

Boundary element methods for acoustics

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Abstract

In these lecture notes we introduce the boundary element method and describe how it can be used to approximate the solution to time harmonic acoustic problems. We also highlight some of the difficulties involved in applying the method, and give references to some current research aimed at resolving these issues.

Contents

1	Introduction	2
2	Boundary Integral Equations	7
2.1	What are integral equations?	7
2.1.1	The classification of integral equations	9
2.2	Fundamental Solutions in Acoustics	10
2.3	Direct Boundary Integral Equations in Acoustics	12
2.3.1	An integral equation for an interior problem	12
2.3.2	Other boundary conditions and exterior problems	18
2.3.3	The 2D Case	21
2.4	Indirect Boundary Integral Equations	22
3	Simple Boundary Element Methods for Acoustics Problems	26
3.1	Computational Cost	29
4	More Sophisticated Ideas and Further Reading	32

Chapter 1

Introduction

Many physicists and engineers are interested in the reliable simulation of processes in which acoustic waves are scattered by obstacles, with applications arising in areas as diverse as sonar (see figure 1.1), road, rail or aircraft noise, or building acoustics. Unless the geometry of the scattering object is

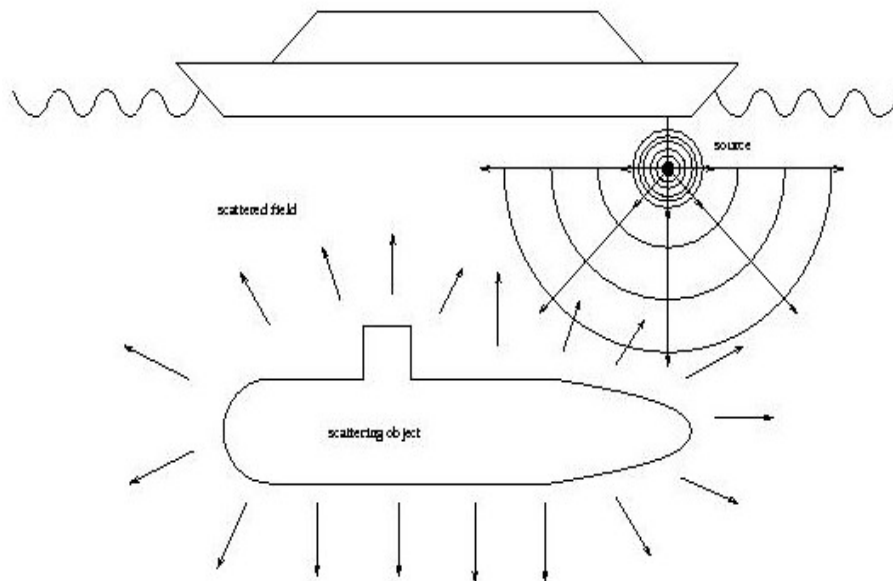


Figure 1.1: Typical acoustic scattering problem

particularly simple, the analytical solution of scattering problems is usually impossible, and hence numerical schemes are required.

Throughout these notes $P(\mathbf{x}, t)$ will denote the pressure at time t at the point whose position vector is \mathbf{x} . We will use Cartesian coordinates (Oxy) for

2D problems, $Oxyz$ for 3D problems). Thus, in 3D problems, \mathbf{x} will be the vector $\mathbf{x} = (x, y, z)$, with x, y, z the three components of \mathbf{x} . In 2D problems $\mathbf{x} = (x, y)$ will have just two components.

In a homogeneous medium at rest the function P satisfies the wave equation

$$\Delta P - \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} = 0, \quad (1.1)$$

where c is the speed of sound and

$$\Delta = \nabla^2$$

is a shorthand for the Laplacian (e.g. $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in 2D). In these notes we will consider, for simplicity, only the (very important!) case of time harmonic ($e^{-i\omega t}$ time dependence¹) acoustic propagation and scattering. (Of course, more general variations as a function of time can be obtained by Fourier synthesis, by combining harmonic time dependences for different frequencies.) Thus the pressure is given by

$$P(\mathbf{x}, t) = \Re (u(\mathbf{x})e^{-i\omega t}), \quad (1.2)$$

where $\omega = 2\pi f$ is the angular frequency, f the frequency (measured in Hz), $i = \sqrt{-1}$, and \Re denotes the real part. The function u , which is complex-valued in general, we will call the *complex acoustic pressure* (but often just the *pressure* for short). Note that we can write (1.2) more explicitly as

$$P(\mathbf{x}, t) = A(\mathbf{x}) \cos(\varphi(\mathbf{x}) - \omega t), \quad (1.3)$$

where $A(\mathbf{x}) = |u(\mathbf{x})|$, $\varphi(\mathbf{x}) = \arg u(\mathbf{x})$, making clear the physical interpretation of u , that the modulus of $u(\mathbf{x})$, $|u(\mathbf{x})|$, is the amplitude of the time harmonic pressure fluctuation at \mathbf{x} , while $\arg u(\mathbf{x})$ determines the phase of the oscillation at \mathbf{x} .

Frequently we are interested in Sound Pressure Level predictions. Since the root mean square of a time harmonic field is its amplitude divided by $\sqrt{2}$, the SPL at \mathbf{x} is given by

$$SPL = 20 \log_{10} \left(\frac{|u(\mathbf{x})|}{\sqrt{2} u_{ref}} \right) \text{ dB},$$

¹Our assumed time dependence is that used in [19]. Assuming time dependence $e^{i\omega t}$ is (perhaps unfortunately!) an equally valid convention. For $e^{i\omega t}$ time dependence, used in some publications, all the formulas in these notes remain valid as long as one replaces all complex numbers by their complex conjugate, in particular replaces each i by $-i$.

where u_{ref} is the usual reference pressure. An important point for numerical calculation in general is that accurate prediction of *SPL* requires small *relative* errors in the computation of $|u|$. Of course, this then implies very small *absolute* errors at points \mathbf{x} where $|u(\mathbf{x})|$ is small. For many applications such regions may be very important, for example if one is calculating the shielding performance of a noise barrier, when one is interested in accurate predictions (with small dB error and so small relative error) in the shadow zone. Thus very accurate numerical methods are of interest for a number of acoustic applications.

Substituting (1.2) into (1.1), we see that u satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0, \quad \text{in } D \subset \mathbb{R}^d, \quad (1.4)$$

where $d = 1, 2$ or 3 is the dimension of the problem we are considering, D denotes the *domain of propagation*, the region in which the wave propagates, which is either a subset of the plane (\mathbb{R}^2) if we are solving a 2D problem, or is a subset of \mathbb{R}^3 if we are solving a fully 3D problem. (Occasionally, especially for instructional purposes, we wish to consider also 1D problems, in which case the domain D is a subset of \mathbb{R} , the real line, i.e. D is an interval of the form (a, b) with $a < b$.) The positive constant k is the *wave number*, given by

$$k := \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}.$$

Here we have introduced $\lambda = c/f$, the wavelength of plane waves of frequency f . Clearly, k is proportional to the frequency and inversely proportional to λ , with SI units m^{-1} .

The rest of these notes will discuss how to compute, by the boundary element method, solutions to (1.4) that also satisfy physically relevant boundary conditions on the boundary of the domain of propagation. We denote the boundary of D by ∂D and will focus on the most commonly relevant boundary condition, namely the *impedance boundary condition*

$$\frac{\partial u}{\partial n} + ik\beta u = g, \quad \text{on } \partial D. \quad (1.5)$$

Let us spend a few moments explaining this boundary condition. First of all, in this equation, and throughout, $\partial/\partial n$ denotes the normal derivative on the boundary, i.e. the rate of increase in the direction \mathbf{n} , where $\mathbf{n}(\mathbf{x})$ denotes the unit normal at $\mathbf{x} \in \partial D$, **directed into**² D . Explicitly, in terms of the

²In writing about and coding the boundary element method one has to take great care about directions of normals. Many authors will take the unit normal in the opposite direction, which changes the sign of the normal derivative.

gradient of u ,

$$\frac{\partial u}{\partial n}(\mathbf{x}) = \mathbf{n}(\mathbf{x}) \cdot \nabla u(\mathbf{x}), \quad (1.6)$$

i.e. the normal derivative is the scalar product of the gradient and the unit normal.

The function g on the right hand side of the equation is identically zero in acoustic scattering problems (problems where we are given an incident wave and a stationary scatterer and have to compute the resulting acoustic field), but is non-zero for radiation problems (where the motion of a radiating structure is given and we have to calculate the acoustic field radiated).

In (1.5) β is the relative surface admittance which, in general, is a function of position on the boundary (and also a function of frequency). The simplest case is when the boundary is *acoustically rigid* or *sound hard*. This is the case when no flow is possible across ∂D and $\beta = 0$ so that (1.5) simplifies to the so-called *sound hard* or *Neumann* boundary condition

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial D. \quad (1.7)$$

More generally, β may be non-zero, its value at position \mathbf{x} on ∂D given by

$$\beta(\mathbf{x}) := \frac{Z_0}{Z_s(\mathbf{x})}$$

where $Z_0 = \rho c$ is the impedance of the medium of propagation (air, water, etc.), ρ its density, and $Z_s(\mathbf{x})$ is the *surface impedance* at \mathbf{x} . The surface impedance is defined by the equation

$$Z_s(\mathbf{x}) = \frac{u(\mathbf{x})}{-\mathbf{v}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})}, \quad \text{for } \mathbf{x} \in \partial D,$$

where $\mathbf{v}(\mathbf{x})$ is the velocity at \mathbf{x} due to the acoustic field. Thus Z_s is the ratio on the surface of the pressure to the normal velocity into the surface. The impedance boundary condition is appropriate whenever, to a good approximation, this ratio is independent of the acoustic field. This is the case for sound hard surfaces (where the ratio is always ∞), for many naturally occurring surfaces in outdoor noise propagation, and for many man-made acoustically absorbing surfaces at lower frequencies. For more details see Chapter 8 of [19] by Keith Attenborough.

