\Gamma_H}$. Thus, on Γ_H , $\frac{\partial u}{\partial x_n}$ has Fourier transform

$$i\sqrt{k^2 - \xi^2} \hat{F}_H(\xi),$$

i.e. on Γ_H

$$\frac{\partial u}{\partial x_n} = -Tu$$

where the **Dirichlet to Neumann map** T is defined by $T = \mathcal{F}^{-1} M_z \mathcal{F}$, with M_z multiplication by $z(\xi) := i\sqrt{k^2 - \xi^2}$.

Sobolev spaces on Γ_H , the Trace Operator, and the Dirichlet to Neumann Operator

Let $H^s(\Gamma_H) = H^s(\mathbb{R}^{n-1})$ denote the usual Sobolev spaces.

Thus, for $s \in \mathbb{R}$, $H^s(\Gamma_H)$ denotes the completion of $C_0^\infty(\Gamma_H)$ in the norm $\|\cdot\|_{H^s(\Gamma_H)}$ defined by

$$\|\phi\|_{H^s(\Gamma_H)} := \left(\int_{\mathbb{R}^{n-1}} (k^2 + \xi^2)^s |\mathcal{F}\phi(\xi)|^2 d\xi \right)^{1/2}.$$

THEOREM *The Dirichlet to Neumann operator*

$T : H^{1/2}(\Gamma_H) \rightarrow H^{-1/2}(\Gamma_H)$ and is bounded with norm $\|T\| = 1$.

*There exists a unique bounded linear operator $\gamma_- : V_H \rightarrow H^{1/2}(\Gamma_H)$ (the **trace operator**) such that $\gamma_- u = u|_{\Gamma_H}$ for $u \in C(\overline{S_H})$. It holds that $\|\gamma_-\| \leq 1$.*

The Variational Formulation cont ...

Thus, for all $v \in V_H$,

$$-(g, v) = b(u, v) := \int_{S_H} (\nabla u \cdot \nabla \bar{v} - k^2 u \bar{v}) \, dx + \int_{\Gamma_H} \gamma_- \bar{v} T \gamma_- u \, ds.$$

THE VARIATIONAL PROBLEM *Find* $u \in V_H$ *such that*

$$b(u, v) = -(g, v), \quad v \in V_H.$$

THEOREM *The sesquilinear form* b *is bounded with*

$$\begin{aligned} |b(u, v)| &\leq \int_{S_H} (|\nabla u| |\nabla v| + k^2 |u| |v|) \, dx \\ &\quad + \|\gamma_- v\|_{H^{-1/2}(\Gamma_H)} \|T \gamma_- u\|_{H^{1/2}(\Gamma_H)} \leq 2 \|u\|_H \|v\|_H. \end{aligned}$$

THEOREM (Equivalence of BVP and Variational Problem) *If u is a solution of the boundary value problem then $u|_{S_H}$ satisfies the variational problem. Conversely, if u satisfies the variational problem, $F_H := \gamma_- u$, and the definition of u is extended to D by setting*

$$u(x) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\mathbb{R}^{n-1}} \exp(i[(x_n - H)\sqrt{k^2 - \xi^2} + \tilde{x} \cdot \xi]) \hat{F}_H(\xi) d\xi,$$

for $x \in U_H$, then the extended function satisfies the boundary value problem.

Analysis of the Variational Formulation for Small Wavenumber

LEMMA For all $\phi \in H^{1/2}(\Gamma_H)$,

$$\Re \int_{\Gamma_H} \bar{\phi} T \phi ds \geq 0.$$

Proof. Clear from $T = \mathcal{F}^{-1} M_z \mathcal{F}$ and the Plancherel identity which gives

$$\int_{\Gamma_H} \bar{\phi} T \phi ds = \int_{\mathbb{R}^{n-1}} \bar{\hat{\phi}}(\xi) z(\xi) \phi(\xi) d\xi.$$

LEMMA (**Friedrichs Inequality**) For $u \in V_H$,

$$\|u\|_2 \leq \frac{H - f_-}{\sqrt{2}} \left\| \frac{\partial u}{\partial x_n} \right\|_2 \leq \frac{H - f_-}{\sqrt{2}} \|u\|_H.$$

Analysis of the Variational Formulation for Small Wavenumber cont ...

LEMMA (**Ellipticity for κ small**) For all $u \in V_H$, $\Re b(u, u) \geq \frac{1 - \kappa^2/2}{1 + \kappa^2/2} \|u\|_H^2$.

Proof. By the above lemmas, $\|u\|_H^2 \geq k^2(2/\kappa^2 + 1)\|u\|_2^2$ and

$$\Re b(u, u) \geq \|\nabla u\|_2^2 - k^2\|u\|_2^2 = \|u\|_H^2 - 2k^2\|u\|_2^2.$$

LEMMA (**Lax-Milgram**) Suppose the sesquilinear form $b : V \times V \rightarrow \mathbb{C}$ is bounded and elliptic, i.e.

$$|b(u, v)| \leq C\|u\|\|v\|, \quad \Re b(u, u) \geq \alpha\|u\|^2, \quad u, v \in V,$$

for some positive constants C and α . Then, for every $G \in V^*$, the variational problem

$$b(u, v) = G(v), \quad v \in V,$$

has exactly one solution $u \in V$ and $\|u\|_V \leq \alpha^{-1}\|G\|$.

