Finite element methods for acoustic scattering

S Langdon[1] & S N Chandler-Wilde[2]
Department of Mathematics
University of Reading
Whiteknights PO Box 220
Berkshire RG6 6AX
UK

July 18, 2007

[1]email address: s.langdon@reading.ac.uk
[2]email address: s.n.chandler-wilde@reading.ac.uk

Abstract

In these lecture notes we introduce the finite element method and describe how it can be used to approximate the solution to certain problems of acoustic scattering. We also highlight some of the difficulties involved, and briefly summarise some current research aimed at resolving these issues.

Contents

Chapter 1

Introduction

Many physicists and engineers are interested in the reliable simulation of processes in which acoustic waves are scattered by obstacles, with applications arising in areas as diverse as sonar, (see figure 1.1), road, rail or aircraft noise, or building acoustics. Unless the geometry of the scattering object is

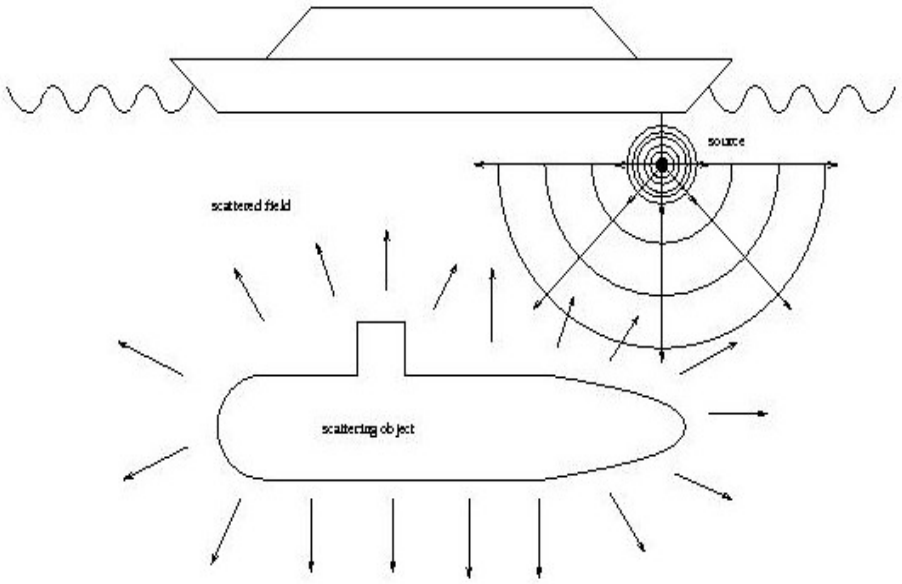

Figure 1.1: Typical acoustic scattering problem

particularly simple, the analytical solution of scattering problems is usually impossible, and hence numerical schemes are required.

Acoustic pressure $P(\mathbf{x}, t)$ in a homogeneous media is modelled by the

wave equation

$$\Delta P - \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = 0, \tag{1.1}$$

where c is the speed of sound. Considering for simplicity only the problem of time harmonic acoustic scattering, the pressure is given by

$$P(\mathbf{x}, t) = u(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}, \tag{1.2}$$

where ω is the frequency. Substituting (1.2) into (1.1), we have to solve the Helmholtz equation

$$\Delta u + k^2 u = 0, \quad \text{in } D \subset \mathbb{R}^d, \quad d = 1, 2, 3, \tag{1.3}$$

where the *wavenumber* $k := \omega/c$ is a physical parameter, proportional to the frequency of the incident wave. We supplement (1.3) with appropriate boundary conditions, for example the impedance boundary condition

$$\frac{\partial u}{\partial \mathbf{n}} + \mathrm{i}ku = g, \quad \text{on } \partial D, \tag{1.4}$$

where ∂D is the boundary of D and $\partial/\partial\mathbf{n}$ is the normal derivative. The simplest situation to model occurs when the computational domain D is bounded and simply connected (the extra complications arising in the case that D is an unbounded domain are discussed in §3.1).

In these notes we describe the numerical solution of (1.3) by the finite element method, which is renowned both for its versatility, being applicable to a wide range of problems on difficult geometries in one, two and three dimensions, and for its mathematical rigour. In particular, the finite element method lends itself easily to a rigorous error analysis, allowing one to establish a degree of definiteness about the accuracy of the numerical solution before the calculations begin, and also easing the development of adaptive algorithms, which can be used to achieve a high degree of accuracy with a minimal computational cost.

An outline of the notes is as follows. In chapter 2 we present the finite element method for the solution of (1.3)–(1.4). We begin by making some definitions in §2.1 and then proceed in §2.2 by demonstrating the implementation of the Galerkin finite element method via a simple one dimensional example. In §2.3 we present some error estimates for the method, as applied to this simple problem, allowing us to discuss the relationship between accuracy and computational cost, and in §2.4 we demonstrate in broad terms how this approach can be extended to two and three dimensional problems.

In chapter 3 we discuss some difficulties in applying the finite element method to the solution of acoustic scattering problems. In §3.1 we consider

the case that the computational domain D is an unbounded domain, in which case one needs to consider with great care the question of what happens at infinity. In §3.2 we consider the case that the wavenumber k is large, in which case standard schemes deteriorate in accuracy.

Finally, in chapter 4 we present some conclusions, and give some ideas for further reading.

Chapter 2

The finite element method

2.1 Some function space definitions

For a domain $\Omega \in \mathbb{R}^d$, $d = 1, 2, 3$, we define the function space $L_2(\Omega)$ of square integrable functions on Ω by saying that $f \in L_2(\Omega)$ if and only if

$$\|f\| := \left(\int_\Omega |f(x)|^2 \mathrm{d}x \right)^{1/2} < \infty.$$

For example, $x \in L_2(0,1)$, but $1/x \notin L_2(0,1)$.

