# The Klein-Gordon equation in a domain with time-dependent boundary

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## 1 Introduction

In the past ten years, an alternative method for solving boundary value problems for linear partial differential equations (PDEs), proposed by Fokas [4], has been applied successfully to several examples. This approach is motivated by the inverse scattering transform, a method introduced originally for the analysis of the Cauchy problem for integrable nonlinear PDEs, and can be considered a generalisation of the Fourier transform. Indeed, for linear initial value problems, this approach and the usual Fourier transform approach are equivalent. However, this is no longer the case in the presence of boundaries, when the approach of Fokas yields in general a representation of the solution of the problem, in terms of the given data, as an integral along a complex contour.

This method of analysis of linear boundary value problems is based on replacing the classic method of separation of variables by the observation that any linear PDE in two variables can be written as the compatibility condition of a system of two linear eigenvalue ODEs. This pair of ODEs, called the *Lax pair* in analogy with the nonlinear integrable case, involves an additional complex parameter. It can be shown that the solution of the boundary value problem can be expressed in terms of the particular solution of this system which is bounded for all values of the parameter. The analysis of the Lax pair depends essentially on analyticity considerations, usually through the formulation of a Riemann-Hilbert problem. Indeed, complex analysis techniques and results are essential in this methodology.
This transform approach has been used to solve boundary value problems posed in fixed domains for linear evolution PDEs in one spatial dimension, involving space derivatives of any order [6], and of linear elliptic problems in two variables [5, 9].

Recently, the authors used this approach to study boundary value problems for linear hyperbolic PDEs. Two physically important linear hyperbolic PDEs are the wave and the Klein-Gordon equations:

$$q_{tt} - q_{xx} = 0, \;\; (wave) \quad q_{tt} - q_{xx} + q = 0 \;\; (Klein-Gordon). \tag{1.1}$$

We focus here on the Klein-Gordon equation (1.1b). The canonical form of this equation is the linearised sine-Gordon equation $q_{tx}+q=0$, for which boundary value problems have already been considered using the approach used here, starting with the work presented in [4]. Note however that the change of variable that reduces equation (1.1b) to its canonical form leads to non physical boundary conditions.
The classical problem for equation (1.1) posed on the half line $x>0$ has been considered by the authors in [15]. Our approach yields the solution as an integral along the real line, and is essentially equivalent to the Fourier transform method. However, the inversion of the integral formula for the unknown boundary value via the Fourier transform requires a series of normalisations and changes of variables. In contrast, our approach yields algorithmically the solution expressed in the most suitable variables, hence the inversion is straightforward. In addition, this example indicates how the method can be generalised to treat more general hyperbolic boundary value problems.

In this paper, we turn our attention to the general case of boundary value problems for linear hyperbolic equations such as (1.1) posed on a time dependent domain of the form

$$\mathcal{D}=\{x>l(t), \quad t>0\}, \quad l''(t)>0, \ \ 0<l'(t)<1, \tag{1.2}$$

where $l(t)$ is a given, smooth function describing the moving boundary. Analogous problems have been studied by one of the authors in the case of evolution PDEs [8, 12]. In these papers, it was shown that the treatment of such moving boundary value problems presents an important difference with the case of fixed domains. Indeed, in this case analyticity requirements must be relaxed, and the Riemann-Hilbert problem must be generalised to a Dbar problem.
An example of moving boundary value problems for equation (1.1(a)) has been considered in [14], where we show that for this equation we can still complete the analysis in terms of a Riemann-Hilbert problem, hence analyticity and the analogy with the classical Fourier transform are retained. This is consistent with the fact that this problem can be solved by Fourier transform. However, we show presently that for the solution of boundary value problems for the Klein Gordon equation posed on a domain of the form (1.2) we need to solve a Dbar problem.
The assumption of convexity for the function $l(t)$ could be replaced by an assumption of concavity, provided that the first derivative is monotonic (the speed of the boundary can decrease or increase, but not alternate between the two regimes). The bounds on $l'(t)$ constrain the speed at which the boundary moves compared with the speed of the wave solution. Namely, we require the boundary to move with a positive speed not exceeding the speed of the wave (which in our coordinates is equal to 1).
Our approach can be used to solve a general moving boundary value problem for the Klein Gordon equation in domains such as (1.2). To give a concrete illustration of the steps involved, we solve the following boundary value problem:

$$q_{tt}-q_{xx}+q=0, \qquad x>l(t), \ \ t>0, \tag{1.3}$$

$$q(x,0)=q_0(x), \ \ q_t(x,0)=q_1(x), \quad x>l(0), \tag{1.4}$$

$$q(l(t),t)=f_0(t), \qquad t>0. \tag{1.5}$$

We are not presently concerned with issues of regularity, hence we assume that all prescribed functions decay at infinity and are as smooth as needed to perform all operations required. The motivation for studying such boundary value problems comes from the consideration of certain free boundary value problem, of interest in engineering, modelled by the wave or Klein-Gordon equation in domains with a time-dependent boundary [3, 11]. To our knowledge, such problems have not been studied analytically in the mathematical literature.