In the case when the domain D is unbounded, the complete mathematical formulation of the problem has to include some condition which encapsulates, in a mathematical way, the idea that the acoustic field, or at least some part of it (e.g. the part which is reflected from the scattering obstacle), is travelling

outwards, towards infinity. The usual conditions imposed are the so-called *Sommerfeld radiation conditions*, that

$$u(\mathbf{x}) = O(r^{-(d-1)/2}), \quad (1.8)$$

$$\frac{\partial u}{\partial r}(\mathbf{x}) - iku(\mathbf{x}) = o(r^{-(d-1)/2}), \quad (1.9)$$

as $r \rightarrow \infty$. In this equation d is the dimension (2 or 3) and r is the radial direction, precisely $r = |\mathbf{x}|$, the distance of \mathbf{x} from the origin, so that, in terms of the gradient of u ,

$$\frac{\partial u}{\partial r}(\mathbf{x}) = \mathbf{r} \cdot \nabla u(\mathbf{x}),$$

where $\mathbf{r} = \mathbf{x}/|\mathbf{x}|$ is a unit vector in the direction of \mathbf{x} . The ‘big O’ and ‘little o’ notations in (1.8) and (1.9) have the following meanings: equation (1.8) says that the pressure, $u(\mathbf{x})$, must decrease, as we go to infinity, *at least as fast as* $r^{-(d-1)/2}$; equation (1.9) says that the left hand side of this equation, namely $\partial u/\partial r - iku$ must decrease *faster than* $r^{-(d-1)/2}$.

The physical basis of these conditions is as follows. The first condition implies that $|u|^2$ decreases like r^{-1} in 2D, like r^{-2} in 3D. But this is exactly what one expects from energy considerations: the energy is spread over a ever larger and larger cylinder of circumference $2\pi r$ in 2D, is spread over the surface of a sphere of radius $4\pi r^2$ in 3D. The second condition says that $\partial u/\partial r - iku$ should be much smaller than $r^{-(d-1)/2}$, and so much smaller than u , when r is large. This makes sense as far away the wave travelling outwards appears locally like a plane wave travelling in the direction \mathbf{r} , i.e. it has the form

$$u(\mathbf{x}) = Ae^{ikr}$$

where A is the local amplitude. But, for such an acoustics field it holds that $\partial u/\partial r - iku = 0$ exactly.

Chapter 2

Boundary Integral Equations

In the introduction we have formulated the problem we are going to study in mathematical terms as what is called a *boundary value problem*, i.e. a partial differential equation (the Helmholtz equation) which is to be solved subject to the requirement that certain boundary conditions hold (e.g. the impedance boundary condition and, when the domain is unbounded, also the Sommerfeld radiation condition, which can be viewed as a boundary condition at infinity). The boundary element method is a numerical method for solving this problem but it is applied not to the problem directly, but to a reformulation of the problem as a *boundary integral equation*.

In a moment we will go ahead and reformulate our acoustic problem as a boundary integral equation. But, conscious that many students will not have met integral equations before in any context, we pause to briefly answer the question ...

2.1 What are integral equations?

Integral equations are a similar sort of thing to differential equations. In a differential equation, e.g.

$$\frac{d^2y(s)}{ds^2} + \frac{1}{s} \frac{dy(s)}{ds} + y(s) = 0, \quad (2.1)$$

there is an unknown function $y(s)$ to be found and the differential equation is so-called because derivatives of the unknown function appear in the equation. An *integral equation* is an equation in which the unknown function appears

under an integral sign. The following are examples of integral equations:

$$y(s) = s + \int_0^1 (s-t)y(t)dt, \quad (2.2)$$

$$y(s) = \sin s + \int_0^s (s-t)y(t)dt, \quad (2.3)$$

$$s^2 = \int_0^{2\pi} \ln |s-t|y(t)ds, \quad (2.4)$$

$$y(s) = e^s + \int_0^1 e^{s+t}y(t)dt, \quad (2.5)$$

$$y(s) = \int_0^1 e^{s+t}y(t)dt. \quad (2.6)$$

In each of the above equations y is the unknown function to be found, and each equation is an integral equation because the unknown function appears under the integral sign (as well as outside the integral sign in all except example (2.4)).

Most integral equations cannot be solved analytically (i.e. it is not possible to express the solution in terms of known functions). Several of the above examples are exceptions to this rule however. For example let us solve equation (2.5). Equation (2.5) can be written

$$y(s) = e^s + e^s \int_0^1 e^t y(t) dt. \quad (2.7)$$

Let p denote the value of the definite integral $\int_0^1 e^t y(t) dt$. Then (2.7) can be written as

$$y(s) = e^s(1 + p). \quad (2.8)$$

Multiplying (2.8) by e^s ,

$$\begin{aligned} e^s y(s) &= e^{2s}(1 + p) \\ \Rightarrow p &= \int_0^1 e^s y(s) ds = (1 + p) \int_0^1 e^{2s} ds = (1 + p) \frac{e^2 - 1}{2} \\ &\Leftrightarrow p = \frac{e^2 - 1}{3 - e^2}. \end{aligned} \quad (2.9)$$

Thus, substituting back in (2.8),

$$y(s) = \frac{2}{3 - e^2} e^s. \quad (2.10)$$

This is the unique solution of equation (2.5).

2.1.1 The classification of integral equations

One-dimensional/two-dimensional integral equations

In all the examples (2.2)-(2.6) given above the unknown function y is a function of just one variable and the integrals are single integrals. Thus the integral equations (2.2)-(2.6) are all classified as one-dimensional integral equations. The integral equation

$$f(x, y) = e^{xy} + \int_0^1 \int_0^1 \frac{f(\xi, \eta)}{\{(x - \xi)^2 + (y - \eta)^2\}^{1/2}} d\xi d\eta, \quad (2.11)$$

in which the unknown function f depends on two variables and the integral is a double integral, is a two-dimensional integral equation. (Similarly one defines 3-dimensional, 4-dimensional, etc. integral equations.) In our applications in acoustics we will meet **one-dimensional integral equations** when we tackle **2D** problems, and **two-dimensional integral equations** when we tackle **3D** problems.

Linear/nonlinear integral equations

The most general linear one-dimensional integral equation is the equation

$$\lambda y(s) = g(s) + \int_a^b K(s, t)y(t)dt, \quad a \leq s \leq b, \quad (2.12)$$

in which λ , a , b are known constants (with $-\infty \leq a < b \leq +\infty$), g and K are known functions, and y is the unknown function to be determined. All integral equations which do not fit the pattern (2.12) are termed nonlinear. All of equations (2.2)-(2.6) are linear and all of the boundary integral equations we will meet later in this chapter are linear.

Linear one-dimensional integral equations can further be classified as follows:

(i) If $g(x) \equiv 0$ then equation (2.12) is termed *homogeneous*, otherwise the equation is *inhomogeneous* (cf. the corresponding terminology for linear ordinary differential equations). Only equation (2.6) of (2.2)-(2.6) is homogeneous.

(ii) If $\lambda = 0$ then equation (2.12) is said to be a *first kind integral equation*. If $\lambda \neq 0$ then it is said to be a *second kind integral equation*. Equation (2.4) of (2.2)-(2.6) is the only first kind equation.

(iii) If a and b are finite, and either K is continuous or K is weakly singular, then an important theorem – Fredholm’s alternative – applies to

equation (2.12)¹. For 1D integral equations $K(s, t)$ is said to be *weakly singular* if it is continuous for $s, t \in [a, b]$, $s \neq t$, and if, for some constants $\alpha \in (0, 1)$, and $c > 0$, $|K(s, t)| \leq c|s - t|^{-\alpha}$ for $s \neq t$.

In addition to the above classification note that the function K in (2.12) is referred to as the *kernel* and that the function g is referred to as the *inhomogeneous* or *forcing* term.

2.2 Fundamental Solutions in Acoustics

The boundary element method can be viewed as some sort of half-way house between analytical and numerical methods. By this I mean that the method depends crucially on our ability first of all to obtain solutions, by analytical means, to certain simple specific acoustic problems.

Most crucially we need to be able to write down explicitly the acoustic field due to a point (time harmonic) source of sound in free space. If the source is at \mathbf{x}_0 then the acoustic pressure at \mathbf{x} is a multiple (depending on the strength and phase of the source) of a known function $G(\mathbf{x}, \mathbf{x}_0)$, of the two positions. That is, the acoustic field is

$$u(\mathbf{x}) = AG(\mathbf{x}, \mathbf{x}_0) \quad (2.13)$$

where A is a (real or complex) constant. Explicitly, the function G is defined in the 3D case by (see [19, Section 1.7.5])

$$G(\mathbf{x}, \mathbf{x}_0) := -\frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{4\pi|\mathbf{x}-\mathbf{x}_0|}, \quad \mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^3, \quad (2.14)$$

the factor 4π included in order to make G the field due to a source of *unit strength* in a certain sense². Note that this definition makes sense for all points $\mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^3$ with $\mathbf{x} \neq \mathbf{x}_0$, and, clearly, $G(\mathbf{x}, \mathbf{x}_0) \rightarrow \infty$ as $\mathbf{x} \rightarrow \mathbf{x}_0$, i.e. as the observation position \mathbf{x} approaches the source. When $r = |\mathbf{x}|$ is large this function looks locally like a plane wave of amplitude $1/(4\pi r)$ travelling in the direction r . A straightforward explicit calculation shows that u given by (2.13) satisfies the Sommerfeld radiation conditions (1.8) and (1.9).

These observations hold true also in the 2D case when the corresponding function G is given by

$$G(\mathbf{x}, \mathbf{x}_0) := -\frac{i}{4}H_0^{(1)}(k|\mathbf{x}-\mathbf{x}_0|), \quad \mathbf{x}, \mathbf{x}_0 \in \mathbb{R}^2. \quad (2.15)$$

¹Fredholm's alternative says that, if $\lambda \neq 0$, then the integral equation (2.12) has exactly one solution if and only if the corresponding homogeneous equation (i.e. (2.12) with $g \equiv 0$) only has the trivial solution $y \equiv 0$.

²In the sense that equation (154) in [19] is satisfied.

In physical terms the 2D solution of the Helmholtz equation given by (2.13) with G given by (2.15) is the field at \mathbf{x} due to a *coherent line source* of sound which is perpendicular to the 2D plane and passes through it at the point \mathbf{x}_0 . The solution is a little more complex than the 3D solution for a point source in that it involves $H_0^{(1)}$, which denotes the *Hankel function of the first kind of order zero*.