The function $\|\cdot\|$ is a *norm*, and has the properties that

$$\begin{aligned} \|f\| &= 0 \quad \text{if and only if } f = 0, \\ \|f+g\| &\leq \|f\| + \|g\|, \quad \text{for all } f, g \in L_2(\Omega), \\ \|\alpha f\| &= \alpha \|f\|, \quad \text{for all } \alpha \in \mathbb{R},\ f \in L_2(\Omega). \end{aligned}$$

We define further the *Sobolev space* $H^1(\Omega)$ by saying that $f \in H^1(\Omega)$ if and only if

$$\|\nabla f\|^2 + \|f\|^2 < \infty.$$

Finally, we say that $f \in H^1_{(0}(\Omega)$ if $f \in H^1(\Omega)$ and $f(0) = 0$.

These function spaces will be very useful when setting up our finite element method.

2.2 One dimensional model problem

In order to illustrate the ideas behind the finite element method, we begin by applying it to the solution of a simple one dimensional model problem. This appears as example 4.2.1 in [14, p.107], where a more rigorous mathematical treatment can be found.

The problem we consider is that of propagation of a time-harmonic plane wave along the x-axis;

$$\begin{aligned} -\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - k^2u &= f, \quad \text{in } \Omega := (0,1), && (2.1)\\ u(0) &= 0, && (2.2)\\ \frac{\mathrm{d}u}{\mathrm{d}x}(1) - \mathrm{i}ku(1) &= 0. && (2.3) \end{aligned}$$

where $f \in L_2(\Omega)$. We will consider a more realistic two dimensional scattering problem in §2.4.

It is straightforward to show that the exact solution to (2.1)–(2.3) is given by

$$u(x) = \frac{\mathrm{e}^{\mathrm{i}kx}}{k}\int_0^x \sin(ks)f(s)\,\mathrm{d}s + \frac{\sin(kx)}{k}\int_x^1 \mathrm{e}^{\mathrm{i}ks}f(s)\,\mathrm{d}s, \qquad (2.4)$$

(see problem sheet) which is periodic with period $\lambda := 2\pi/k$, the *wavelength*. For more complicated problems, and particularly for higher dimensional problems, it will not be possible to determine the exact solution in this way.

The first step to setting up the finite element method is to rewrite the problem (2.1)–(2.3) in its weak form. We begin by multiplying (2.1) by a test function $v \in H^1_{(0}(\Omega)$ and integrating to get

$$-\int_0^1 u''(x)v(x) - k^2u(x)v(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x.$$

Integrating the first term by parts,

$$[-u'(x)v(x)]_0^1 + \int_0^1 u'(x)v'(x)\,\mathrm{d}x - k^2\int_0^1 u(x)v(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x.$$

Now using the boundary condition (2.3) and the fact that $v(0) = 0$, we have our weak formulation;
Find $u \in H^1_{(0}(\Omega)$ such that

$$\int_0^1 u'(x)v'(x)\,\mathrm{d}x - k^2\int_0^1 u(x)v(x)\,\mathrm{d}x - \mathrm{i}ku(1)v(1) = \int_0^1 f(x)v(x)\,\mathrm{d}x, \quad (2.5)$$

holds for all $v \in H^1_{(0}(\Omega)$.

To solve (2.5), we begin by defining the *finite element mesh*

$$X_h := \{x_i : 0 = x_0 < x_1 < x_2 < \ldots < x_N = 1\},$$

on $\Omega = (0,1)$, and we define the *mesh size*

$$h := \max_{1\le i\le N}(x_i - x_{i-1}).$$

The intervals $\tau_i = (x_{i-1}, x_i)$ are called the *finite elements*, and we say that the mesh is *uniform* if all of the elements have the same size $h = 1/N$.

Next, we define the *basis functions*. We denote by $S_h(0,1) \subset H^1_{(0}(0,1)$ the space of continuous piecewise linear functions, with nodal values at the points of X_h, satisfying the boundary condition (2.2) at $x = 0$. Then a set of basis functions for the space $S_h(0,1)$ is the set of *hat functions* defined for $j = 1, \ldots, N-1$ by

$$\chi_j(x) = \begin{cases} \frac{1}{h}(x - x_{j-1}), & x \in [x_{j-1}, x_j], \\ \frac{1}{h}(x_{j+1} - x), & x \in [x_j, x_{j+1}], \\ 0 & \text{elsewhere,} \end{cases}$$

and for $j = N$ by

$$\chi_N(x) = \begin{cases} \frac{1}{h}(x - x_{N-1}), & x \in [x_{N-1}, 1], \\ 0 & \text{elsewhere,} \end{cases}$$

where $x_j = jh$, $j = 0, \ldots, N$, with $h = 1/N$. Some of these are illustrated in figure 2.1.

To construct our approximate solution for (2.5) we then proceed by replacing the requirement that $u, v \in H^1_{(0}(0,1)$ with the requirement that $U, v \in S_h(0,1) \subset H^1_{(0}(0,1)$, where U is our approximation to u. This is our *Galerkin finite element method*;
Find $U \in S_h(0,1)$ such that

$$\int_0^1 U'(x)v'(x)\,\mathrm{d}x - k^2\int_0^1 U(x)v(x)\,\mathrm{d}x - \mathrm{i}kU(1)v(1) = \int_0^1 f(x)v(x)\,\mathrm{d}x, \quad (2.6)$$

holds for all $v \in S_h(0,1)$.

We are now looking for a function that lies in a finite dimensional vector space, so we can write U as a linear sum of the basis functions,

$$U(x) = \sum_{j=1}^{N} u_j\chi_j(x), \quad (2.7)$$

where u_j are unknown coefficients which we must find. Substituting into (2.6) we have

$$\sum_{j=1}^{N}\left[\int_0^1 \chi_j'(x)v'(x)\,\mathrm{d}x - k^2\int_0^1 \chi_j(x)v(x)\,\mathrm{d}x\right]u_j - \mathrm{i}ku_Nv(1) = \int_0^1 f(x)v(x)\,\mathrm{d}x, \quad (2.8)$$