The paper in divided in two parts. In one part, independently of the boundary value problem, we derive an inversion theorem for a certain integral transform. This transform is in some sense a generalisation of the Fourier transform but cannot be obtained directly by manipulating the Fourier transform formulas. The inversion result, and its derivation, are similar to the transform derived to solve moving boundary value problems for evolution equations [8].

In the second part, we solve the boundary value problem (1.3). This solution involves two distinct steps: the derivation of formal integral representations of the solution, and the characterisation, in terms of the given data of the problem, of the unknown boundary value involved in this representation.

The details of the computations that lead to these results are mainly presented in section 2. In this section, we introduce two Lax pairs associated with the PDE. One of the ODEs appearing in these Lax pairs motivates the definition of the following generalisation of the usual Fourier transform

$$F(k) = \int_0^\infty e^{-ik_+ s - ik_- l(s)} f(s) ds. \quad k \in \mathbb{C}^*, \tag{1.6}$$

Here $l(t)$ is the same as in (1.2), $f(t)$ is a the smooth function defined on $[0, \infty)$, decaying as $t \to \infty$, and

$$k_- = \frac{1}{2}\left(k - \frac{1}{k}\right), \quad k_+ = \frac{1}{2}\left(k + \frac{1}{k}\right). \tag{1.7}$$

Here, and throughout the paper, $\mathbb{C}^*$ denotes the usual extended complex plane:

$$\mathbb{C}^* = \mathbb{C} \cup \{\infty\}. \tag{1.8}$$

We prove that this trasform can be inverted implicitly, through the solution of a linear Volterra integral equation. Namely, we have the following result.

**Theorem 1.1** *Let $f(t)$ be an arbitrary smooth, decaying function defined for $t \geq 0$. Let the function $F(k)$ be defined in terms of the function $f(t)$ by (1.6), where we assume that $l(t)$ is a given convex function of $t$ with $l'(0) = 0$ and $l'(t) \leq 1$. Then the function $f(t)$ can be expressed in terms of the function $F(k)$ by*

$$f(t) = -\frac{1 + l'(t)}{2\pi} \int_{\partial D_2^-} e^{ik_+ t + ik_- l(t)} F(k) dk - \frac{1 + l'(t)}{2\pi} \int_0^t K(s,t) f(s) ds. \tag{1.9}$$

*The kernel $K(s,t)$ in (1.9) is a well-defined integral kernel given by*

$$K(s,t) = \int_0^\pi i\rho(s) e^{i\theta} E(\rho(s)e^{i\theta}, s, t) d\theta + 2Re \int_0^{\rho(s)} E(k,t,s) dk, \tag{1.10}$$

*where*

$$\rho(t) = \sqrt{\frac{1-l'(t)}{1+l'(t)}}, \tag{1.11}$$

$$E(k,t,s) = \mathrm{e}^{ik_+(t-s)+ik_-(l(t)-l(s))}, \tag{1.12}$$

*and the $t$-dependent domain $D_2^-$ is defined by*

$$D_2^- = \{k \in (\mathbb{C}^*)^- : \ |k| > \rho(t)\}. \tag{1.13}$$

The appearence of a Volterra integral equation reflects the fact that the spectral problem associated with the PDE on the domain $\mathcal{D}$ given by (1.2) is solved via a Dbar, rather than a Riemann-Hilbert, problem.
We prove this inversion formula in detail, as its derivation forms the basis for all computations presented in the sequel.
In section 3, we derive two equivalent integral representation of the solution of the boundary value problem (1.3), and compare them. Both representations are formal, as they involve the unknown function $q_x(l(t),t)$. Hence the main result of this section is the proof that this unknown function can be characterised uniquely as the solution of a Volterra integral equation. Namely, we prove the following.

**Proposition 1.1** *Let $q(x,t)$ be the solution of the boundary value problem (1.3). The boundary value $q_x(l(t),t)$ is the unique solution of the linear Volterra integral equation*

$$q_x(l(t),t) = \frac{1}{2\pi((l'(t)-1)}H(t) + \frac{1}{2\pi((l'(t)-1)}\int_0^t (1-l'(s)^2)K(s,t)q_x(l(s),s)ds. \tag{1.14}$$

*In this expression, $H(t)$ is a function defined explicitly in terms of the given initial and boundary conditions as*

$$H(t) = \int_{\partial D_2^-} \mathrm{e}^{ik_+t+ik_-l(t)} \left\{ \int_0^\infty \mathrm{e}^{-ik_-y}(q_t + ik_+q)(y,0)dy - \right. \tag{1.15}$$
$$\left. - \int_0^\infty \mathrm{e}^{-ik_+s-ik_-l(s)}[l'(s)[q(l(s),s)]' + (ik_+l'(s) + ik_-)q(l(s),s)]ds \right\} dk,$$

*the domain $D_2^-$ is given by (1.13), and the kernel $K(s,t)$ is given by (1.10). The expressions $\rho(t)$, $E(\lambda,t,s)$ are defined by (1.11)-(1.12).*

The proof of this proposition is based on the inversion theorem of section 2.

