Inverse Problems for Acoustic and EM Waves using the Point Source Method

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Outline

1. Introduction: approaches to inverse scattering
   - Setting for reconstruction problem
   - Iterative Methods
   - Decomposition and Sampling Methods

2. Field reconstructions via the point source method (PSM)
   - The PSM for 3d acoustics
   - The PSM for 3d electromagnetics

3. Shape reconstruction via the singular sources method (SSM)
   - The SSM for 3d acoustics
   - The SSM for medium reconstructions

4. Reduce need of data via the no response test (NRT)
   - The idea of the no response test
   - Numerical examples for the NRT
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Setting for reconstruction problem

Partial Differential Equation (Acoustic, Electromagnetic, Elastic)

Boundary Condition on Object

Remote Measurements

Incident Wave
Example: acoustic scattering problem

- **Bounded scatterer** in three dimensions with boundary of class $C^2$
- Incident field $u^i(x) = e^{i\kappa x \cdot d}$, i.e. plane wave with direction of incidence $d \in S$
- Scattered field $u^s$ solves Helmholtz equation
  \[ \Delta u^s + \kappa^2 u^s = 0 \]  
  in $\mathbb{R}^3 \setminus \overline{D}$ and satisfies the Sommerfeld radiation condition
  \[ r \left( \frac{\partial u^s}{\partial r} - i\kappa u^s \right) \to 0, \quad r = |x| \to \infty. \]  
- On the boundary $\Gamma := \partial D$ the total field $u = u^i + u^s$ vanishes, i.e. we have the Dirichlet boundary condition
  \[ u|_{\Gamma} = 0 \]
**Measured data**

Measured data are either the scattered field $u^s$ on some surface $\Lambda$ or the far field pattern $u^\infty$ defined by

$$u^s(x) = \frac{e^{i\kappa|x|}}{|x|} \left\{ u^\infty(\hat{x}) + O\left(\frac{1}{x}\right) \right\}, \quad \hat{x} := x/|x| \quad (4)$$

uniformly on $\mathbb{S}$ for $|x| \to \infty$.

In the above example the far field pattern is calculated via

$$u^\infty(x) = \frac{1}{4\pi} \int_{\Gamma} \frac{\partial e^{-i\kappa\hat{x} \cdot y}}{\partial \nu(y)} \varphi(y) ds(y), \quad \hat{x} \in \mathbb{S}. \quad (5)$$
The inverse problem

Task: reconstruct the shape and properties of the unknown scatterer!
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   - Numerical examples for the NRT
Consider the operator equation

\[ F(\Gamma) = 0 \]

with

\[ F(\Gamma) := u^\infty[\Gamma] - u^\infty_{\text{meas}} \]

for \( u^i \) fixed.

Treat it as general nonlinear operator equation!

Example: Federal project on humanitarian mine detection
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Decomposition Methods

+ First reconstruct $u^s$ from $u^\infty$.

+ Then use $u = u^i + u^s$ and the boundary condition to find $\Gamma$.

Decompose the problem into a *linear ill-posed problem* and a *nonlinear well-posed problem*. 
Sampling Methods: principle

Define some indicator function which characterizes the unknown scatterer $D$ and evaluate it on a sampling grid to find the shape $\Gamma = \partial D$.

Sampling principle I: point sampling
Definition: approaches to inverse scattering
Field reconstructions via the point source method (PSM)
Shape reconstruction via the singular sources method (SSM)
Reduce need of data via the no response test (NRT)

Setting for reconstruction problem
Iterative Methods
Decomposition and Sampling Methods

Sampling Methods: principle 2

Define some indicator function on a set of domains $G$ to discriminate $D \subset G$ from $D \not\subset G$ and sample the area with the unknown scatterer!