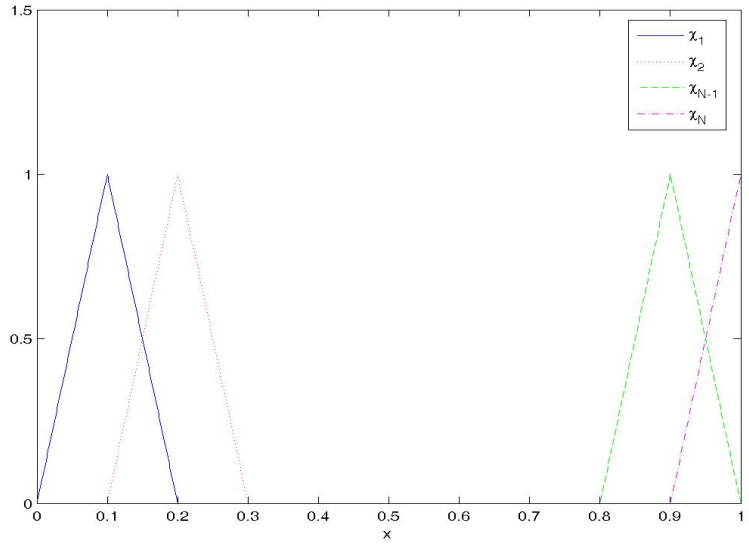

Figure 2.1: Some hat functions

which holds for all $v \in S_h(0,1)$. In particular, (2.8) must hold for each $v = \chi_m$, the basis functions for $S_h(0,1)$. Substituting $v = \chi_m$, $m = 1, \ldots, N$, into (2.8) gives us our linear system, a set of N equations for the N unknown coefficients u_j, $j = 1, \ldots, N$;

$$\sum_{j=1}^{N} \left[\int_0^1 \chi_j'(x)\chi_m'(x)\,\mathrm{d}x - k^2 \int_0^1 \chi_j(x)\chi_m(x)\,\mathrm{d}x \right] u_j - \mathrm{i}k u_N \chi_m(1) = \int_0^1 f(x)\chi_m(x)\,\mathrm{d}x, \tag{2.9}$$

for $m = 1, \ldots, N$.

In order to set up the linear system to solve on a computer, we then need to determine the *coefficient matrix*, by evaluating for all $j, m = 1, \ldots, N$ each term in (2.9). It is a simple exercise in integration (see problem sheet) to show that

$$\int_0^1 \chi_j'(x)\chi_m'(x)\,\mathrm{d}x = \begin{cases} 0, & \text{if } |j-m| > 1, \\ -1/h, & \text{if } |j-m| = 1, \\ 2/h, & \text{if } j = m \neq N, \\ 1/h, & \text{if } j = m = N, \end{cases}$$

$$\int_0^1 \chi_j(x)\chi_m(x)\,\mathrm{d}x = \begin{cases} 0, & \text{if } |j-m|>1, \\ h/6, & \text{if } |j-m|=1, \\ 2h/3, & \text{if } j=m\neq N, \\ h/3, & \text{if } j=m=N. \end{cases}$$

Noting also that

$$\chi_m(1) = \begin{cases} 0, & \text{if } m\neq N, \\ 1, & \text{if } m=N, \end{cases}$$

the linear system is then

$$(A - k^2 B - \mathrm{i}kC)\mathbf{u} = \mathbf{f}, \tag{2.10}$$

where

$$A := \begin{bmatrix} \frac{2}{h} & -\frac{1}{h} & 0 & \ddots & 0 \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \ddots & 0 \\ 0 & -\frac{1}{h} & \frac{2}{h} & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & -\frac{1}{h} \\ 0 & 0 & 0 & -\frac{1}{h} & \frac{1}{h} \end{bmatrix}, \; B := \begin{bmatrix} \frac{2h}{3} & \frac{h}{6} & 0 & \ddots & 0 \\ \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & \ddots & 0 \\ 0 & \frac{h}{6} & \frac{2h}{3} & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & \frac{h}{6} \\ 0 & 0 & 0 & \frac{h}{6} & \frac{h}{3} \end{bmatrix},$$

$$C := \begin{bmatrix} 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; \mathbf{u} := \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{bmatrix} \; \mathbf{f} := \begin{bmatrix} \int_0^1 f(x)\chi_1(x)\,\mathrm{d}x \\ \int_0^1 f(x)\chi_2(x)\,\mathrm{d}x \\ \int_0^1 f(x)\chi_3(x)\,\mathrm{d}x \\ \vdots \\ \int_0^1 f(x)\chi_N(x)\,\mathrm{d}x \end{bmatrix}.$$

Immediately we see that the matrix to be inverted is *sparse* - almost all entries are zero. This is in marked contrast to the *boundary element method*, for which the matrix to be inverted is smaller in size, but *dense*, i.e. most entries are nonzero. We also remark that the finite element matrix $A - k^2 B - \mathrm{i}kC$ is *tridiagonal* - all entries are zero except for those on the main diagonal, and on the diagonal either side of the main diagonal. The *bandwidth* of the matrix, defined as the width of the band of nonzero entries, is equal to three. Very efficient schemes exist for inverting sparse matrices, and in particular matrices with a small bandwidth.

To compute our approximation U to u all that remains is to solve the linear system (2.10), and then having computed the coefficients u_j, $j = 1, \ldots, N$ we can use the formula (2.7).

2.3 Error estimates

When solving second order elliptic partial differential equations such as (2.1) using a Galerkin finite element method, such as (2.6), it is often possible to prove an error estimate of the form

$$\frac{\|u - U\|}{\|u\|} \leq Ch^2, \tag{2.11}$$

for h sufficiently small, where the constant C is independent of h (see [14, p.137] for the derivation of such an estimate for the problem (2.1)–(2.3)). This is known as an *asymptotic error estimate* because of the condition that it holds only for h sufficiently small (equivalently for N sufficiently large). We know that if we keep taking N to be larger and larger then eventually we will achieve a small error, but the question of exactly how large N has to be to achieve a certain prescribed error (e.g. 1%) is not always clear.

However, for the Helmholtz problem this is not the whole story. The constant C in (2.11) will depend on a number of factors - it may depend on f, and it may depend on the exact solution u, but most importantly it will depend on the wavenumber k. The wavenumber k is proportional to the frequency of the incident wave, and thus represents the oscillatory nature of the exact solution. The larger k is, the bigger the oscillations in the exact solution. Note that $\mathrm{e}^{\mathrm{i}kx}$ is an elementary solution of the Helmholtz equation in one dimension, and is periodic with period $\lambda := 2\pi/k$, thus the period decreases as k increases.