More generally $H_n^{(1)}$ denotes the *Hankel function of the first kind of order n* . The Hankel functions are a type of Bessel function; indeed $H_n^{(1)}$ is a solution of Bessel's equation of order n (for which see e.g. p. 130 in [19]). $H_n^{(1)}$ is defined in terms of the more standard Bessel function J_n and Y_n (see e.g. [19, p. 131]), by

$$H_n^{(1)}(z) = J_n(z) + iY_n(z).$$

Thus, for $z > 0$, J_n and Y_n are the real and imaginary parts of $H_n^{(1)}$. Graphs of J_0 and Y_0 (the real and imaginary parts of $H_0^{(1)}$) are shown as Fig. 6.2 in [19] and a lot of information about Bessel functions is given in [1, Chapter 9]. Helpfully, the Bessel functions are built into Matlab (`besselh(n,z)` computes $H_n^{(1)}(z)$). For small z a good approximation is

$$H_0^{(1)}(z) = 1 + i \left(\frac{2}{\pi} \left\{ \ln \left(\frac{1}{2}z \right) + \gamma \right\} \right) + O(z^2 \ln z), \quad (2.16)$$

where $\gamma \approx 0.57721566490153$ is Euler's constant (computed by `-psi(1)` in Matlab). For large z a good approximation is

$$H_0^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z-\pi/4)} + O(z^{-3/2}). \quad (2.17)$$

In fact, this approximation gives a good qualitative picture of $H_0^{(1)}(z)$ already for $z \geq 1$, and implies that, when

$$k|\mathbf{x} - \mathbf{x}_0| = 2\pi \frac{|\mathbf{x} - \mathbf{x}_0|}{\lambda}$$

is sufficiently large,

$$G(\mathbf{x}, \mathbf{x}_0) \approx \frac{e^{i\pi/4}}{4} \sqrt{\frac{2}{\pi k}} \frac{e^{ik|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|^{1/2}}.$$

We term the function $G(\mathbf{x}, \mathbf{x}_0)$ that we have just written down a *fundamental solution of the Helmholtz equation*, which just means that it is a solution of the Helmholtz equation appropriate to a point source excitation. We also call $G(\mathbf{x}, \mathbf{x}_0)$ the *free-field Green's function*, meaning that it is the (unique) solution to the problem of a point source in free space.

We will see later that it can be very helpful, to extend the range of problems which can be tackled efficiently, to also know explicitly Green's functions (i.e. solutions for point source excitation) for more complicated problems.

2.3 Direct Boundary Integral Equations in Acoustics

There are two main routes to obtaining boundary integral equation formulations for acoustic problems. The first route, the so-called **direct method**, is to obtain integral equation formulations **via Green's theorem**.

Green's first theorem, a simple consequence of the divergence theorem (see [19, p. 25]), is that

$$-\int_{\partial V} \phi \frac{\partial \psi}{\partial n} ds = \int_V [\phi \Delta \psi + \nabla \phi \cdot \nabla \psi] d\mathbf{x}. \quad (2.18)$$

In this equation V is a **bounded** 2- or 3-dimensional domain and ∂V its boundary. The integral on the left hand side is an integral over the boundary, a line integral in 2D, a surface integral in 3D. The ' ds ' on the left hand side denotes a 'little element of arc-length' in 2D, 'a little surface area element' in 3D. $\frac{\partial \psi}{\partial n}$ is the normal derivative on the boundary, the normal directed **into** V . On the right hand side the integral is over the domain V , an area integral in 2D, a volume integral in 3D, with $d\mathbf{x}$ denoting a 'little element of area' in 2D, a 'little element of volume' in 3D.

A simple consequence of Green's first theorem is Green's second theorem [19, p. 25], that

$$\int_{\partial V} \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) ds = \int_V [\phi \Delta \psi - \psi \Delta \phi] d\mathbf{x}. \quad (2.19)$$

Note that both these equations only hold provided certain conditions on both the domain V and the functions ϕ and ψ are satisfied. Sufficient conditions for Green's second theorem to be valid are that ϕ and ψ and their first and second order partial derivatives are continuous in V and up to its boundary ∂V . The condition on the domain V is that its boundary not behave too wildly; a boundary consisting of a finite number of smooth patches (e.g. the boundary of a polyhedron) is sufficiently well-behaved.

2.3.1 An integral equation for an interior problem

We now obtain our first integral equation formulation of an acoustics problem. This first problem we consider is a **3D interior** problem, the problem

of computing the 3D acoustic field generated in a bounded region D (say the interior of a room, of a car, etc.) due to a source of sound somewhere in the region D . We model the boundary (the walls of the room/the interior surface of the car, etc.) by an impedance boundary condition, suitable for a surface which is rigid, sound absorbing, or a mixture of both. Thus we seek to compute the acoustic field u in a bounded domain D due to a point source located at the point \mathbf{x}_0 in the domain. The acoustic field u is assumed to satisfy:

- the Helmholtz equation $\Delta u + k^2 u = 0$ in D (except at \mathbf{x}_0);
- the impedance boundary condition $\frac{\partial u}{\partial \mathbf{n}} + ik\beta u = 0$ on ∂D ;
- that near \mathbf{x}_0 the field is close to that in the free-field case, precisely that the difference,

$$u^s(\mathbf{x}) := u(\mathbf{x}) - G(\mathbf{x}, \mathbf{x}_0),$$

between u and the free-field solution, is continuous together with its gradient ∇u^s in a neighbourhood of \mathbf{x}_0 ;

- that, except at \mathbf{x}_0 , u and its partial derivatives up to second order are continuous³ in D and up to the boundary ∂D .

In physical terms this problem models the acoustic field inside a room or some interior space (e.g. the interior of a car) generated by a source at the point \mathbf{x}_0 . The impedance boundary condition models a rigid surface wherever $\beta = 0$ and an absorbing surface treatment wherever β has positive real part.

To obtain an integral equation formulation an application of Green's second theorem is made. Let \mathbf{x} denote any point in D or on its boundary (except \mathbf{x}_0). For the moment we are going to treat both \mathbf{x} and \mathbf{x}_0 as fixed positions and, keeping these points fixed, we are going to integrate over V with respect to another position vector \mathbf{y} which will serve as our variable of integration. We apply Green's 2nd theorem with

$$V = D_\epsilon := \{\mathbf{y} \in D : |\mathbf{y} - \mathbf{x}_0| \geq \epsilon \text{ and } |\mathbf{y} - \mathbf{x}| \geq \epsilon\},$$

³This assumption is valid if the boundary ∂D is smooth enough, but is not valid if ∂D has corners or edges. To cope with the corner and edge case we have to relax these assumptions, but not so much that we can no longer justify applying Green's theorem. In 2D, if the boundary is piecewise smooth, i.e. smooth except for corners, the following 'relaxing' works: assume that u is continuous in D and continuous up to the boundary ∂D , *including* corner points, while the partial derivatives of u are continuous in D and continuous up to the boundary ∂D , *except at* corner points.

i.e. V is D with a small circles (in 2D) or balls (in 3D) of radius ϵ cut out from around \mathbf{x} and \mathbf{x}_0 . We apply Green's 2nd theorem to the functions ϕ and ψ defined by

$$\phi(\mathbf{y}) := u(\mathbf{y}), \quad \psi(\mathbf{y}) := G(\mathbf{y}, \mathbf{x}),$$

so that ϕ is just the solution u to the problem that we are considering and ψ is the acoustic field due to a source at \mathbf{x} in the free-field case. Note that $\phi(\mathbf{y})$ (by virtue of our assumptions about the behaviour of u) is singular at $\mathbf{y} = \mathbf{x}_0$ (the source position) but is continuous with continuous partial derivatives up to 2nd order in $V = D_\epsilon$ and up to its boundary *because we have excluded \mathbf{x}_0 and a little ball around \mathbf{x}_0 from the domain D_ϵ* . Similarly, $\psi(\mathbf{y})$ is singular at $\mathbf{y} = \mathbf{x}$ but is smooth elsewhere and so smooth in D_ϵ *because we have excluded \mathbf{x} and a little ball around \mathbf{x} from the domain D_ϵ* .

Since the conditions of Green's theorem are satisfied equation (2.19) holds. To proceed further we note that ϕ and ψ are both solutions to the Helmholtz equation in D , except at \mathbf{x}_0 and \mathbf{x} , respectively. Since these points are excluded from D_ϵ we have that, in D_ϵ ,

$$\Delta\phi + k^2\phi = \Delta u + k^2u = 0$$

and

$$\Delta\psi + k^2\psi = 0.$$

Thus

$$\phi\Delta\psi - \psi\Delta\phi = -\phi k^2\psi + \psi k^2\phi = 0 \text{ in } D_\epsilon,$$

and so the integral on the right hand side of (2.19) vanishes. Hence

$$\int_{\partial V} \left(\psi \frac{\partial\phi}{\partial n} - \phi \frac{\partial\psi}{\partial n} \right) ds = 0.$$

For the time being we **focus on the case when $\mathbf{x} \in D$** and note that, as long as ϵ is small enough, $\partial V = \partial D_\epsilon$, the boundary of D_ϵ , consists of three separate pieces: the boundary ∂D of D ; the boundary γ_ϵ of the ball of radius ϵ centred on \mathbf{x} ; the boundary Γ_ϵ of the ball of radius ϵ centred on \mathbf{x}_0 . Thus the integral over ∂V is really the sum of three integrals over these separate parts of the boundary. Thus the previous equation implies that

$$\int_{\partial D} + \int_{\gamma_\epsilon} + \int_{\Gamma_\epsilon} = 0 \tag{2.20}$$

where $\int_{\partial D}$, \int_{γ_ϵ} , and \int_{Γ_ϵ} are a short-hand for the integrals of $\psi \frac{\partial\phi}{\partial n} - \phi \frac{\partial\psi}{\partial n}$ over ∂D , γ_ϵ , and Γ_ϵ , respectively.