Sampling principle 2: domain sampling
Sampling Methods

1. Linear sampling method (Colton-Kirsch)
2. Factorization method (Kirsch)
3. **Singular sources method (P.)**
4. Probe method (Ikehata)
5. Enclosure method (Ikehata)
7. Range test (Kusiak/P./Sylvester)
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Decomposition Methods: Point source method

1. Green's formula (using Dirichlet boundary condition)

\[ u^s(x) = \int_{\Gamma} \Phi(x, y) \frac{\partial u}{\partial \nu}(y) ds(y) \]  

(6)

2. Approximation of the point source (on test domain)

\[ \Phi(x, y) \approx \int_S e^{i\kappa y \cdot d} g_x(d) ds(d) \]  

(7)

3. Insert and exchange order of integration:

\[ u^s(x) \approx \int_{\Gamma} \left( \int_S e^{i\kappa y \cdot d} g_x(d) ds(d) \right) \frac{\partial u}{\partial \nu}(y) ds(y) \]

\[ = \int_S \left( \int_{\Gamma} e^{i\kappa y \cdot d} \frac{\partial u}{\partial \nu}(y) ds(y) \right) g_x(d) ds(d) \]

\[ = 4\pi \int_S u^\infty(-d) g_x(d) ds(d). \]  

(8)
PSM: acoustics 3d, field reconstruction

PSM: Field reconstruction by Ben Hassen, Erhard and P. 2005
PSM: Shape reconstruction by Ben Hassen, Erhard and P. 2005
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PSM: electromagnetics 3d, field reconstruction

Point source method, electromagnetics
(Stratton-Chu formula and approximation of the point source.)

\[ E_s(x) = \text{curl} \int_{\Gamma} \nu(y) \times E_s(y) \Phi(x, y) \, ds(y) \]
\[ - \frac{1}{i \kappa} \text{curl curl} \int_{\Gamma} \nu(y) \times H_s(y) \Phi(x, y) \, ds(y), \quad x \in \mathbb{R}^3 \setminus \overline{D}, \quad (9) \]

Approximation of Point source by superposition, exchange of the order of integration ⇒ Reconstruction formula

\[ E_s(x) \approx 4\pi \int_{S} E_{\infty}(-d)g_x(d) \, ds(d), \quad x \in \mathbb{R}^3 \setminus \overline{D} \quad (10) \]
PSM: electromagnetics 3d, field reconstruction

Original and Reconstruction of the modulus of the total electromagnetic field.
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Singular sources method, idea

scatterers → singular source → measurements
Basic property used for the singular sources method

\[ |\Phi^s(z, z)| \to \infty \text{ for } z \to \Gamma. \]  \hspace{1cm} (11)

1. Use PSM to reconstruct \( \Phi^s(z, z) \) on a sampling grid from the knowledge of \( u^\infty(\hat{x}, d) \) for \( \hat{x}, d \in \mathbb{S} \).
2. Use the blow-up (11) to find the unknown shape as level curves of \( |\Phi^s(z, z)| \).
Singular sources method, derivation

Approximation of the point source (on test domain)

\[ \Phi(x, y) \approx \int_S e^{i \kappa \cdot d} g_y(d) ds(d) \quad (12) \]

leads to the far field approximation

\[ \Phi^\infty(\hat{x}, y) \approx \int_S u^\infty(\hat{x}, d) g_y(d) ds(d) \quad (13) \]

Using the point source method to reconstruct \( \Phi^s(\cdot, y) \) from \( \Phi^\infty(\cdot, y) \) leads to the formula

\[ \Phi^s(x, y) \approx 4\pi \int_S \int_S u^\infty(-\hat{x}, d) g_y(d) ds(d) g_x(\hat{x}) ds(\hat{x}) \quad (14) \]
Singular sources method

Neumann BC, Indicator function and Reconstruction.
Image by F. Ben Hassen 2005.
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Singular sources method: T shape

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The singular sources method (SSM) can also be used to detect **piecewise constant inhomogeneous media**! The strength of the blow-up of the scattered field of multipoles is proportional to the jump in the refractive index.