This has to be resolved by the numerical model by using a fixed number of elements per wavelength. If an insufficient number of elements is used, the wave will not be well modelled (see figure 2.2), and the "rule of thumb" in the literature (see e.g. [14, 24]) is that ten elements per wavelength are required. As $k \to \infty$, the number of elements required to maintain accuracy thus grows at least linearly with respect to k in one dimension, with the cost growing at a faster rate in higher dimensions, and this leads to prohibitive computational cost for large values of k.

However, when k is very large this approach alone is not sufficient, due to pollution errors (see e.g. [14, 4]). These arise due to the wavelength not being modelled exactly, and propagate through the numerical solution (see figure 2.3).

In particular, for the exact problem we have discussed here (2.1)–(2.3), Ihlenburg shows [14, p.127] that the relative error satisfies the bound

$$\frac{\|(u - U)'\|}{\|u'\|} \leq C_1 hk + C_2 k^3 h^2, \tag{2.12}$$

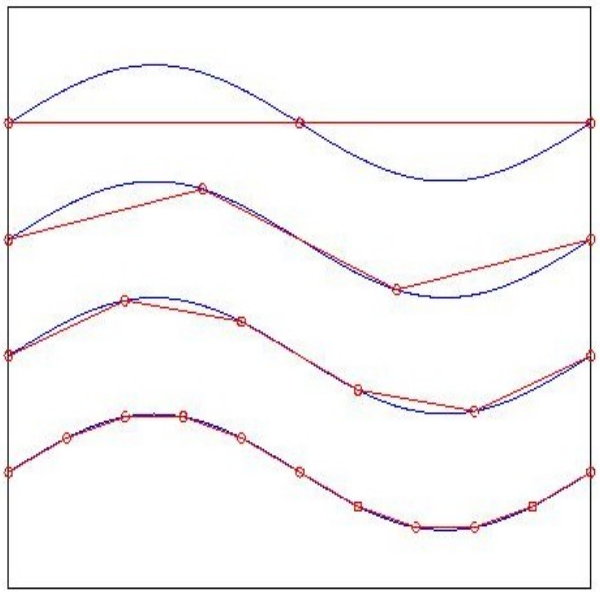

Figure 2.2: Modelling a single wave with 2, 3, 5, 10 elements

where the constants C_1 and C_2 are independent of both h and k. Thus in our numerical approximation, if we fix hk to be a constant then the error will grow as $k \to \infty$, due to the second term on the right hand side of (2.12).

As a result of this, if we want to fix the error to be constant for all values of k then we need to increase N in such a way that h^2k^3 remains constant, and this will lead to significantly increased computing times when k is large. We discuss some recent research in this area, and some possible schemes for reducing the computational cost when k is large, in §3.2.

2.4 Higher dimensions

When we move to higher dimensions the formulation of the finite element method becomes a little more complicated. For simplicity, we proceed by demonstrating the application of the finite element method to a simple two dimensional problem. The extension to three dimensional problems follows in a similar way.

Consider then

$$\Delta u + k^2 u = 0, \quad \text{in } \Omega, \tag{2.13}$$

$$\frac{\partial u}{\partial n} + \beta u = g, \quad \text{on } \Gamma, \tag{2.14}$$

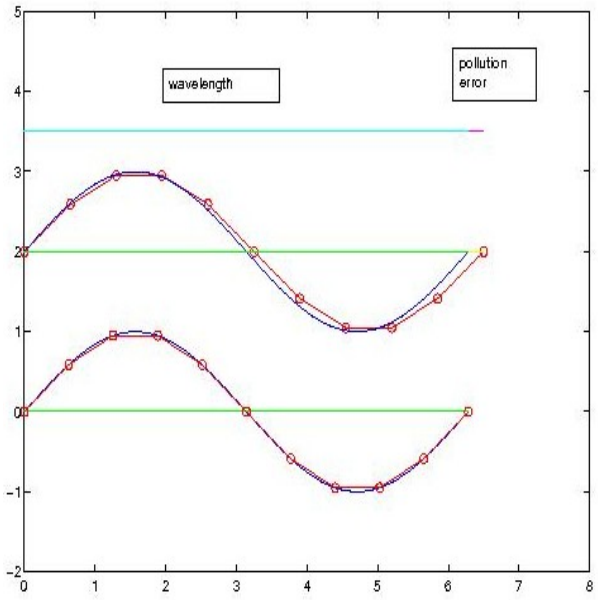

Figure 2.3: Pollution error

where $k \in \mathbb{R}$, $\beta \in \mathbb{C}$ are constants, $\partial/\partial n$ denotes the outward normal derivative, and $\Omega \in \mathbb{R}^2$ is a bounded domain with boundary Γ. Multiplying (2.13) by a test function $v \in H^1(\Omega)$ and integrating we get

$$\int_\Omega v\Delta u \, \mathrm{d}x + \int_\Omega k^2 uv \, \mathrm{d}x = 0. \tag{2.15}$$

Applying the divergence theorem (see e.g. [19, p.26]) we get

$$\int_\Omega v\Delta u \, \mathrm{d}x = \int_\Gamma v\frac{\partial u}{\partial n} \, \mathrm{d}s - \int_\Omega \nabla v.\nabla u \, \mathrm{d}x,$$

where $\mathrm{d}x$ is the element of area in $\mathbb{R}^2$ and $\mathrm{d}s$ is the element of arc length on Γ. Substituting into (2.15) and recalling the boundary conditions (2.14) we have the weak formulation;
find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u.\nabla v - k^2 uv \, \mathrm{d}x + \beta \int_\Gamma uv \, \mathrm{d}s = \int_\Gamma vg \, \mathrm{d}s, \quad \text{for all } v \in H^1(\Omega). \tag{2.16}$$