Analysis of the Variational Formulation for Small Wavenumber cont ...

THEOREM *If $\kappa < \sqrt{2}$ then the variational problem has exactly one solution and*

$$\kappa \|u\|_H \leq \frac{\kappa}{\sqrt{2}} \frac{1 + \kappa^2/2}{1 - \kappa^2/2} \|g\|_2,$$

where $\|g\|_2 = \left\{ \int_{S_H} |g|^2 dx \right\}^{1/2}$.

Analysis of the Variational Formulation for Arbitrary Wavenumber

LEMMA (**Generalised Lax-Milgram**) *Suppose the sesquilinear form $b : V \times V \rightarrow \mathbb{C}$ is bounded and that the **inf-sup** condition*

$$\inf_{0 \neq u} \sup_{0 \neq v} \frac{|b(u, v)|}{\|u\| \|v\|} \geq \beta$$

holds for some $\beta > 0$ and also

$$\sup_{0 \neq v} \frac{|b(v, u)|}{\|v\|} > 0$$

for every $u \neq 0$. Then, for every $G \in V^$, the variational problem*

$$b(u, v) = G(v), \quad v \in V,$$

has exactly one solution $u \in V$ and $\|u\|_V \leq \beta^{-1} \|G\|$. Conversely, if $\|u\|_V \leq \beta^{-1} \|G\|$ holds whenever u and G satisfy the variational problem, then the inf-sup condition holds.

Analysis of the Variational Formulation for Arbitrary Wavenumber cont ...

LEMMA *For $u, v \in V_H$, $b(v, u) = b(\bar{u}, \bar{v})$, so that the transposed inf-sup condition follows from the inf-sup condition.*

Analysis of the Variational Formulation for Arbitrary Wavenumber cont ...

LEMMA For $u, v \in V_H$, $b(v, u) = b(\bar{u}, \bar{v})$, so that the transposed inf-sup condition follows from the inf-sup condition.

LEMMA Suppose there exists $\tilde{C} > 0$ such that, for all $u \in V_H$ and $g \in L^2(S_H)$ satisfying

$$b(u, v) = -(g, v), \quad v \in V_H,$$

it holds that $\|u\|_H \leq k^{-1} \tilde{C} \|g\|_2$.

Then, for all $u \in V_H$ and $G \in V_H^*$ satisfying

$$b(u, v) = G(v), \quad v \in V_H,$$

it holds that

$$\|u\|_H \leq \left(1 + 2\tilde{C}\right) \|G\|.$$

Analysis of the Variational Formulation for Arbitrary Wavenumber cont ...

LEMMA *If ∂D is the graph of a C^∞ function then, for all $u \in V_H$ and $g \in L^2(S_H)$ satisfying*

$$b(u, v) = -(g, v), \quad v \in V_H,$$

it holds that $\|u\|_H \leq k^{-1} \tilde{C} \|g\|_2$ with $\tilde{C} = \frac{\kappa}{\sqrt{2}} (\kappa + 1)^2$

Proof. A clever integration by parts and use of the equivalence of the BVP and variational problem.

COROLLARY *The variational problem has exactly one solution whenever ∂D is the graph of a C^∞ function. Moreover,*

$$k \|u\|_H \leq \frac{\kappa}{\sqrt{2}} (\kappa + 1)^2 \|g\|_2.$$

Analysis of the Variational Formulation for Arbitrary Wavenumber cont ...

LEMMA *If the Secondary Geometric Assumption holds then, for all $u \in V_H$ and $g \in L^2(S_H)$ satisfying*

$$b(u, v) = -(g, v), \quad v \in V_H,$$

it holds that $\|u\|_H \leq k^{-1} \tilde{C} \|g\|_2$ with $\tilde{C} = \frac{\kappa}{\sqrt{2}} (\kappa + 1)^2$

Proof. Approximation of ∂D by a C^∞ boundary.

COROLLARY *The variational problem has exactly one solution whenever the SGA holds. Moreover,*

$$k \|u\|_H \leq \frac{\kappa}{\sqrt{2}} (\kappa + 1)^2 \|g\|_2.$$

MAIN THEOREM *If*

$$\kappa := k(H - f_-) < \sqrt{2}$$

or the Secondary Geometric Assumption (SGA) holds, then the BVP has exactly one solution. Moreover

$$k\|u\|_H \leq \frac{\kappa}{\sqrt{2}} \frac{1 + \kappa^2/2}{1 - \kappa^2/2} \|g\|_2,$$

where $\|g\|_2 = \left\{ \int_{S_H} |g|^2 dx \right\}^{1/2}$, in the case that $\kappa < \sqrt{2}$, while

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whenever the SGA holds.

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- Some open questions

Some Open (Mathematical) Questions

- Variational formulation (VF) and analysis: extension to other boundary conditions, electromagnetic case
- VF: extension to other function spaces (e.g. weighted spaces of L^2 functions, cf. Arens & Hohage 2005)
- VF: analysis of finite element methods: discrete inf-sup condition
- Integral equation methods (IEM): extension to 3D, Lipschitz surfaces with no restriction on surface slope and elevation
- IEM: explicit estimates for condition number as a function of the wavenumber and the coupling parameter (cf. VF above and Jerison & Kenig identities for Laplace case)
- BOTH: what can we say in the non-normal, surface wave case (e.g. Neumann b.c.)