We proceed by taking the limit as $\epsilon \rightarrow 0$. Considering \int_{γ_ϵ} first, we note that, for \mathbf{y} on γ_ϵ ,

$$\psi(\mathbf{y}) = G(\mathbf{y}, \mathbf{x}) = -\frac{e^{ik\epsilon}}{4\pi\epsilon}.$$

Further, where $\nabla_{\mathbf{y}}$ denotes the gradient with respect to the components of \mathbf{y} and $R := |\mathbf{y} - \mathbf{x}|$, it follows by the chain rule that

$$\nabla\psi(\mathbf{y}) = \nabla_{\mathbf{y}}G(\mathbf{y}, \mathbf{x}) = -\frac{1}{4\pi} \frac{d}{dR} \left(\frac{e^{ikR}}{R} \right) \nabla_{\mathbf{y}}R = -\frac{1}{4\pi} \frac{e^{ikR}(ikR - 1)}{R^3} (\mathbf{y} - \mathbf{x}) \quad (2.21)$$

so that, on γ_ϵ ,

$$\frac{\partial\psi}{\partial n}(\mathbf{y}) = \mathbf{n}(\mathbf{y}) \cdot \nabla\psi(\mathbf{y}) = \frac{\mathbf{y} - \mathbf{x}}{R} \cdot \nabla\psi(\mathbf{y}) = -\frac{1}{4\pi} \frac{e^{ik\epsilon}(ik\epsilon - 1)}{\epsilon^2}. \quad (2.22)$$

Thus

$$\int_{\gamma_\epsilon} \left[\psi \frac{\partial\phi}{\partial n} - \phi \frac{\partial\psi}{\partial n} \right] ds = -\frac{1}{4\pi} \left[\frac{e^{ik\epsilon}}{\epsilon} \int_{\gamma_\epsilon} \frac{\partial\phi}{\partial n} ds - \frac{e^{ik\epsilon}(ik\epsilon - 1)}{\epsilon^2} \int_{\gamma_\epsilon} \phi ds \right].$$

Now, in the limit as $\epsilon \rightarrow 0$, as γ_ϵ homes in on the centre of the ball, \mathbf{x} , we see that

$$\int_{\gamma_\epsilon} \phi ds \sim \phi(\mathbf{x}) \int_{\gamma_\epsilon} ds = 4\pi\epsilon^2\phi(\mathbf{x})$$

and, similarly, since $|\frac{\partial\phi}{\partial n}| = |\mathbf{n} \cdot \nabla\phi| \leq |\nabla\phi|$, which is bounded in a neighbourhood of \mathbf{x} , we have that

$$\int_{\gamma_\epsilon} \frac{\partial\phi}{\partial n} ds = O(\epsilon^2)$$

as $\epsilon \rightarrow 0$. Thus, in the limit $\epsilon \rightarrow 0$,

$$\frac{e^{ik\epsilon}}{\epsilon} \int_{\gamma_\epsilon} \frac{\partial\phi}{\partial n} ds = O(\epsilon) \rightarrow 0$$

while

$$\frac{e^{ik\epsilon}(ik\epsilon - 1)}{\epsilon^2} \int_{\gamma_\epsilon} \phi ds \rightarrow -4\pi\phi(\mathbf{x})$$

so that

$$\int_{\gamma_\epsilon} \left[\psi \frac{\partial\phi}{\partial n} - \phi \frac{\partial\psi}{\partial n} \right] ds \rightarrow \phi(\mathbf{x}). \quad (2.23)$$

A similar but slightly lengthier calculation leads to the conclusion that

$$\int_{\Gamma_\epsilon} \left[\psi \frac{\partial\phi}{\partial n} - \phi \frac{\partial\psi}{\partial n} \right] ds \rightarrow -\psi(\mathbf{x}_0) \quad (2.24)$$

as $\epsilon \rightarrow 0$. Thus, taking the limit $\epsilon \rightarrow 0$ in (2.20) we get that

$$\int_{\partial D} \left[\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right] ds + \phi(\mathbf{x}) - \psi(\mathbf{x}_0) = 0.$$

Remembering the definitions of ϕ and ψ we see that this equation can be rearranged slightly as an explicit formula for $u(\mathbf{x}) = \phi(\mathbf{x})$, that

$$u(\mathbf{x}) = G(\mathbf{x}_0, \mathbf{x}) + \int_{\partial D} \left[G(\mathbf{y}, \mathbf{x}) \frac{\partial u}{\partial n}(\mathbf{y}) - u(\mathbf{y}) \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \right] ds(\mathbf{y}), \quad (2.25)$$

this equation valid for all $\mathbf{x} \in D$ with $\mathbf{x} \neq \mathbf{x}_0$.

A few words are in order about this important formula. First of all some clarification of some of the notations may be helpful: the (\mathbf{y}) in $\partial n(\mathbf{y})$ in the normal derivative $\frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})}$ is necessary to indicate that we are computing partial derivatives with respect to the \mathbf{y} (and not with respect to the \mathbf{x}) variables. Thus, explicitly,

$$\frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})}$$

denotes the rate of increase of $G(\mathbf{y}, \mathbf{x})$ as \mathbf{y} moves off the boundary in the direction $\mathbf{n}(\mathbf{y})$. Similarly, the (\mathbf{y}) in $ds(\mathbf{y})$ is there to indicate that the integration is taking place with respect to the \mathbf{y} variables; the integrand is a function of two variables, of \mathbf{x} and \mathbf{y} , but the integration is only with respect to \mathbf{y} (with \mathbf{x} fixed).

Secondly (and more importantly), what does this formula give us? This formula, sometimes called a *Green's representation formula*, is an explicit representation for the solution u in terms of the explicitly known Green's function G and in terms of the so-called *Cauchy data*, the values of u and its normal derivative $\partial u / \partial n$ on the boundary. Thus, **once we know the values of u and $\partial u / \partial n$ on ∂D , equation (2.25) will give us an explicit (albeit complicated!) formula for computing the solution throughout the domain D .** In fact, the situation is even better than that: we need only find u on the boundary, for once we know u on ∂D then we know $\partial u / \partial n$ immediately from the impedance boundary condition that

$$\frac{\partial u}{\partial n} + ik\beta u = 0. \quad (2.26)$$

Of course, we have not used the impedance boundary condition at all in obtaining (2.25). An important consequence is that (2.25) must hold **irrespective of what boundary conditions apply on ∂D** . We will make use of this impedance boundary condition shortly. But first we consider

how the derivation of (2.25) differs when \mathbf{x} is on the boundary ∂D instead of inside D . A version of equation (2.20) holds in this case, namely

$$\int_{\partial D'} + \int_{\gamma'_\epsilon} + \int_{\Gamma_\epsilon} = 0, \quad (2.27)$$

the last term in this equation as before, the primes added to the first two terms indicating that they have changed somewhat. Precisely, $\int_{\partial D'}$ denotes the integral over ∂D but with the part of ∂D within distance ϵ of \mathbf{x} omitted. In the limit as $\epsilon \rightarrow 0$ this makes no difference. The second term is affected more significantly: γ'_ϵ denotes that part of the surface of the ball of radius ϵ centred on \mathbf{x} that **lies within** D . In the limit as $\epsilon \rightarrow 0$ we find that

$$\int_{\gamma'_\epsilon} \phi ds \sim \phi(\mathbf{x}) \int_{\gamma'_\epsilon} ds \sim \Omega(\mathbf{x}) \epsilon^2 \phi(\mathbf{x})$$

where

$$\Omega(\mathbf{x}) := \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \int_{\gamma'_\epsilon} ds$$

is the solid angle (in the range 0 to 4π) subtended at \mathbf{x} by the domain D . At most points on ∂D (all points except edges and corners) it holds that

$$\Omega(\mathbf{x}) = 2\pi,$$

the area of a hemisphere of unit radius. Thus, taking the limit as $\epsilon \rightarrow 0$ in (2.27), we arrive at a modified version of (2.25) that holds when $\mathbf{x} \in \partial D$, that

$$\frac{\Omega(\mathbf{x})}{4\pi} u(\mathbf{x}) = G(\mathbf{x}_0, \mathbf{x}) + \int_{\partial D} \left[G(\mathbf{y}, \mathbf{x}) \frac{\partial u}{\partial n}(\mathbf{y}) - u(\mathbf{y}) \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \right] ds(\mathbf{y}), \quad (2.28)$$

with $\frac{\Omega(\mathbf{x})}{4\pi} = \frac{1}{2}$ at most points on ∂D . Since we have not used the boundary condition in deriving (2.28), this equation also **holds irrespective of the boundary condition**.

Making use now of the boundary condition (2.26) to replace $\partial u / \partial n$ by $-ik\beta u$ in (2.25) and (2.28), we arrive at versions of these equations which are satisfied by the solution u to our particular acoustics problem, that

$$u(\mathbf{x}) = G(\mathbf{x}_0, \mathbf{x}) - \int_{\partial D} \left[ik\beta(\mathbf{y})G(\mathbf{y}, \mathbf{x}) + \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \right] u(\mathbf{y}) ds(\mathbf{y}), \quad x \in D, \quad (2.29)$$

and

$$\frac{\Omega(\mathbf{x})}{4\pi} u(\mathbf{x}) = G(\mathbf{x}_0, \mathbf{x}) - \int_{\partial D} \left[ik\beta(\mathbf{y})G(\mathbf{y}, \mathbf{x}) + \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \right] u(\mathbf{y}) ds(\mathbf{y}), \quad x \in \partial D. \quad (2.30)$$

This last equation is **our first integral equation for an acoustics problem**, the unknown function to be found being u (the pressure) on the boundary ∂D . Referring to our classification in section 2.1.1, we see that this is a second kind equation, as the unknown u appears on the left hand side, outside the integral, as well as under the integral sign. Since the integration is a surface integral, over the **boundary** of the region D , we call this integral equation a **boundary integral equation**. The **boundary integral equation method** for this problem consists of two steps: first solve the integral equation (2.30) to find u on ∂D ; after this first step everything on the right hand side of (2.29) is known and so this equation can be used to compute u anywhere in D .

So far, all our work has been analytical, manipulating formulae. But to carry out each of these two steps for practical problems numerical methods must be used: this is where the boundary element method and some serious scientific computing come in!

2.3.2 Other boundary conditions and exterior problems

Before we go on to discuss numerical solution, we will introduce integral equation formulations for a wider range of problems. Suppose first of all that we wish to solve the same problem as in section 2.3.1 but with the simpler boundary condition that

$$u = 0 \text{ on } \partial D. \quad (2.31)$$

Equations (2.25) and (2.28) hold irrespective of the boundary condition. Substituting $u = 0$ on ∂D in these equations we get that

$$u(\mathbf{x}) = G(\mathbf{x}_0, \mathbf{x}) + \int_{\partial D} G(\mathbf{y}, \mathbf{x}) \frac{\partial u}{\partial n}(\mathbf{y}) ds(\mathbf{y}), \quad x \in D, \quad (2.32)$$

and that

$$0 = G(\mathbf{x}_0, \mathbf{x}) + \int_{\partial D} G(\mathbf{y}, \mathbf{x}) \frac{\partial u}{\partial n}(\mathbf{y}) ds(\mathbf{y}), \quad x \in \partial D. \quad (2.33)$$

This last equation is our second boundary integral equation: the unknown function is $\partial u/\partial n$ on the boundary ∂D . This equation is a *first kind integral equation* as the unknown function only appears under the integral sign. Once (2.33) has been solved to find $\partial u/\partial n$ then (2.32) can be used to compute u throughout D .