As for the one dimensional problem the Galerkin finite element method then consists of replacing the space $H^1(\Omega)$ in (2.16) with a finite dimensional

approximation space $V \subset H^1(\Omega)$, and our approximation $U \in V$ to $u \in H^1(\Omega)$ is then defined by;

$$\int_\Omega \nabla U.\nabla v - k^2 U v \,\mathrm{d}x + \beta \int_\Gamma U v \,\mathrm{d}s = \int_\Gamma v g \,\mathrm{d}s, \quad \text{for all } v \in V. \tag{2.17}$$

Defining V to be the linear span of the basis functions χ_j, $j = 1, \ldots, N$, we can then write $U \in V$ as

$$U(x) := \sum_{j=1}^{N} u_j \chi_j(x),$$

where the coefficients u_j are to be determined, and substituting into (2.17) we have

$$\sum_{j=1}^{N} \left[\int_\Omega \nabla \chi_j.\nabla v - k^2 \chi_j v \,\mathrm{d}x + \beta \int_\Gamma \chi_j v \,\mathrm{d}s \right] u_j = \int_\Gamma v g \,\mathrm{d}s, \quad \text{for all } v \in V. \tag{2.18}$$

Since this equation holds for all $v \in V$ it must hold in particular for $v = \chi_m$, $m = 1, \ldots, N$, and thus we have the linear system

$$\sum_{j=1}^{N} \left[\int_\Omega \nabla \chi_j.\nabla \chi_m - k^2 \chi_j \chi_m \,\mathrm{d}x + \beta \int_\Gamma \chi_j \chi_m \,\mathrm{d}s \right] u_j = \int_\Gamma \chi_m g \,\mathrm{d}s, \tag{2.19}$$

for $m = 1, \ldots, N$, or equivalently

$$A\mathbf{u} = \mathbf{f},$$

where

$$A := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \vdots & a_{2N} \\ a_{31} & a_{32} & a_{33} & \vdots & a_{3N} \\ \ldots & \ldots & \ldots & \ddots & \ldots \\ a_{N1} & a_{N2} & a_{N3} & \vdots & a_{NN} \end{bmatrix}, \ \mathbf{u} := \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{bmatrix} \ \mathbf{f} := \begin{bmatrix} \int_\Gamma \chi_1 g \,\mathrm{d}s \\ \int_\Gamma \chi_2 g \,\mathrm{d}s \\ \int_\Gamma \chi_3 g \,\mathrm{d}s \\ \vdots \\ \int_\Gamma \chi_N g \,\mathrm{d}s \end{bmatrix},$$

with the matrix entries a_{jm} defined by

$$a_{jm} := \int_\Omega \nabla \chi_j.\nabla \chi_m - k^2 \chi_j \chi_m \,\mathrm{d}x + \beta \int_\Gamma \chi_j \chi_m \,\mathrm{d}s.$$

As for the one dimensional problem we can then compute U in three steps;

1. Compute each of the matrix entries a_{jm} and each of the right hand side entries $\int_\Gamma \chi_m g \, \mathrm{d}s$.

2. Solve the linear system $A\mathbf{u} = \mathbf{f}$.

3. Form our approximation $U(x) = \sum_{j=1}^{N} u_j \chi_j(x)$.

There are three main difficulties associated with moving to higher dimensions.

2.4.1 Size of the linear system

As the dimension grows, so does the size of the linear system needed to achieve a prescribed level of accuracy. For example, suppose we seek a solution accurate to 1%, and we know that the error is bounded by h^2, with $h := 1/N$ and N the number of degrees of freedom in each direction. To achieve 1% accuracy we would thus need to choose $N = 10$, giving an error of $1/N^2 = 1\%$. So in one dimension, we would need 10 elements, giving a matrix of size 10×10. However, in two dimensions choosing $N = 10$ gives 100 elements, and the matrix is then of size 100×100. In three dimensions the matrix would be of size 1000×1000. So the size of the linear system grows as the dimension increases. In practice, for practical problems we would need to take N to be a great deal larger than 10 in order to achieve 1% accuracy and this often leads to impractically large systems. Often it is not just the solution of these systems that causes problems, even storing them can become impossible.

In order to solve the very large systems, it will not usually be possible to do a direct solve - iterative approaches are needed. These will be faster if the *bandwidth* of the system is smaller - the bandwidth is the width of the band of nonzero diagonals. As we saw in §2.2, in one dimension the matrix has a bandwidth equal to 3 - this is no problem. However, in two dimensions the bandwidth is of order N - although there are only maybe 5 or 9 nonzero diagonals, the furthest of these from the main diagonal will be a distance N away. In three dimensions, the bandwidth is of order N^2. This means that the cost of achieving an accurate iterative solve grows with dimension, and everything becomes more expensive.

For further details on iterative solution of large sparse linear systems, we refer to [26, 13] and the references therein.

2.4.2 Mesh generation

In 1D mesh generation is very easy, as it is just a case of dividing a line up into sections.

In 2D things becomes a lot more difficult, especially if the geometry of the computational domain is complicated. However, for 2D mesh generation many good codes, both commercial and publicly available, can be used to generate meshes. A lot of these are based on the Delaunay triangulation algorithm, which has been shown to be very effective in 2D. For example, a mesh generation algorithm is available from
`http://www.cs.cmu.edu/~quake/triangle.html`
which can be used to generate meshes for a wide range of geometries in 2D, including the "snail's shell" type geometry shown in figure 2.4. This

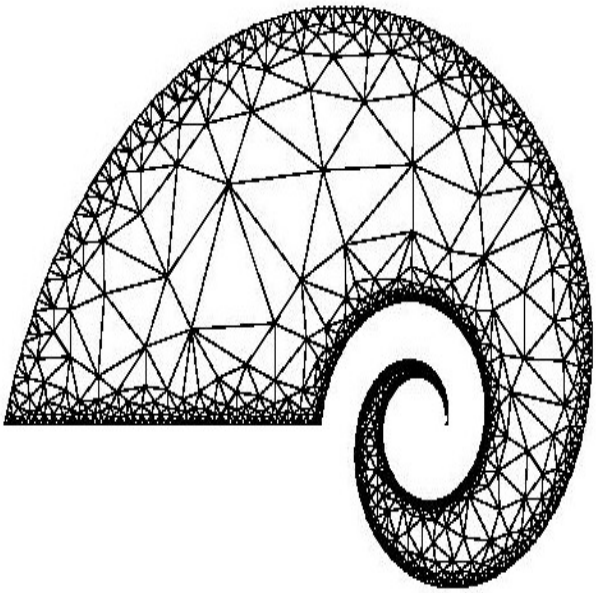

Figure 2.4: 2D mesh on a complicated geometry

type of geometry would be very hard to model by any means other than a triangulation of the type shown, and demonstrates the versatility of the finite element method.