# 2 The Lax pair formulation of the Klein-Gordon equation and a Fourier-type transform

We derived in [15] the Lax pair formulation for equation (1.3), a pair of linear ODE whose compatibility condition is the Klein-Gordon equation. These ODEs, after a uniformising change of variable, are the following:

$$\begin{cases} M_x(x,t,k) - ik_- M(x,t,k) = q(x,t), \\ \\ M_{tt}(x,t,k) + ik_+^2 M(x,t,k) = q_x(x,t) + ik_- q(x,t). \end{cases} \tag{2.16}$$

where $M(x,t,k)$ is a real function, and $k_\pm$ are defined by (1.7).
The pair of ODEs (2.16) can be written as a first order matrix system, which reduces by diagonalisation to the two first order pairs

$$\begin{cases} \nu_x - ik_-\nu = q_t + ik_+ q \\ \nu_t - ik_+\nu = q_x + ik_- q, \end{cases} \tag{2.17}$$

and

$$\begin{cases} \mu_x - ik_-\mu = q_t - ik_+ q \\ \mu_t + ik_+\mu = q_x + ik_- q. \end{cases} \tag{2.18}$$

The second of these pairs is obtained from the first by the transformation $k \to -\frac{1}{k}$. Each one of the pairs (2.17), (2.18) has the PDE as its compatibility condition. Hence a representation for $q(x,t)$ could be derived, by asymptotic considerations, from the analysis of either one of them, as we show in section 3. However, by considering the solution of both of them explicitly, we can also easily obtain a representation for $q(x,t)$:

$$M(x,t,k) = \frac{\nu(x,t,k) - \mu(x,t,k)}{2ik_+}, \quad q(x,t) = M_x - ik_- M. \tag{2.19}$$

## 2.1 The global relation

In what follows, we consider the first order Lax pair (2.17). This Lax pair is equivalent to the condition

$$[\mathrm{e}^{-ik_-x - ik_+t}(q_t + ik_+q)]_t - [\mathrm{e}^{-ik_-x-ik_+t}(q_x + ik_-q)]_x = 0.$$

From this, using Green's theorem in the simply connected domain $\mathcal{D}$ defined by (1.2), we obtain the global relation:

$$\int_{\partial\mathcal{D}} \mathrm{e}^{-ik_-x-ik_+t}[(q_t + ik_+q)dx + (q_x + ik_-q)dt] = 0.$$

Using the definition of $\mathcal{D}$, and the decay of the solution at infinity, the global relation can be written explicitly as follows:

$$\int_{l(0)}^{\infty} \mathrm{e}^{-ik_-x}(q_t + ik_+q)(x,0)dx = \int_0^{\infty} \mathrm{e}^{-ik_-l(t)-ik_+t}[l'(t)(q_t + ik_+q) + (q_x + ik_-q)](l(t),t)dt. \tag{2.20}$$

Note that $q(x,0) = q_0(x)$, $q_t(x,0) = q_1(x)$, and $q(l(t),t) = f_0(t)$. Hence, since

$$f_0'(t) = q_t(l(t),t) + l'(t)q_x(l(t),t),$$

we can write the global relation as

$$\int_0^\infty \mathrm{e}^{-ik_-l(t)-ik_+t}(1-l'(t)^2)q_x(l(t),t)dt = \int_{l(0)}^\infty \mathrm{e}^{-ik_-x}(q_1+ik_+q_0)(x)dx \tag{2.21}$$
$$-\int_0^\infty \mathrm{e}^{-ik_-l(t)-ik_+t}[l'(t)f_0'(t)+i(l'(t)k_+k_-)f_0(t)]dt.$$

We shall use this relation to characterise the unknown boundary functions $q_x(l(t),t)$, the "Neumann datum" at the boundary. However, in the expression (2.21) there appears an integral transform of this function, rather than the function itself.
In the next section, we prove an inversion formula for this particular transform.

## 2.2 A Fourier-type transform

In this section, we give the proof of Theorem (1.1).
Let $\nu(x,t,k)$ be the solution of (2.17b). Then the function $N(t,k)=\nu(l(t),t,k)$ satisfies

$$N_t - i(k_+ + l'(t)k_-)N = [q_x + l'(t)q_t + i(k_- + l'(t)k_+)q](l(t),t), \quad k\in\mathbb{C}^*. \tag{2.22}$$

Motivated by this remark, we consider the ODE

$$N_t - i(k_+ + l'(t)k_-)N = f(t), \quad k\in\mathbb{C}^* \tag{2.23}$$

where $f(t)$ is a given, smooth function of $t$.
Before proving the theorem, we perform the spectral analysis of the ODE (2.23). This yields a formal representation for $f(t)$.