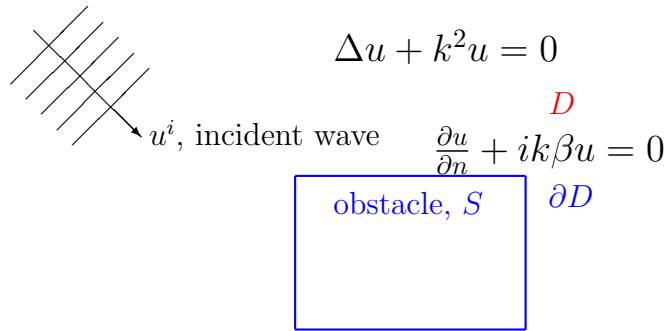


Figure 2.1: Schematic diagram of an exterior/scattering problem.

We turn now to consider **exterior problems**. Specifically let us consider the following scattering problem, illustrated schematically in Figure 2.1. A wave is incident on some obstacle of finite size, the surface of which is adequately modelled by an impedance boundary condition. We suppose that the obstacle that scatters the incident acoustic wave is situated in a homogeneous medium at rest, and is sufficiently far from other objects such that it is an adequate model to regard the domain of the acoustic field as being the whole of \mathbb{R}^3 outside the obstacle, so that $D = \mathbb{R}^3 \setminus S$, where S is the bounded region occupied by the obstacle (and \setminus is the set ‘take away’ sign). (This might be an adequate model, for example, of scattering of an incident wave by an object suspended in an anechoic chamber.) We assume that the incident acoustic field is that due to a point source of sound at \mathbf{x}_0 , somewhere in D .

Thus, we seek to compute the acoustic field u in the unbounded domain $D = \mathbb{R}^3 \setminus S$ due to a point source located at the point \mathbf{x}_0 in D . The acoustic field u is assumed to satisfy:

- the Helmholtz equation $\Delta u + k^2 u = 0$ in D (except at \mathbf{x}_0);
- the impedance boundary condition $\frac{\partial u}{\partial n} + ik\beta u = 0$ on ∂D ;
- that, near \mathbf{x}_0 , the field is close to that in the free-field case, precisely that the difference,

$$u^s(\mathbf{x}) := u(\mathbf{x}) - G(\mathbf{x}, \mathbf{x}_0),$$

between u and the free-field solution, is continuous together with its gradient ∇u^s in a neighbourhood of \mathbf{x}_0 ;

- that, except at \mathbf{x}_0 , u and its partial derivatives up to second order are continuous in D and up to the boundary ∂D ;

- the Sommerfeld radiation conditions, (1.8) and (1.9).

To derive a boundary integral equation formulation for this problem we will make use of our results for the case considered in section 2.3.1 where D is bounded. In order to do this we introduce, for $R > 0$, the set D_R which is the part of the unbounded set D which lies within distance R of the origin, in symbols

$$D_R := \{\mathbf{y} \in D : |\mathbf{y}| < R\}.$$

We assume that R is chosen large enough so that both \mathbf{x}_0 and S (the scattering obstacle) are inside D_R . Then, because the set D_R is bounded, and because equations (2.25) and (2.28) hold irrespective of the boundary conditions on ∂D , they hold for this problem, but not, directly, for the unbounded region D but for the region D_R . Thus equations (2.25) and (2.28) hold for this problem, as long as we replace D by D_R , with R sufficiently large. Now ∂D_R (the boundary of D_R) consists of two disconnected pieces, namely ∂D , the boundary of D (and of the obstacle S) and Γ^R , the surface of the sphere of radius R centred on the origin. Thus equation (2.25) applied to this problem tells us that, for $\mathbf{x} \in D_R$,

$$\begin{aligned} u(\mathbf{x}) &= G(\mathbf{x}_0, \mathbf{x}) + \int_{\partial D} \left[G(\mathbf{y}, \mathbf{x}) \frac{\partial u}{\partial n}(\mathbf{y}) - u(\mathbf{y}) \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \right] ds(\mathbf{y}) \\ &\quad + \int_{\Gamma^R} \left[G(\mathbf{y}, \mathbf{x}) \frac{\partial u}{\partial n}(\mathbf{y}) - u(\mathbf{y}) \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \right] ds(\mathbf{y}), \end{aligned} \quad (2.34)$$

this equation holding for all sufficiently large R .

We finish the derivation of the representation we want for u by taking the limit as $R \rightarrow \infty$. We note that, since $\partial/\partial n = -\partial/\partial r$ on Γ^R , it holds that

$$\begin{aligned} &\int_{\Gamma^R} \left[G(\mathbf{y}, \mathbf{x}) \frac{\partial u}{\partial n}(\mathbf{y}) - u(\mathbf{y}) \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \right] ds(\mathbf{y}) = \\ & - \int_{\Gamma^R} \left[G(\mathbf{y}, \mathbf{x}) \left(\frac{\partial u}{\partial r}(\mathbf{y}) - ik u(\mathbf{y}) \right) - u(\mathbf{y}) \left(\frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial r(\mathbf{y})} - ik G(\mathbf{y}, \mathbf{x}) \right) \right] ds(\mathbf{y}). \end{aligned}$$

It follows from the Sommerfeld radiation conditions, which are satisfied both by u and by G , that the integrand on Γ_R decreases at the rate $o(R^{-2})$ as $R \rightarrow \infty$. Since the area of Γ^R increases only proportional to R^2 it follows that the integral over Γ^R vanishes in the limit as $R \rightarrow \infty$. Thus, taking the limit as $R \rightarrow \infty$ in (2.34), it follows that

$$u(\mathbf{x}) = G(\mathbf{x}_0, \mathbf{x}) + \int_{\partial D} \left[G(\mathbf{y}, \mathbf{x}) \frac{\partial u}{\partial n}(\mathbf{y}) - u(\mathbf{y}) \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \right] ds(\mathbf{y}), \quad (2.35)$$

for $x \in D$. Similarly, applying (2.28) to the region D_R and then taking the limit as $R \rightarrow \infty$, we get that

$$\frac{\Omega(\mathbf{x})}{4\pi} u(\mathbf{x}) = G(\mathbf{x}_0, \mathbf{x}) + \int_{\partial D} \left[G(\mathbf{y}, \mathbf{x}) \frac{\partial u}{\partial n}(\mathbf{y}) - u(\mathbf{y}) \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \right] ds(\mathbf{y}), \quad (2.36)$$

for $x \in \partial D$.

Note that:

- (i) these equations are identical to equations (2.25) and (2.28) (though now D denotes an unbounded rather than a bounded region);
- (ii) as we have not used the boundary condition on ∂D yet, these equations hold irrespective of the boundary condition.

Using the impedance boundary condition to replace $\frac{\partial u}{\partial n}$ by $-ik\beta u$, equations (2.35) and (2.36) become

$$u(\mathbf{x}) = G(\mathbf{x}_0, \mathbf{x}) - \int_{\partial D} \left[ik\beta(\mathbf{y})G(\mathbf{y}, \mathbf{x}) + \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \right] u(\mathbf{y}) ds(\mathbf{y}), \quad x \in D, \quad (2.37)$$

and

$$\frac{\Omega(\mathbf{x})}{4\pi} u(\mathbf{x}) = G(\mathbf{x}_0, \mathbf{x}) - \int_{\partial D} \left[ik\beta(\mathbf{y})G(\mathbf{y}, \mathbf{x}) + \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \right] u(\mathbf{y}) ds(\mathbf{y}), \quad x \in \partial D. \quad (2.38)$$

This latter equation is our third boundary integral equation. Once we solve this to determine u on ∂D then equation (2.37) enables us to compute u anywhere in D .

Note that equations (2.37) and (2.38) are identical in form to equations (2.29) and (2.30). In particular, the boundary integral equations (2.38) and (2.30) are almost the same equation; precisely, if the interior region D in (2.30) coincides with S , the region occupied by the obstacle in (2.38), so that ∂D is the same surface in the two cases, then the two integral equations are identical, except that the normal direction is directed into the bounded region D in (2.30) and out of the bounded region S in (2.38). This minor difference makes it very easy to tweak a computer code for an interior problem so as to solve an exterior problem, or vice versa.

2.3.3 The 2D Case

We assume for simplicity and to make things specific in the previous two sections that the problems we consider are 3D. But a very similar derivation of integral equation formulations holds in the 2D case and, in fact, all the integral equation formulations and representation formulas we have derived

are valid also for the 2D case, provided we make the small change that, in (2.30) and (2.38), we replace $\Omega(\mathbf{x})/(4\pi)$ by $\Omega(\mathbf{x})/(2\pi)$ and understand $\Omega(\mathbf{x})$ now to mean the angle (rather than solid angle) subtended at $\mathbf{x} \in \partial D$ by the domain D . At most points (all points which are not corners of the domain), it holds that

$$\frac{\Omega(\mathbf{x})}{2\pi} = \frac{1}{2}$$

as before.

In the 3D case, to compute the kernels of the integral equations explicitly, we can use equation (2.21). Using the definition of G in the 2D case, equation (2.15), and the fact that the derivative of $H_0^{(1)}(z)$ is $H_0^{(1)'}(z) = -H_1^{(1)}(z)$, we see that the corresponding equation in the 2D case is

$$\nabla_{\mathbf{y}}G(\mathbf{y}, \mathbf{x}) = -\frac{i}{4} \frac{d}{dR} H_0^{(1)}(kR) \nabla_{\mathbf{y}}R = \frac{ik}{4} \frac{H_1^{(1)}(kR)}{R} (\mathbf{y} - \mathbf{x}), \quad (2.39)$$

with $R = |\mathbf{y} - \mathbf{x}|$.

2.4 Indirect Boundary Integral Equations

In the previous sections we have looked at the derivation of so-called direct boundary integral equation formulations, obtained from Green's theorem. There is another whole class of integral equation formulations, called *indirect* formulations, obtained in the following way.

To explain these formulations we need to introduce the so-called *single- and double-layer potentials* which in fact, as we will see in a moment, we have already met in particular cases.

Suppose that D is a domain with boundary ∂D ; we assume that the boundary ∂D is finite in extent (which is the case if D is bounded or if D is the exterior of a bounded set). Given a function φ defined on ∂D , we call the function v_s , defined by

$$v_s(\mathbf{x}) := \int_{\partial D} G(\mathbf{x}, \mathbf{y}) \varphi(\mathbf{y}) ds(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^d, \quad (2.40)$$

($d = 2, 3$ the dimension) the *single-layer potential with density φ* . In physical terms $G(\mathbf{x}, \mathbf{y})$ is the field at \mathbf{x} due to a source at \mathbf{y} in free space. The single-layer potential is the field in free space due to a sum (well, the limit of a sum, i.e. an integral) of sources smeared over the boundary ∂D , with $\varphi(\mathbf{y})$ denoting the density (the magnitude and phase) of those sources at the point $\mathbf{y} \in \partial D$.