P.: "A new non-iterative singular sources method for the reconstruction of piecewise constant media." Num.Math 2004
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PSM and SSM revisited

- **PSM** uses measurements for *one wave*, but needs to know the boundary condition for shape reconstruction (only impenetrable scatterers)
- **SSM** uses measurements for *many incident waves*, does not need to know the boundary condition

Is there a one-wave method which does not need to know the boundary condition?
The idea of the no response test

**Setting.** Consider scattering of time-harmonic acoustic waves by some sound-soft obstacle (Dirichlet boundary condition).

**Preparation step I:** The far field pattern can be expressed as:

\[
\begin{align*}
  u^\infty(\hat{x}) &= \frac{1}{4\pi} \int_\Gamma e^{-i\kappa \hat{x}\cdot y} \frac{\partial u}{\partial \nu}(y) ds(y) \\
  &\quad \text{for } \hat{x} \in \mathbb{S}.
\end{align*}
\]
Preparation step II:
Consider a superposition of plane waves (a Herglotz wave function)

$$v[g](y) := \int_S e^{-i\kappa y \cdot \hat{x}} g(\hat{x}) ds(\hat{x}), \quad y \in \mathbb{R}^m$$  \hspace{1cm} (16)

with density $g \in L^2(S)$.

Choosing appropriate densities $g$ we can make $v[g]$ arbitrarily small on some given test domain $\overline{G}$ where on compact subsets of $\mathbb{R}^m \setminus \overline{G}$ the function $v[g]$ becomes arbitrarily large.
The idea of the no response test

Basic derivation step:
Multiply $u^\infty$ by $g \in L^2(\mathbb{S})$, integrate over $\mathbb{S}$ and exchange the order of integration to obtain

$$
\mu(g) := \int_{\mathbb{S}} g(\hat{x}) u^\infty(\hat{x}) \, ds(\hat{x})
$$

$$
= \int_{\mathbb{S}} g(\hat{x}) \left( 4\pi \int_{\Gamma} e^{-i\kappa \hat{x} \cdot y} \frac{\partial u}{\partial \nu}(y) \, ds(y) \right) \, ds(\hat{x})
$$

$$
= 4\pi \int_{\Gamma} \left( \int_{\mathbb{S}} e^{-i\kappa \hat{x} \cdot y} g(\hat{x}) \, ds(\hat{x}) \right) \frac{\partial u}{\partial \nu}(y) \, ds(y)
$$

$$
= 4\pi \int_{\Gamma} \nu[g](y) \frac{\partial u}{\partial \nu}(y) \, ds(y).
$$

response
NRT sampling

Sampling for the no response test
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Reconstruction example from inverse scattering

Boat-like sound-soft obstacle:
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Reduce need of data via the no response test (NRT)

Numerical proof of concept

Boat-like sound-soft obstacle2:
Boat-like sound-hard obstacle:
Numerical proof of concept

Rectangular penetrable obstacle:
Magnetic tomography

Images by L. Kühn, Dissertation Thesis Göttingen 2005
Summary

- We have surveyed three categories of methods for inverse shape reconstruction problems:
  - iterative
  - decomposition
  - sampling

- Recent results on the point source method PSM in 3d acoustics and electromagnetics have been shown.

- We presented the singular sources method SSM for obstacle reconstructions in three dimensions.

- The no response test has been introduced with its basic idea and numerical examples for acoustics and magnetic tomography.
For Further Reading

Potthast, R.
*Point-sources and Multipoles in Inverse Scattering.*

G. Nakamura, M. Sini and R. Potthast
*Unification of the probe and singular sources methods for the inverse boundary value problem by the no-response test.*
CPDE, to appear.

Potthast, R.
*A new non-iterative singular sources method for the reconstruction of piecewise constant media.*
Numerische Mathematik 98 (2004), 703-730.