**Proposition 2.1** *Let $f(t)$ be a smooth decaying function defined for $t\geq 0$. Let the function $F(k)$ be defined in terms of the function $f(t)$ by (1.6), where we assume that $l(t)$ is a given convex function of $t$ with $l'(0)=0$ and $l'(t)\leq 1$. Then the function $f(t)$ can be expressed in terms of the function $F(k)$ by*

$$\begin{aligned} f(t) &= -\frac{1+l'(t)}{4\pi}\left\{\int_{\partial D_2^-} E(\lambda,t,0)F(\lambda)d\lambda + \int_{-1}^{1}\left[\int_0^{S(|\lambda|)} E(\lambda,t,s)f(s)ds\right]d\lambda\right. \\ &+ \left.\int\int_{D_3}\frac{\partial S(|\lambda|)}{\partial\bar\lambda}E(\lambda,t,S(|\lambda|))f(S(|\lambda|)\,d\lambda\wedge d\bar\lambda\right\}. \end{aligned} \tag{2.24}$$

*where $S(|k|)$ is defined as*

$$S(|k|) = \begin{cases} s & for\ \ |k|=\rho(s), 0\leq s<\infty, \\ \infty & for\ \ |k|\leq\rho(\infty), \end{cases} \tag{2.25}$$

*with $\rho(t)$, $E(k,t,s)$ given by (1.11), $D_3$ defined below by (2.30) and $D_2^-$ defined by (1.13).*

**Proof**: We seek solutions of the ODE (2.23) that are bounded in $k$. Two particular solutions of this ODE are given by

$$N_1(t,k) = \int_0^t \mathrm{e}^{ik_+(t-s)+ik_-(l(t)-l(s))} f(s)ds, \tag{2.26}$$

$$N_2(t,k) = -\int_t^\infty \mathrm{e}^{ik_+(t-s)+ik_-(l(t)-l(s))} f(s)ds. \tag{2.27}$$

To determine where each of these two functions is bounded as a function of $k$, we consider the exponential of the integrand. The real part of the exponent of this exponential is

$$\begin{aligned} Re(ik_+(t-s)+ik_-(l(t)-l(s)) &= -(t-s)\mathrm{Im}[k(1+l'(\tau)+\frac{1}{k}(1-l'(\tau))] \\ &= -(t-s)k_2[1+l'(\tau)-\frac{1}{|k|^2}(1-l'(\tau))], \end{aligned}$$

where $l'(\tau) = \frac{l(t)-l(s)}{t-s}$ so that $\tau$ is a point in the interval between $s$ and $t$.
Since $l'(t) < 1$ and $l''(t) > 0$ for $0 < \tau < t$ we have

$$0 = l'(0) < l'(\tau) < l'(t) \Rightarrow \frac{1-l'(t)}{1+l'(t)} < \frac{1-l'(\tau)}{1+l'(\tau)} < \frac{1-l'(0)}{1+l'(0)} = 1$$

and similarly for $t < \tau < \infty$ we have

$$l'(t) < l'(\tau) < l'(\infty) \Rightarrow \frac{1-l'(\infty)}{1+l'(\infty)} < \frac{1-l'(\tau)}{1+l'(\tau)} < \frac{1-l'(t)}{1+l'(t)},$$

where $l'(\infty) = \lim_{\sigma\to\infty} l'(\sigma)$. Since by assumption $l'(\infty) \leq 1$, we have

$$\frac{1-l'(\infty)}{1+l'(\infty)} \geq 0.$$

$\mathbf{N_1}$: In this case, $t - s \geq 0$. Hence the solution $N_1(t,k)$ is asymptotically bounded as a function of $k$ if

$$\textit{either}\;\; k_2 \geq 0 \text{ and } |k|^2 \geq \frac{1-l'(\tau)}{1+l'(\tau)} \quad \textit{or}\;\; k_2 \leq 0 \text{ and } |k|^2 \leq \frac{1-l'(\tau)}{1+l'(\tau)}$$

$\mathbf{N_2}$: In this case, $t - s \leq 0$, hence the solution $N_2(t,k)$ is asymptotically bounded as a function of $k$ if

$$\textit{either}\;\; k_2 \geq 0 \text{ and } |k|^2 \leq \frac{1-l'(\tau)}{1+l'(\tau)} \quad \textit{or}\;\; k_2 \leq 0 \text{ and } |k|^2 \geq \frac{1-l'(\tau)}{1+l'(\tau)}$$