Similarly, the function v_d , defined by

$$v_d(\mathbf{x}) := \int_{\partial D} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} \varphi(\mathbf{y}) ds(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^d, \quad (2.41)$$

is called the *double-layer potential with density* φ . In physical terms it is not difficult to see that

$$\frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} = \mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y})$$

is the field due to a dipole source in free space, located at $\mathbf{y} \in \partial D$ and oriented so that the two equal and opposite sources making up the dipole lie on the line through \mathbf{y} in the direction $\mathbf{n}(\mathbf{y})$, so that they can be thought of as lying close to and on opposite sides of ∂D . The double-layer potential is the field in free space due to a continuous distribution of such dipoles over the surface ∂D .

I have said that we have already met v_s and v_d . Inspecting (2.25) we see that it is a representation for u in D as the sum of the fields in free space due to: a source at \mathbf{x}_0 ; a single-layer potential with density $\partial u / \partial n$; a double-layer potential with density $-u$.

The single- and double-layer potentials have many properties that are attractive for someone wanting to solve the Helmholtz equation. Since they are the fields due to sources smeared on ∂D , it holds that:

- they are smooth functions on both sides of ∂D , in fact are infinitely differentiable, their partial derivatives of all orders being continuous (except, as we shall see, on ∂D);
- they satisfy the Helmholtz equation in $\mathbb{R}^d \setminus \partial D$;
- they satisfy the Sommerfeld radiation conditions.

On and across the boundary ∂D (where the sources are located) their behaviour is more complicated. The following theorem is a basic, very important, and standard result (see e.g. Colton & Kress [10]):

THEOREM (on jump relations). *Suppose that the boundary ∂D is sufficiently smooth (it is enough that the normal direction $\mathbf{n}(\mathbf{y})$ and curvature vary continuously on ∂D). Suppose also that the density φ varies continuously (on ∂D). Then:*

- *The single-layer potential v_s is continuous in \mathbb{R}^d .*

- The double-layer potential $v_d(\mathbf{x})$ has well-defined limits as \mathbf{x} approaches ∂D from either side, but its value jumps as it crosses ∂D . Precisely, for every $\mathbf{x} \in \partial D$, the limits

$$v_{d\pm}(\mathbf{x}) := \lim_{h \rightarrow 0^+} v_d(\mathbf{x} \pm h\mathbf{n}(\mathbf{x}))$$

exist and are given by

$$v_{d\pm}(\mathbf{x}) = v_d(\mathbf{x}) \mp \frac{1}{2}\varphi(\mathbf{x}) = \int_{\partial D} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} \varphi(\mathbf{y}) ds(\mathbf{y}) \mp \frac{1}{2}\varphi(\mathbf{x}), \quad \mathbf{x} \in \partial D. \quad (2.42)$$

- The normal derivative of the single-layer potential is well-defined on either side of ∂D as the limit

$$\frac{\partial u_{s\pm}}{\partial n}(\mathbf{x}) := \lim_{h \rightarrow 0^+} \mathbf{n}(\mathbf{x}) \cdot \nabla v_s(\mathbf{x} \pm h\mathbf{n}(\mathbf{x})),$$

but the normal derivative jumps across ∂D , as made clear by the explicit formula

$$\frac{\partial u_{s\pm}}{\partial n}(\mathbf{x}) = \int_{\partial D} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{x})} \varphi(\mathbf{y}) ds(\mathbf{y}) \pm \frac{1}{2}\varphi(\mathbf{x}), \quad \mathbf{x} \in \partial D. \quad (2.43)$$

- The normal derivative of the double-layer potential is continuous across ∂D , i.e.

$$\frac{\partial u_{d+}}{\partial n}(\mathbf{x}) = \frac{\partial u_{d-}}{\partial n}(\mathbf{x});$$

precisely, it holds that

$$\mathbf{n}(\mathbf{x}) \cdot (\nabla v_d(\mathbf{x} + h\mathbf{n}(\mathbf{x})) - \nabla v_d(\mathbf{x} - h\mathbf{n}(\mathbf{x}))) \rightarrow 0$$

as $h \rightarrow 0$, for every $\mathbf{x} \in \partial D$.

The properties of the layer-potentials listed immediately before the above theorem make them very attractive as **candidate solutions** for acoustic problems. By this I mean that our aim in solving the problems discussed above is to find a function which satisfies the Helmholtz equation in a region D , the Sommerfeld radiation conditions (in the case that D is unbounded), and a particular boundary condition on ∂D . Automatically v_s and v_d satisfy the first two of these conditions. The **indirect boundary integral equation method** is to seek the solution to a particular problem in the form of a single- or double-layer potential, and to try to choose the density of that potential in such a way that the boundary condition on ∂D is satisfied.

We will illustrate this method by one example. Let us look for a solution of the problem of section 2.3.1, assuming for a moment that the boundary ∂D is smooth, in the form

$$u(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_0) + v_s(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_0) + \int_{\partial D} G(\mathbf{x}, \mathbf{y}) \varphi(\mathbf{y}) ds(\mathbf{y}). \quad (2.44)$$

In this equation v_s is the single-layer potential with density φ . This single-layer potential satisfies the Helmholtz equation. The expression (2.44) satisfies the impedance boundary value problem provided

$$\frac{\partial v_{s+}}{\partial n}(\mathbf{x}) + ik\beta(\mathbf{x})v_s(\mathbf{x}) = -\frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial n(\mathbf{x})} - ik\beta(\mathbf{x})G(\mathbf{x}, \mathbf{x}_0), \quad x \in \partial D. \quad (2.45)$$

In this equation we use the notation of the theorem: $\partial v_{s+}/\partial n$ is the normal derivative on the '+' side of ∂D , i.e. the side towards which $\mathbf{n}(\mathbf{x})$ points, which is the side of ∂D on which D lies. Letting $g(\mathbf{x})$ be a shorthand for the (known) function on the right hand side of (2.45), and substituting for v_s and for $\partial v_{s+}/\partial n$ from (2.40) and (2.43), we see that (2.45) holds if and only if

$$\int_{\partial D} \left(\frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{x})} + ik\beta(\mathbf{x})G(\mathbf{x}, \mathbf{y}) \right) \varphi(\mathbf{y}) ds(\mathbf{y}) + \frac{1}{2}\varphi(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial D. \quad (2.46)$$

This is a second kind *indirect boundary integral equation* for the unknown density φ . We solve (2.46) to find φ and then compute the solution u using (2.44).

Chapter 3

Simple Boundary Element Methods for Acoustics Problems

The boundary element method (BEM) is obtained by putting together two components: the formulation of the problem as a boundary integral equation and the finite element method (FEM) of numerical solution. Put another way, the BEM consists of applying a finite element method discretisation to a boundary integral equation formulation of a problem.

To make this idea concrete we will describe in this chapter a simple, specific implementation of the boundary element method. We start by dividing the boundary ∂D into N small pieces (the *boundary elements*, which we will denote by $\gamma_1, \gamma_2, \dots, \gamma_N$). We assume that the boundary ∂D is piecewise smooth, consisting of a number of smooth pieces and choose this discretisation so that each element γ_j is a smooth piece of ∂D , typically a triangle or curvilinear triangle, rectangle or curvilinear rectangle (see e.g. Figure 3.1).

To make things specific, we assume that the boundary integral equation (2.30) is to be approximated by the BEM. The idea of the method is to approximate the solution of the integral equation, that is the pressure, u , on the boundary ∂D , by a simple function, usually a polynomial, in each element. In this chapter we will consider only the simplest possible approximation of this type where we approximate the solution by a polynomial of degree 0 on each element, i.e. we approximate u in element γ_j by a constant u_j .

Making this approximation, since $\int_{\partial D} = \int_{\gamma_1} + \dots + \int_{\gamma_N}$, equation (2.29)

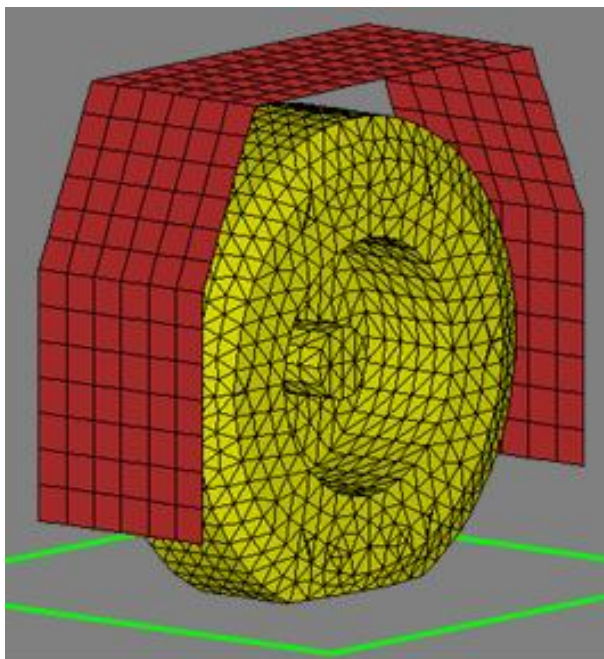


Figure 3.1: Typical BEM surface mesh

becomes

$$u(\mathbf{x}) \approx G(\mathbf{x}_0, \mathbf{x}) - \sum_{j=1}^N u_j \int_{\gamma_j} \left[ik\beta(\mathbf{y})G(\mathbf{y}, \mathbf{x}) + \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \right] ds(\mathbf{y}), \quad \mathbf{x} \in D, \quad (3.1)$$

and equation (2.30) becomes

$$\frac{\Omega(\mathbf{x})}{4\pi} u(\mathbf{x}) \approx G(\mathbf{x}_0, \mathbf{x}) - \sum_{j=1}^N u_j \int_{\gamma_j} \left[ik\beta(\mathbf{y})G(\mathbf{y}, \mathbf{x}) + \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \right] ds(\mathbf{y}), \quad \mathbf{x} \in \partial D. \quad (3.2)$$