In 3D, things get much more difficult again. The Delaunay triangulation is not particularly effective for 3D mesh generation, and in fact the success of most 3D algorithms is measured not by their good performance but rather by the percentage of elements generated that have big problems, e.g. negative volume. However, some excellent codes are available, such as NETGEN,

downloadable from
http://www.hpfem.jku.at/netgen,
which was used to generate the meshes in figure 2.5 and 2.6.

There are many resources on the web dealing with mesh generation. Links to many of the people working in this area, and many publicly available and commercial codes for mesh generation can be found on some of the following webpages;
http://www-users.informatik.rwth-aachen.de/...
~roberts/meshgeneration.html
http://www.engr.usask.ca/~macphed/finite/fe_resources/mesh.html
http://www.andrew.cmu.edu/user/sowen/mesh.html

2.4.3 Design of the approximation space

Rather than using piecewise polynomial basis functions, many schemes such as the *generalised finite element method* (see e.g. [2]) use basis functions chosen specifically to model the behaviour of the solution.

For one dimensional problems it is straightforward to get a good handle on the behaviour of the solution. Plane waves can only travel in two directions, so they can easily be incorporated into the approximation space in order to improve the accuracy of the scheme at large wavenumbers (see also §3.2).

In two dimensions, plane waves can travel in all directions on a plane, making the behaviour much harder to model, and in three dimensions the number of possible directions increases again. Thus the design of an appropriate approximation space becomes much more difficult as the dimension of the problem increases.

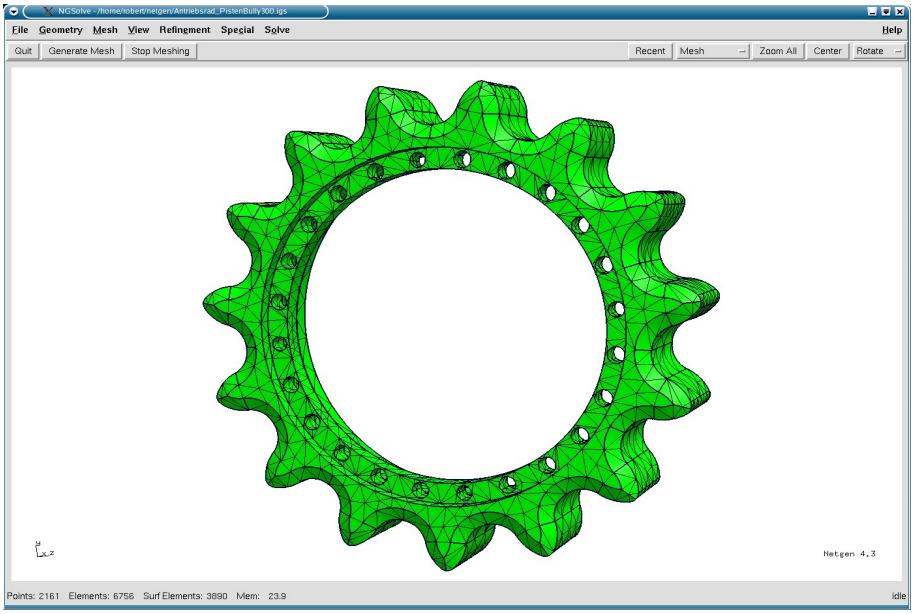

Figure 2.5: 3D mesh

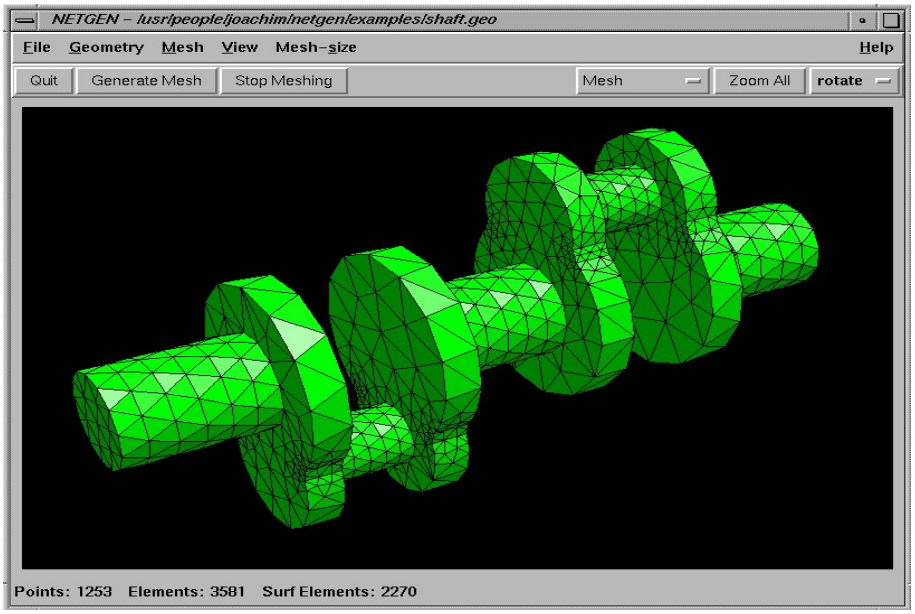

Figure 2.6: 3D mesh

Chapter 3

Current research on finite element methods for acoustics

Although finite element methods have been around for a considerable time, there is still much work to be done on their application to acoustic scattering problems, which present a unique set of difficulties. Here, we focus particularly on two of these.