Hence, recalling the definition (1.11) of the function $\rho(t)$, the functions $N_i(t,k)$ are bounded, and indeed analytic, functions of $k$ in the region $D_i$ given by

$$\begin{aligned} D_1 &= \{k \in (\mathbb{C}^*)^+ : |k|^2 \geq 1\} \cup \{k \in (\mathbb{C}^*)^- : |k|^2 \leq \rho(t)^2\} & (2.28) \\ D_2 &= \{k \in (\mathbb{C}^*)^+ : |k|^2 \leq \rho(\infty)^2\} \cup \{k \in (\mathbb{C}^*)^- : |k|^2 \geq \rho(t)^2\}. & (2.29) \end{aligned}$$

It follows that for $k$ in the region $D_3$ defined by

$$D_3 = \left\{ k \in (\mathbb{C}^*)^+ : \ \rho(\infty) < |k|^2 < 1 \right\}, \tag{2.30}$$

We have not yet defined a solution of (2.23) bounded with respect to $k$. To this end, let

$$N_3 = \int_{S(|k|)}^{t} \mathrm{e}^{ik_+(t-s)+ik_-(l(t)-l(s))} f(s)ds \tag{2.31}$$

where the function $S(|k|)$ is defined on $D_3$ by (2.25). Note that since $S(|k|)$ is a function of both $k$ and $\bar{k}$, the function $N_3(t,k,\bar{k})$. is not an analytic function of $k$. However, we claim that $N_3$ is bounded in $D_3$. Indeed:

- If $S(|k|) < t$:

  then we need $|k|^2 > \frac{1-l'(\tau)}{1+l'(\tau)}$, which follows from

$$S(|k|) < s < \tau \Rightarrow |k|^2 > \frac{1-l'(s)}{1+l'(s)} > \frac{1-l'(\tau)}{1+l'(\tau)}.$$

- If $S(|k|) > t$:

  then we need $|k|^2 < \frac{1-l'(\tau)}{1+l'(\tau)}$, which follows from

$$\tau < s < S(|k|) \Rightarrow |k|^2 < \frac{1-l'(s)}{1+l'(s)} < \frac{1-l'(\tau)}{1+l'(\tau)}.$$

Integration by parts of the equations defining the $N_j$'s implies the following asymptotic behaviour

$$N_j = O\left(\frac{1}{k}\right), \quad k \in \ D_j, \quad k \to \infty, \quad j = 1,2,3. \tag{2.32}$$

Hence equations (2.26), (2.27) and (2.31) define a function $N(t,k,\bar{k})$ in terms of $f(t)$. This function is bounded for every $k \in \mathbb{C}^*$. Using this boundedness, it is possible to find an alternative representation for this function using the Dbar formula [1],

$$N(t,k,\bar{k}) = \frac{1}{2\pi i}\int_{\gamma(t)} \frac{\text{“jumps''}}{\lambda - k} d\lambda + \frac{1}{2\pi i}\int\int_{D_3} \frac{\partial N(t,\lambda,\bar{\lambda})}{\partial\bar{\lambda}} \frac{d\lambda\wedge d\bar{\lambda}}{\lambda - k}, \quad 0 < t < \infty,\, k \in \mathbb{C}^*, \tag{2.33}$$

where $d\lambda \wedge d\bar{\lambda} = -2id\lambda_R d\lambda_I$ if $\lambda = \lambda_R + i\lambda_I$, and $\gamma(t)$ denotes the contour along which the function $N$ has a “jump” discontinuity and $D_3$ is the domain where $\partial N/\partial\bar{k} \neq 0$. The “jumps” are given by the expressions

$$\begin{aligned}
N_1 - N_2 &= \int_0^\infty E(k,t,s)f(s)ds, \quad k \in D_1 \cap D_2, \\
N_1 - N_3 &= \int_0^{S(|k|)} E(k,t,s)f(s)ds, \quad k \in \mathbb{R} \cap D_1 \cap D_3 \\
N_3 - N_2 &= \int_{S(|k|)}^\infty E(k,t,s)f(s)ds, \quad k \in D_3 \cap D_2,
\end{aligned}$$

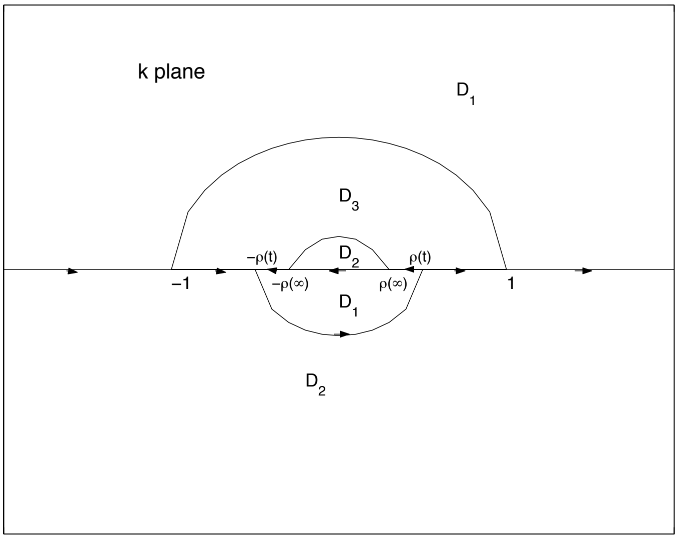


Figure 1: The domains $D_1$, $D_2$ and $D_3$ in the $k$-plane. Here $l'(0) = 0$, hence $\rho(0) = 1$.