To obtain a set of equations to determine values for the constants u_j , the simplest method is the *collocation method*. We pick a point \mathbf{x}_i in each element γ_i (\mathbf{x}_i is typically the centroid of the element) and require that equation (3.2) holds *exactly* at each of these so-called *collocation points*. Assuming that each point \mathbf{x}_i is on a smooth part of the boundary (so that $\frac{\Omega(\mathbf{x})}{4\pi} = \frac{1}{2}$), we thus require that

$$\frac{1}{2} u_i = G(\mathbf{x}_0, \mathbf{x}_i) - \sum_{j=1}^N u_j \int_{\gamma_j} \left[ik\beta(\mathbf{y})G(\mathbf{y}, \mathbf{x}_i) + \frac{\partial G(\mathbf{y}, \mathbf{x}_i)}{\partial n(\mathbf{y})} \right] ds(\mathbf{y}), \quad i = 1, \dots, N. \quad (3.3)$$

This is a set of N simultaneous linear equations to determine the N unknowns u_1, \dots, u_N . We can write this set of equations in matrix form. Let B denote the order N matrix whose ij th entry is

$$b_{ij} := \int_{\gamma_j} \left[ik\beta(\mathbf{y})G(\mathbf{y}, \mathbf{x}_i) + \frac{\partial G(\mathbf{y}, \mathbf{x}_i)}{\partial n(\mathbf{y})} \right] ds(\mathbf{y}), \quad (3.4)$$

and let I denote the order N identity matrix. Let \mathbf{u} denote the column vector whose j th entry is u_j and let \mathbf{b} denote the column vector whose i th entry is $G(\mathbf{x}_0, \mathbf{x}_i)$. Then the above linear system can be written as

$$\frac{1}{2} u_i = G(\mathbf{x}_0, \mathbf{x}_i) - \sum_{j=1}^N b_{ij} u_j, \quad i = 1, \dots, N.$$

i.e. as

$$\frac{1}{2} \mathbf{u} = \mathbf{b} - B\mathbf{u} \quad (3.5)$$

or as

$$A\mathbf{u} = \mathbf{b} \quad (3.6)$$

where

$$A := \frac{1}{2}I + B.$$

The first step in this simple boundary element method is to compute the matrix A and then solve (3.6) to obtain the vector \mathbf{u} . The second step is to use (3.1) to compute values of $u(\mathbf{x})$ at points \mathbf{x} of interest in D .

In order for this scheme to be completely explicit, we need to indicate how one would compute the entries of the matrix B , and how one would evaluate the integrals in (3.1). A simple, approximate scheme which is adequate for rough approximations and small problems in the case when the boundary is smooth is the following one. Assume that each element is small enough so that

$$\beta(\mathbf{y}) \approx \beta(\mathbf{x}_j), \quad G(\mathbf{y}, \mathbf{x}) \approx G(\mathbf{x}_j, \mathbf{x}), \quad \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} \approx \mathbf{n}(\mathbf{x}_j) \cdot \nabla_{\mathbf{y}} G(\mathbf{x}_j, \mathbf{x}),$$

for $\mathbf{y} \in \gamma_j$. Then

$$\int_{\gamma_j} \frac{\partial G(\mathbf{y}, \mathbf{x})}{\partial n(\mathbf{y})} ds(\mathbf{y}) \approx \mathbf{n}(\mathbf{x}_j) \cdot \nabla_{\mathbf{y}} G(\mathbf{x}_j, \mathbf{x}) \int_{\gamma_j} ds(\mathbf{y}) = \mathbf{n}(\mathbf{x}_j) \cdot \nabla_{\mathbf{y}} G(\mathbf{x}_j, \mathbf{x}) A_j$$

where

$$A_j := \int_{\gamma_j} ds$$

is the area of γ_j in 3D, the arc-length of γ_j in 2D. Similarly,

$$\int_{\gamma_j} ik\beta(\mathbf{y})G(\mathbf{y}, \mathbf{x})ds(\mathbf{y}) \approx ik\beta(\mathbf{x}_j)G(\mathbf{x}_j, \mathbf{x})A_j.$$

Thus (3.1) can be approximated by

$$u(\mathbf{x}) \approx G(\mathbf{x}_0, \mathbf{x}) - \sum_{j=1}^N u_j A_j [ik\beta(\mathbf{x}_j)G(\mathbf{x}_j, \mathbf{x}) + \mathbf{n}(\mathbf{x}_j) \cdot \nabla_{\mathbf{y}}G(\mathbf{x}_j, \mathbf{x})], \quad \mathbf{x} \in D. \quad (3.7)$$

We can make the same approximation for the entries of the matrix B , namely

$$b_{ij} \approx A_j [ik\beta(\mathbf{x}_j)G(\mathbf{x}_j, \mathbf{x}_i) + \mathbf{n}(\mathbf{x}_j) \cdot \nabla_{\mathbf{y}}G(\mathbf{x}_j, \mathbf{x}_i)],$$

at least provided $i \neq j$, as this expression does not make sense (is infinite) when $i = j$. The crudest approximation, but not completely hopeless as it only affects one term in a large sum, is to approximate b_{ii} by zero. Thus we approximate

$$b_{ij} \approx \tilde{b}_{ij} := \begin{cases} 0, & i = j, \\ A_j [ik\beta(\mathbf{x}_j)G(\mathbf{x}_j, \mathbf{x}_i) + \mathbf{n}(\mathbf{x}_j) \cdot \nabla_{\mathbf{y}}G(\mathbf{x}_j, \mathbf{x}_i)], & i \neq j. \end{cases} \quad (3.8)$$

This scheme achieves approximately first order convergence for smooth surfaces, i.e. the error reduces approximately in proportion to N^{-1} .

3.1 Computational Cost

We have described a simple numerical scheme. In principle (with powerful enough computers which have enough memory) we can solve large classes of complex acoustical problems with this method. Of course, in practice, computational limitations will become an issue and, therefore, more efficient algorithms become desirable.

We will discuss, briefly, more effective computational algorithms in the next chapter. But let us consider now, with back of the envelope calculations, what the computational limits are likely to be for algorithms of the type we have just discussed.

Comparison with FEM. The BEM has the attraction compared to the FEM that only the surface of the domain is discretised. This means that the number of degrees of freedom (the size of the linear system to be solved) is hugely more for the FEM compared with a BEM discretisation of the same

problem. But, on the other hand, the matrix A in the linear system for BEM (e.g. the matrix in (3.6)) is a **full matrix**, i.e. every element of the matrix is non-zero. By contrast the FEM matrix is very sparse (almost every element is non-zero) which means it can be stored and solved efficiently.

It is impossible to give a general rule as to which of BEM and FEM are better. It all depends on the particular discretisation and solution scheme and their implementation. With this proviso, as a broad guideline, BEM has a tendency to be superior for exterior problems and for problems in which the boundary is complex but smooth.

Memory requirement of BEM. The main memory requirement of a simple-minded BEM implementation is the storage required for the matrix A . This is an $N \times N$ matrix, so it has N^2 elements. Each of these is a complex number. Each double precision real number in IEEE arithmetic requires 8 bytes. Thus A (and so the BEM) requires

$$16N^2 \text{ bytes}$$

of storage. For $N = 1000$ this is approximately 15 Mbytes, for $N = 10,000$ it is 1.5 Gbytes, for $N = 100,000$ over 100 Gbytes. So, on a well-equipped PC, with 2Gbytes of memory, $N = 10,000$ would be feasible, but anything much larger unthinkable (without better algorithms).

On the other hand, the value of N is dictated by the accuracy required and on the sizes of the domain and of the wavelength. A rough, frequently quoted rule of thumb for a reasonable BEM implementation is that 5-10 elements per wavelength are required for ‘engineering accuracy’. Here ‘per wavelength’ means in each direction. Thus in 2D (where the integral equations are one-dimensional), if the length of boundary to be discretised is L , then we need

$$N \approx \frac{5L}{\lambda}.$$

Thus in air, at frequency 1000 Hz, where $\lambda \approx 0.34\text{m}$, $N = 10,000$ corresponds to $L \approx 700\text{m}$.

For 3D problems, if the area to be discretised is L^2 , then we need

$$N \approx \left(\frac{5L}{\lambda}\right)^2.$$

Thus, in air at frequency 1000 Hz, $N = 10,000$ corresponds to $L^2 \approx 50\text{m}^2$.

Computational cost. The main computational costs of the BEM are computing the matrix A , which requires N^2 evaluations of the Green’s function

G and its derivatives, and solving the linear system. Classical direct solution methods (e.g. Gaussian elimination which is approximately what backslash does in Matlab) require about N^3 basic arithmetical operations (multiplications and additions). Potentially this is very expensive for large N , but one tends to run out of memory before N becomes large enough to be problematic. For example, when $N = 3000$, about the limit on my laptop, the solve time in Matlab is only 75 seconds.

For larger values of N iterative solvers are recommended. For example GMRES (preferably with a suitable preconditioner) is appropriate when the matrix A is complex and not Hermitian, and is built in to Matlab (though with a rather inefficient implementation).

Chapter 4

More Sophisticated Ideas and Further Reading

We have given only the briefest of introductions to the boundary element method in these pages. Here is a partial list of some of the more sophisticated ideas that we have neglected, with references for further reading.



Figure 4.1: Multiple-edged outdoor noise barrier design developed using BEM simulations [16].

1. The boundary element method we have described above is very crude in a number of respects:
 - (a) We propose, in (3.8), a very crude approximation for the entries of the matrix: there are much more sophisticated and accurate

(but no less efficient) numerical quadrature schemes which, typically, use something like our simple scheme for most of the matrix entries, those for which $k|\mathbf{x}_i - \mathbf{x}_j| \gg 1$, and use higher order Gaussian quadrature rules for the remaining entries: the entries on the diagonal (the $i = j$ entries) are a special case and there are special approximations for this case. See some of the standard boundary element textbooks for more detail, e.g. [2, 8, 20].

- (b) We use a very crude approximation, as a piecewise constant, for the solution. As you will see in the Matlab class, with this method the BEM solution converges only very slowly to the true solution as N increases (the error is approximately proportional to N^{-1}). If higher order polynomials are used for approximation on each element, and if the element sizes are adjusted correctly (by which we mean that smaller elements are used, in a precisely controlled way, near corners and edges where the solution tends to be singular), then much faster rates of convergence are achievable. Typically, for 2D problems, one can achieve that the error decreases proportional to $N^{-(1+p)}$ if polynomials of degree p are used for approximation. However, this rate of convergence only kicks in once the mesh is sufficiently refined to approximately resolve the oscillatory solution. At the very minimum 2 elements per wavelength are needed in each direction, this figure increasing to 5-10 elements per wavelength if only low order polynomial approximations of the solution are used.
- (c) A related issue is accurate approximation of the boundary through the use of isoparametric elements - see [2, 8, 20].