Firstly, we remark that the finite element method was originally developed for the numerical solution of problems on bounded domains. However, often in acoustic scattering applications the computational domain may be unbounded. In this case, there is an immediate difficulty - how do we discretise an infinite domain? This question is addressed in §3.1.

A second difficulty is that when the wavenumber k becomes large, the accuracy of the "standard" finite element method deteriorates, as alluded to already in §2.3. Various techniques have been developed to get around this difficulty, and these are discussed in §3.2.

3.1 Unbounded domains

In the event that D is an unbounded domain, as is often the case for scattering problems, we also need to supplement (1.3) with a Sommerfeld radiation condition to ensure uniqueness of solution. This corresponds to imposing a condition that no waves are reflected from ∞. So that standing waves cannot occur, we force

$$\frac{1}{R^{(d-1)/2}}\left(\frac{\partial u}{\partial R} - \mathrm{i}ku\right) \to 0, \quad \text{as } R \to \infty, \quad d = 1, 2, 3. \tag{3.1}$$

Solutions of exterior Helmholtz problems that also satisfy (3.1) are known as *radiating* solutions.

There is though a further difficulty. Clearly we cannot discretise an unbounded domain with finite elements. To get around this, many clever schemes have been suggested for the application of finite element methods to unbounded domains, with the big question being "what to do at infinity"?

One approach is to replace the unbounded domain with a bounded one, by introducing an artificial boundary as shown in figure 3.1. The finite

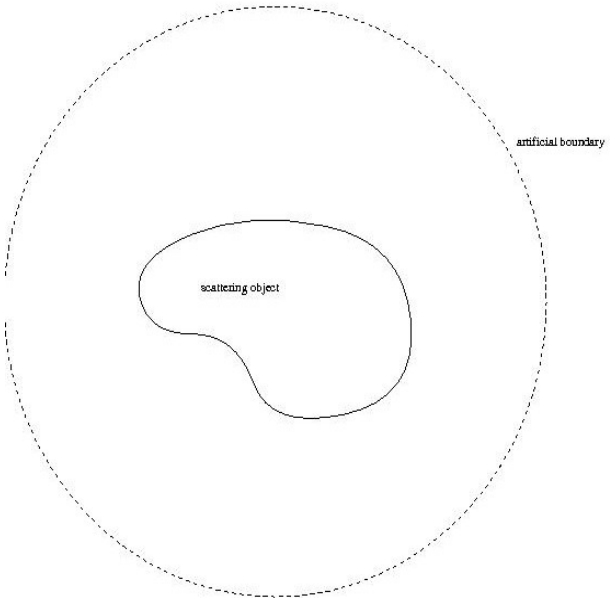

Figure 3.1: Artificial boundary to replace a problem on an unbounded domain with one on a bounded domain

element discretisation of the region exterior to the scatterer is then carried out only in the (small) annular domain enclosing the scatterer. In this case a problem on an unbounded domain has been replaced with one on a bounded domain, the difficulty comes in choosing the boundary conditions on the artificial boundary in such a way that the solution of the modified problem is a sufficiently close approximation to the solution of the original problem.

The main tool in the choice of this boundary condition is a coupling of the finite element solution to some discrete representation of the analytical solution, with the *absorbing boundary condition* (ABC) chosen in such a way that there is no reflection of scattered waves (see for example [14, chapter 3]).

To find the behaviour in the region exterior to the artificial boundary, one can either use integral representations (see e.g. [10, 11]), or separation of variables, looking for solutions in the form of plane wave solutions (see e.g. [14, §2.1]).

For example, consider again the one dimensional problem (2.1) of §2.2, and suppose now that we wish to solve this problem on an unbounded domain. The exact solution of (2.1) is given by

$$u(x) = A\mathrm{e}^{\mathrm{i}kx} + B\mathrm{e}^{-\mathrm{i}kx},$$

with the constants A and B to be determined by the boundary data. The corresponding time-dependent solution is then

$$P(x,t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} + B\mathrm{e}^{-\mathrm{i}(kx+\omega t)},$$

where the first term on the right hand side represents the outgoing wave, travelling from 0 to ∞, and the second term on the right hand side represents the incoming wave, travelling from ∞ to 0. Applying at any point $x = x_0$ the boundary condition

$$\frac{\omega}{k}\frac{\partial P}{\partial x}(x_0) + \frac{\partial P}{\partial t}(x_0) = 0$$

eliminates the incoming wave. This is thus a *nonreflecting boundary condition* (NRBC) at x_0 (see problem sheet).

In higher dimensions, the plane waves $\mathrm{e}^{\mathrm{i}k\mathbf{x}.\mathbf{d}}$ are particular solutions of the two or three dimensional Helmholtz equation, where the direction vector $\mathbf{d}$ represents a particular direction in which the plane wave is travelling. For example, for $d = 3$ the plane wave $\mathrm{e}^{\mathrm{i}k(\alpha x+\beta y+\gamma z)}$ solves (1.3) provided $\alpha^2 + \beta^2 + \gamma^2 = k^2$. The difficulty here comes in determining α, β, γ; if we know the directions of the plane waves in the exact solution then a NRBC can be deduced for the higher order problem. However, in general the directions are not known, and thus instead one has to construct an ABC as an approximation to the NRBC.

Other techniques for solving problems on exterior domains include the use of *Perfectly Matched Layers* (see e.g. [5]), or infinite elements (see e.g. [1, 6, 12]). For a full review of the many schemes available we refer to [14, chapter 3].

3.2 Large wavenumbers

We have already discussed some of the difficulties encountered when k is large in §2.3.

Various approaches have recently been developed to get around these difficulties. Rather than using piecewise linear basis functions, using higher order piecewise polynomials (the *hp* approach) can lead to a big improvement

in the accuracy of the method (see e.g. [15, 16, 25]). In particular, if the approximation space consists of piecewise polynomials of order p, then we can replace the error estimate (2.12) with an estimate of the form

$$\frac{\|(u-U)'\|}{\|u'\|} \leq C_1 \left(\frac{hk}{2p}\right)^p + C_2 k \left(\frac{hk}{2p}\right)^{2p}, \tag{3.2}$$

(see [14, p.154]). As for the h-version of this estimate, the first term represents the *approximation error* and the second term the *pollution error*. Note that taking $p = 1$ gives (2.12) again. Using an hp approach thus significantly improves the accuracy of the finite element method - increasing p can lead to a dramatic reduction in the pollution error without needing to decrease h as significantly as for the h version.