where the function $E(k,t,s)$ is given by (1.12), and the orientation of the curves $D_i \cap D_j$ is always counterclockwise when seen from inside $D_i$, see figure 1.
Note that the definition of $S(|k|)$ implies that, for $k \in (\mathbb{C}^*)^+$ such that $|k| = \rho(0) = 1$, $S(|k|) = 0$, hence $N_1 = N_3$ along this semicircle. Similarly, $N_2 = N_3$ along the semicircle $|k| = \rho(\infty)$ in the upper half plane. Hence along these two semicircles there is no jump discontinuity.
We now compute the Dbar derivative of $N$ for $k \in D_3$. Since the only representation of $N(t,k)$ which is not analytic is $N_3$, we have

$$\frac{\partial N}{\partial \bar{k}} = \frac{\partial N_3}{\partial \bar{k}} = -\frac{\partial S(|k|)}{\partial \bar{k}} E(k,t,S(|k|)) f(S(|k|)). \tag{2.34}$$

Hence, recalling the definition (1.6) of $F(\lambda)$, and writing the various contours explicitly, we can write the expression (2.33) as

$$\begin{aligned} N(t,k,\bar{k}) &= \frac{1}{2\pi i}\left(\int_{-\infty}^{-1} + \int_{1}^{\infty} - \int_{-\rho(\infty)}^{\rho(\infty)} + \int_{|k|=\rho(t),\, Im k \le 0}\right) E(\lambda,t,0)F(\lambda)\frac{d\lambda}{\lambda - k} \\ &- \frac{1}{2\pi i}\left(\int_{-\rho(t)}^{-\rho(\infty)} + \int_{\rho(\infty)}^{\rho(t)}\right)\left[\int_0^{S(|\lambda|)} E(\lambda,t,s)f(s)ds\right]\frac{d\lambda}{\lambda - k} \\ &+ \frac{1}{2\pi i}\left(\int_{-1}^{-\rho(t)} + \int_{\rho(t)}^{1}\right)\left[\int_{S(|\lambda|)}^{\infty} E(\lambda,t,s)f(s)ds\right]\frac{d\lambda}{\lambda - k} \\ &- \frac{1}{2\pi i}\int\int_{D_3} \frac{\partial S(|\lambda|)}{\partial \bar{\lambda}} E(\lambda,t,S(|\lambda|))f(S(|\lambda|))\frac{d\lambda \wedge d\bar{\lambda}}{\lambda - k}. \end{aligned} \tag{2.35}$$

Adding and subtracting from this expression the term

$$\frac{1}{2\pi i}\left(\int_{-1}^{-\rho(t)}+\int_{\rho(t)}^{1}\right)\int_{0}^{S(|\lambda|)}E(\lambda,t,s)f(s)ds$$

and taking into account the definition of $F(\lambda)$, we find the terms

$$\left(\int_{-\infty}^{-\rho(t)}+\int_{\rho(t)}^{\infty}+\int_{|k|=\rho(t),\,Imk\le 0}\right)E(\lambda,t,0)F(\lambda)\frac{d\lambda}{\lambda-k}=-\int_{\partial D_2^-}E(\lambda,t,0)F(\lambda)\frac{d\lambda}{\lambda-k},$$

where $D_2^-$ is given by (1.13). Combining all resulting terms, we then obtain

$$\begin{aligned} N(t,k,\bar{k}) &= -\frac{1}{2\pi i}\int_{\partial D_2^-}E(\lambda,t,0)F(\lambda)\frac{d\lambda}{\lambda-k}-\int_{-\rho(\infty)}^{\rho(\infty)}E(\lambda,t,0)F(\lambda)\frac{d\lambda}{\lambda-k}\\ &- \frac{1}{2\pi i}\left(\int_{-1}^{-\rho(\infty)}+\int_{\rho(\infty)}^{1}\right)\left[\int_{0}^{S(|\lambda|)}E(\lambda,t,s)f(s)ds\right]\frac{d\lambda}{\lambda-k}\\ &- \frac{1}{2\pi i}\int\int_{D_3}\frac{\partial S(|\lambda|)}{\partial\bar{\lambda}}E(\lambda,t,S(|\lambda|))f(S(|\lambda|))\frac{d\lambda\wedge d\bar{\lambda}}{\lambda-k}. \end{aligned} \tag{2.36}$$