2. Our treatment of the different possibilities for integral equation formulations for acoustic problems has been very partial. Very importantly, we have neglected:

- (a) **The non-uniqueness/non-existence problem for exterior problems.** The issue here is that the integral equation formulations for exterior acoustic problems that we have presented fail at certain frequencies which depend on the geometry. For the direct formulations the problem is that, although our argument shows that the integral equation formulation must have a solution, we haven't presented any argument as to why there should be only one solution. A careful study (using the theorem on jump relations) shows that, in fact, equation (2.38), for example, has more

than one solution whenever the wavenumber k is such that a certain homogeneous interior problem in the bounded region S on the other side of ∂D has a non-trivial solution. The specific interior problem is the Helmholtz equation in S together with the homogeneous Neumann condition $\partial u/\partial n = 0$ on ∂D . This non-uniqueness is a significant computational problem. To resolve this problem alternative, more complicated integral equation formulations have been proposed and are widely used: see [10, 20].

- (b) Boundary integral equation formulations for more complex problems, e.g. transmission problems, where the boundary is a boundary between two different media (see [10]), or coupled vibro-acoustic problems.
3. Use of more sophisticated Green's functions. If one uses, instead of the free-field Green's function G , a more complex Green's function that satisfies the same boundary conditions as the solution u on a part of the boundary ∂D , then it is not difficult to see that the integral over that part of the boundary (in equation (2.25), for example) vanishes. Thus the integral equation is only over that part of the boundary where G and u satisfy different boundary conditions, and so only that part of the boundary need be discretised by boundary elements, potentially greatly reducing the size of the linear system. On the other hand, computing the entries in the matrix A will become more expensive if the Green's function is more complex.

A spectacular example of the effectiveness, in terms of reducing computational cost, of this approach, is the use of the Green's function for the problem of acoustic propagation over a homogeneous impedance plane to formulate the problem of noise barriers sitting on an impedance plane (modelling outdoor noise barriers on flat ground) as a boundary integral equation only over the noise barrier and possibly a small part of the ground surface, rather than integrating over the whole surface of the ground, or at least that part of it between the source and the receiver positions of interest. See for example [3, 20] and [4], this latter paper discussing the efficient calculation of this Green's function, necessary for the method to be effective. Figure 4.1, taken from [16], shows an outdoor noise barrier design developed based in large part on BEM simulations.

Another example of where this approach is very effective is in the computation of plane wave scattering by periodic arrays of scatterers. Use of *periodic* Green's functions (e.g. [15]) enables problems of this type

to be reduced to an integration over the surface of a single scatterer.

4. Matrix compression and fast matrix-vector multiplication techniques. We discussed, in the last chapter, the bottleneck caused by the requirement to store the large, full BEM matrix A . In the last ten years effective methods have been developed to compress the information in the matrix, reducing spectacularly the storage required. Most importantly, effective **fast multipole methods** have been developed for acoustic and electromagnetic problems. These methods allow the information in the matrix to be stored with storage which asymptotically, as N increases, is only proportional to N (rather than proportional to N^2 if all the elements of A are stored). Similarly, the linear system can be solved in times nearly proportional to N , rather than proportional to N^3 if Gaussian elimination is used. For up-to-date details of these fantastic (but complicated) algorithms see [9, 11]. Figure 4.2 shows a typical simulation by the group of Chew et al [9], who have achieved solutions with $N > 10^7$ using these schemes and powerful parallel computers.

Alternative matrix compression methods, e.g. [21], have been developed for surface which are almost flat, in the sense that the surface is basically a flat plane with deviations that are not too large in height compared to the wavelength (e.g. not more than 2-5 wavelengths). This is the case for many rough surface scattering problems. Methods of this type are particularly effective when the boundary is completely flat. For example, a method is described for propagation over an inhomogeneous impedance plane in [7] which allows problems with $N > 10^6$ to be solved in minutes on a PC, which contrasts with the limit of $N \approx 10^{2.5}$ for the same problem in 1985, using Gaussian elimination [5, 12].

5. High frequency approximation methods. In the last few years many authors have experimented with using, instead of polynomials as approximating functions, solutions of the Helmholtz equation (e.g. plane waves), or products of polynomials with plane waves. This approach still needs further research but is promising in terms of reducing substantially the size of N required for accurate approximation [17]. Indeed, for specific acoustic scattering problems, it is possible to almost remove the dependence of N on the wave number k , keeping N almost fixed as k increases, while retaining an accurate approximation; see [14, 6] and the references therein.
6. The mathematical theory of the boundary element method. You may have already had enough of mathematical theory in this area, but, if

not, there is much more including: conditions which ensure stability of numerical methods (e.g. that the condition number of the matrix A remains bounded as N increases); results on how fast different BEMs converge, and how this depends on mesh grading; analysis of fast multipole methods (estimating the error due to the matrix compression, and computational complexity); etc. Good starting points for more mathematics are the books of Hackbusch [13] and Colton & Kress [10] together with the forthcoming English translation of the book of Sauter and Schwab [18].

7. Time domain boundary element methods. The lectures today, on both finite and boundary element methods, have been entirely concerned with solving the reduced wave equation or Helmholtz equation (1.4). As mentioned on page 3, this is a perfectly feasible method of tackling time dependent problems: one represents the incident wave as a superposition of time harmonic waves of different frequencies, solves for each time harmonic incident field separately, and then combines the time harmonic total fields together to produce the acoustic field as a function of time. On the other hand, it is also possible to solve the wave equation (1.1) directly, either by finite difference/finite element methods (see the lecture on Friday), or by applying boundary element methods to a boundary integral equation reformulation of the wave equation (1.1), or by a combination of the two. See e.g. [22], or talk to Jonathan Hargeaves who has just completed his PhD on time domain BEM and room acoustic applications.

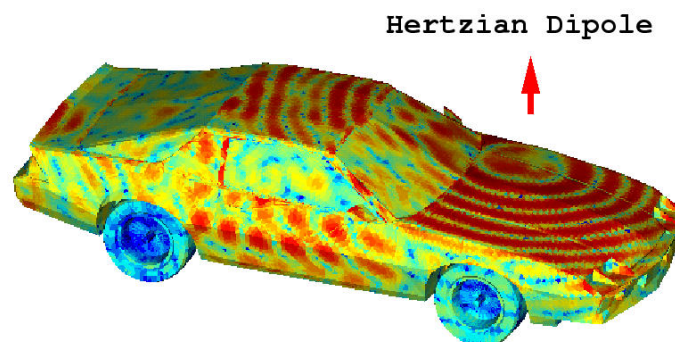


Figure 4.2: Large scale electromagnetic scattering problem solved using fast multipole methods by Chew et al. [9].

Bibliography

- [1] M Abramowitz and I Stegun. *Handbook of Mathematical Functions*. Dover, 1972.
- [2] G. Beer and J.O. Watson. *Introduction to Finite and Boundary Element Methods for Engineers*. Wiley: New York, 1992.
- [3] S. N. Chandler-Wilde. Tyndall medal lecture: the boundary element method in outdoor noise propagation. *Proc. Inst. Acoust.*, 19(8):27–50, 1997.
- [4] S. N. Chandler-Wilde and D. C. Hothersall. Efficient calculation of the Green function for acoustic propagation above a homogeneous impedance plane. *J. Sound Vib.*, 180:705–724, 1995.
- [5] S.N. Chandler-Wilde and D.C. Hothersall. Sound propagation above an inhomogeneous impedance plane. *J. Sound Vib.*, 98:475–491, 1985.
- [6] S.N. Chandler-Wilde and S. Langdon. A Galerkin boundary element method for high frequency scattering by convex polygons. *SIAM J. Numer. Anal.*, 45:610–640, 2007.
- [7] S.N. Chandler-Wilde, M Rahman, and C.R. Ross. A fast two-grid and finite section method for a class of integral equations on the real line with application to an acoustic scattering problem in the half-plane. *Numer. Math.*, 93:1–51, 2002.
- [8] G. Chen and J. Zhou. *Boundary Element Methods*. Academic Press, 1992.
- [9] W. C. Chew, J. M. Song, T. J. Cui, S. Velarnparambil, M. L. Hastriter, and B. Hu. Review of large scale computing in electromagnetics with fast integral equation solvers. *CMES-Comp. Model. Eng.*, 5:361–372, 2004.

- [10] D Colton and R Kress. *Integral Equation Methods in Scattering Theory*. Wiley, 1983.
- [11] E Darve. The fast multipole method: numerical implementation. *J. Comput. Phys.*, 160:195–240, 2000.
- [12] D. Habault. Sound propagation above an inhomogeneous plane. *J. Sound Vib.*, 100:55–67, 1985.
- [13] W. Hackbusch. *Integral Equations: Theory and Numerical Treatment*. Birkhäuser: Berlin, 1989.
- [14] S. Langdon and S.N. Chandler-Wilde. A wavenumber independent boundary element method for an acoustic scattering problem. *SIAM J. Numer. Anal.*, 5:2450–2477, 2006.
- [15] C. M. Linton. The Green’s function for the two-dimensional Helmholtz equation in periodic domains. *J. Eng. Math.*, 33:377–402, 1998.
- [16] P.A. Morgan. *Boundary element modelling and full scale measurement of the acoustic performance of outdoor noise barriers*. PhD thesis, Brunel University, 1999.
- [17] E. Perrey-Debain, O. Lagrouche, P. Bettess, and J. Trevelyan. Plane-wave basis finite elements and boundary elements for three-dimensional wave scattering. *Phil. Trans. R. Soc. Lond. A*, 362:561–577, 2004.
- [18] S. Sauter and C. Schwab. *Randelemente*. Teubner, 2004.
- [19] M.C.M. Wright (editor). *Lecture Notes on the Mathematics of Acoustics*. Imperial College Press: London, 2005.
- [20] L. Wrobel and F. Aliabadi. *The Boundary Element Method, Volume 1: Applications in Thermo-Fluids and Acoustics*. John Wiley: New York, 2002.
- [21] M. Xia, C. H. Chan, S. Li, B. Zhang, and L. Tsang. An efficient algorithm for electromagnetic scattering from rough surfaces using a single integral equation and multilevel sparse-matrix canonical-grid method. *IEEE Trans. Antennas Prop.*, AP-51:1142–1149, 2003.
- [22] A.E. Yilmaz, Z. Lou, E. Michielssen, and J.M. Jin. A single-boundary implicit and FFT-accelerated time-domain finite element-boundary integral solver. *IEEE Trans. Ant. Prop.*, 55:1382–1397, 2007.