Another approach is to use basis functions that are specifically tailored to the problem of high frequency acoustic scattering. This is the idea behind the *generalised finite element method* and the *partition of unity method* (see e.g. [3, 22]), and has been applied to great effect by Bettess et. al. [21, 24, 20] using *plane wave* basis functions. These basis functions take the form $e^{ik\mathbf{x}.\mathbf{d}}$, where $\mathbf{d}$ is a unit vector. Including many such basis functions in the approximation space, with many direction vectors $\mathbf{d}$, can lead to dramatically improved performance of the method, with a reduction in the number of elements required per wavelength from ten to two.

A further difficulty in the case that k is large is that the integrals to be evaluated in order to set up the linear system will be highly oscillatory. Computing these integrals may become more expensive as k increases. Various schemes have recently been developed for the efficient evaluation of highly oscillatory integrals (see e.g. [17, 18]), but this issue is still not fully resolved.

Chapter 4

Conclusions and further reading

In these notes we have attempted to provide a brief introduction to the finite element method, and its use in problems of acoustic scattering. The particular difficulties inherent in this problem, chief amongst them the oscillatory nature of the solution, mean that a naive application of standard schemes may give poor results. We have thus attempted also to explain why using a standard scheme with a piecewise linear approximation space may perform poorly when the frequency is large, and to give a short summary of some alternative approaches which may lead to improved performance.

The main reference we have used in writing these lecture notes is the excellent book by Ihlenburg [14], who deals specifically with the application of the finite element method to problems of acoustic scattering. In [23], Monk deals with the application and analysis of finite element methods to problems of electromagnetic scattering, which share many features and difficulties with acoustic problems.

In addition, there are many excellent books such as [19, 9, 25, 7, 8] providing a clear introduction to the finite element method and its analysis, treating the subject with a far greater mathematical rigour than we have attempted here.

Bibliography

[1] R.J. Astley. Infinite elements for wave problems: A review of current formulations and an assessment of accuracy. *Int. Journ. Num. Meth. Engng.*, 49:951–976, 2000.

[2] I. Babuška, U. Banerjee, and J.E. Osborn. Generalized finite element methods - main ideas, results and perspective. *International Journla of Computational Methods*, 1:67–103, 2004.

[3] I Babuška and J.M. Melenk. The partition of unity method. *Int. J. Numer. Meth. Eng.*, 40:727–758, 1997.

[4] I Babuška and S Sauter. Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers. *SIAM J. Numer. Anal.*, 34:2392–2423, 1997. Reprinted in *SIAM Rev.* **42** (3), 451-484, 2000.

[5] J. Berenger. A perfectly matched layer for the absorption of electromagnetic waves. *J. Comput. Phys.*, 114:185–200, 1994.

[6] P. Bettess. *Infinite elements*. Penshaw Pres, Cleadon, 1992.

[7] D. Braess. *Finite Elements*. Cambridge University Press, Cambridge, 2nd edition, 2001.

[8] S.C. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer, New York, 1994.

[9] P.G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland Publishing Company, 1978.

[10] D Colton and R Kress. *Integral Equation Methods in Scattering Theory*. Wiley, 1983.

[11] D Colton and R Kress. *Inverse Acoustic and Electromagnetic Scattering Theory*. Springer Verlag, 1992.

[12] L. Demkowicz and K. Gerdes. Convergence of the infinite element methods for the Helmholtz equation in separable domains. *Numer. Math.*, 79:11–42, 1998.

[13] G.H. Golub and C.F. Van Loan. *Matrix Computations*. Johns Hopkins University Press, 3rd edition, 1996.

[14] F Ihlenburg. *Finite Element Analysis of Acoustic Scattering*. Springer-Verlag, New York, 1998.

[15] F Ihlenburg and I Babuška. Finite element solution of the Helmholtz equation with high wavenumber. 1. the h version of the FEM. *Comput. Math. Appl.*, 30(9):9–37, 1995.

[16] F Ihlenburg and I Babuška. Finite element solution of the Helmholtz equation with high wavenumber. 2. the $h-p$ version of the FEM. *SIAM J. Numer. Anal.*, 34(1):315–358, 1997.

[17] A. Iserles. On the numerical quadrature of highly-oscillating integrals I: Fourier transforms. *IMA J. Numer. Anal.*, 24:365–391, 2004.

[18] A. Iserles. On the numerical quadrature of highly-oscillating integrals II: Irregular oscillations. *IMA J. Numer. Anal.*, 25:25–44, 2005.

[19] C. Johnson. *Numerical Solution of Partial Differential Equations by the Finite Element Method*. Cambridge University Press, 1987.

[20] O Laghrouche, P Bettess, and R.J. Astley. Modelling of short wave diffraction problems using approximating systems of plane waves. *Int. J. Numer. Meth. Eng.*, 54:1501–1533, 2002.

[21] O. Laghrouche, P. Bettess, E. Perrey-Debain, and J. Trevelyan. Wave interpolation finite elements for Helmholtz problems with jumps in the wave speed. *Comp. Meth. Appl. Mech. Eng.*, 194:367–381, 2005.

[22] J.M. Melenk and I Babuška. The partition of unity finite element method: Basic theory and applications. *Comput. Method Appl. M.*, 139:289–314, 1996.

[23] P. Monk. *Finite Element Methods for Maxwell's Equations*. Oxford Science Publications, 2003.

[24] E. Perrey-Debain, O. Lagrouche, P. Bettess, and J. Trevelyan. Plane-wave basis finite elements and boundary elements for three-dimensional wave scattering. *Phil. Trans. R. Soc. Lond. A*, 362:561–577, 2004.

[25] B.S. Szabó and I. Babuška. *Finite Element Analysis*. J. Wiley, 1991.

[26] L.N. Trefethen and D. Bau. *Numerical Linear Algebra*. SIAM, 1997.