By the definition (2.25) of $S(|k|)$, when $|k|\le\rho(\infty)$ we have $S=\infty$. Hence we can write

$$\int_{-\rho(\infty)}^{\rho(\infty)}E(\lambda,t,0)F(\lambda)\frac{d\lambda}{\lambda-k}=\int_{-\rho(\infty)}^{\rho(\infty)}\int_{0}^{S(|\lambda|)}E(\lambda,t,s)f(s)ds\frac{d\lambda}{\lambda-k}$$

and we can combine the second and third integral in expression (2.36) into the single integral

$$-\int_{-1}^{1}\int_{0}^{S(|\lambda|)}E(\lambda,t,s)f(s)ds\frac{d\lambda}{\lambda-k}.$$

Finally, computing $N_t-i(k_+ + l'(t)k_-)N$ we find that $f(t)$ is given by (2.24).
**QED**

**Proof of Theorem 1.1** To prove this result, we follow the approach of [8] and show that equation (2.24) can be rewritten as equation (1.9).
We start by extending the double integral to the larger semicircle $D_2^+\cup D_3$, the domains defined by (2.29) and (2.30). Since the Dbar derivative of the integrand vanishes for $k\in D_2^+$, this does not change the value of the integral. Then we subdivide the domain $D_2^+\cup D_3$ in two disjoint regions:

$$D_2^+\cup D_3=D_3^{(1)}\cup D_3^{(2)}:\quad D_3^{(1)}=\{k\in D_3:\rho(t)<|k|<1\},\ D_3^{(2)}=\{k\in D_3:0<|k|<\rho(t)\}.$$

In the domain $D_3^{(2)}$ we use the complex form of Green's theorem:

$$\int\int_{D_3^{(2)}}\frac{\partial N_3(k,\bar{k})}{\partial\bar{k}}dk\wedge d\bar{k}=-\int_{\partial D_3^{(2)}}N_3(k,\bar{k})dk, \tag{2.37}$$

The boundary of $D_3^{(2)}$ is composed of the semicircle $|k|^2 = \rho^2(t)$ in $(\mathbb{C}^*)^+$ and of the real interval $[-\rho(t), \rho(t)]$. If $|k|^2 = \rho^2(t)$, by definition $S(|k|) = t$, and hence $N_3 = 0$. It follows that

$$\int_{\partial D_3^{(2)}} N_3(k, \bar{k})dk = \int_{-\rho(t)}^{\rho(t)} \left[ \int_{S(|k|)}^{t} E(k,t,s)f(s)ds \right] dk.$$

We now compute the double integral in the region $D_3^{(1)}$. We start by writing this region and the integral in polar coordinate. This gives

$$D_3^{(1)} = \{ r\mathrm{e}^{i\theta} : \;\; \rho^2(t) < r^2 < 1, \; 0 < \theta < \pi \}$$

and since

$$d\lambda \wedge d\bar{\lambda} = -2irdrd\theta, \qquad \frac{\partial S(|\lambda|)}{\partial \bar{\lambda}} = \frac{\mathrm{e}^{i\theta}}{2} \frac{dS(r)}{dr},$$

we obtain

$$-\int\int_{D_3^{(1)}} \frac{\partial S(|\lambda|)}{\partial \bar{\lambda}} E(\lambda, t, S(|\lambda|)) f(S(|\lambda|)\, d\lambda \wedge d\bar{\lambda} = \int_0^{\pi} i\mathrm{e}^{i\theta} \int_{\rho(t)}^{1} \frac{dS(r)}{dr} E(r\mathrm{e}^{i\theta}, t, S(r)) f(S(r))\, rdrd\theta.$$

Changing variable to $\sigma = S(r)$ and using the definition of $S(r)$, we can write the inner integral as

$$\int_{\rho(t)}^{1} \frac{dS(r)}{dr} E(r\mathrm{e}^{i\theta}, t, S(r)) rdr = -\int_0^t E(\rho(\sigma)\mathrm{e}^{i\theta}, \sigma, t) f(\sigma)\rho(\sigma)d\sigma$$

so that finally

$$\int\int_{D_3} \frac{\partial N_3(k,\bar{k})}{\partial \bar{k}} dk \wedge d\bar{k} = -\int\int_{D_3^{(1)}} \frac{\partial S(|\lambda|)}{\partial \bar{\lambda}} E(\lambda, t, S(|\lambda|)) f(S(|\lambda|)\, d\lambda \wedge d\bar{\lambda} - \int_{\partial D_3^{(2)}} N_3(k,\bar{k})dk$$

$$= -\int_0^{\pi} i\mathrm{e}^{i\theta} \int_0^t E(\rho(\sigma)\mathrm{e}^{i\theta}, \sigma, t) f(\sigma)\rho(\sigma)d\sigma d\theta - \int_{-\rho(t)}^{\rho(t)} \left[ \int_{S(|\lambda|)}^{t} E(\lambda,t,s)f(s)ds \right] d\lambda.$$

Substituting this expression in the representation (2.24) we obtain

$$\begin{aligned} f(t) \;\; &= \;\; \frac{1+l'(t)}{2\pi} \left\{ -\int_{\partial D_2^-} E(\lambda,t,0)F(\lambda)d\lambda - \int_{-1}^{1} \left[ \int_0^{S(|\lambda|)} E(\lambda,t,s)f(s)ds \right] d\lambda \right. \\ &- \;\; \int_0^t \int_0^{\pi} i\mathrm{e}^{i\theta} E(\rho(\sigma)\mathrm{e}^{i\theta}, \sigma, t) f(\sigma)\rho(\sigma)d\theta d\sigma \\ &- \;\; \left. \int_{-\rho(t)}^{\rho(t)} \left[ \int_{S(|\lambda|)}^{t} E(\lambda,t,s)f(s)ds \right] d\lambda \right\}. \end{aligned} \tag{2.38}$$

Noting that $E(-\lambda, t, s) = \overline{E(\lambda, t, s)}$, it follows that

$$\int_{-\alpha}^{\alpha} \left[ \int_{S(|\lambda|)}^{t} E(\lambda,t,s)f(s)ds \right] d\lambda = 2Re \int_0^{\alpha} \left[ \int_{S(|\lambda|)}^{t} E(\lambda,t,s)f(s)ds \right] d\lambda,$$

where $\alpha \in [0,1]$. Using once again the definition of $S(|\lambda|)$ and exchanging order of integration, we have

$$-\int_0^{\rho(t)} \left[ \int_{S(|\lambda|)}^t E(\lambda,t,s) f(s) ds \right] d\lambda = \int_t^{\infty} \int_0^{\rho(s)} E(\lambda,t,s) d\lambda \, f(s) ds.$$

Similarly

$$-\int_0^1 \left[ \int_0^{S(|\lambda|)} E(\lambda,t,s) f(s) ds \right] d\lambda = -\int_0^{\infty} \int_0^{\rho(s)} E(\lambda,t,s) d\lambda \, f(s) ds.$$

Adding the last two integrals we obtain

$$-\int_0^t \int_0^{\rho(s)} E(\lambda,t,s) d\lambda \, f(s) ds.$$

Inserting the above expression into (2.38), we obtain the following Volterra integral equation:

$$f(t) = -\frac{1+l'(t)}{2\pi} \left\{ \int_{\partial D_2^-} E(\lambda,t,0) F(\lambda) d\lambda + \int_0^t \int_0^{\pi} i e^{i\theta} E(\rho(\sigma) e^{i\theta}, \sigma, t) d\theta \rho(\sigma) d\theta \, f(\sigma) d\sigma \right.$$
$$\left. + \, 2Re \int_0^t \left[ \int_0^{\rho(s)} E(\lambda,t,s) d\lambda \right] f(s) ds \right\}, \tag{2.39}$$

which yields (1.9).
**QED**
The analysis of the kernel of the Volterra integral equation (1.9) is similar to the analogous analysis of the kernel of the transform studied in [8], and will be presented elsewhere.

# 3 The solution of moving boundary value problems for the Klein Gordon equation

We now consider boundary value problems for the Klein-Gordon equation (1.3) posed on the domain $\mathcal{D}$ defined by (1.2). We assume, without loss of generality, that $l(0) = l'(0) = 0$.

## 3.1 The formal solution representation

In this section, we derive the integral representation of the solution of (1.3). This can be done in two equivalent ways: by using one of the first-order Lax pairs (2.17-2.18), or by using them both to obtain a solution of the second order pair (2.16).
We define the following functions:

$$\hat{q}_i(k) = \int_0^\infty \mathrm{e}^{-ikx} q_i(x)dx, \quad i=0,1, \tag{3.40}$$

$$Q_1(t,k) = (1-l'(t)^2)q_x(l(t),t) + l'(t)f_0'(t) + i(k_- + l'(t)k_+)f_0(t), \tag{3.41}$$

$$Q_2(t,k) = (1-l'(t)^2)q_x(l(t),t) + l'(t)f_0'(t) + i(k_- - l'(t)k_+)f_0(t), \tag{3.42}$$

$$\hat{Q}_0(k) = \int_0^\infty \mathrm{e}^{-ik_-x}[q_1(x) + q_0'(x) + \frac{i}{k}q_0(x)]dx, \tag{3.43}$$

$$\tilde{Q}_1(t,k) = (1-l'(t)^2)q_x(l(t),t) + (1+l'(t))f_0'(t) - \frac{i(1-l'(t))}{k}f_0(t), \tag{3.44}$$

where $q_0(x)$, $q_1(x)$, $f_0(t)$ are the given initial and boundary conditions given in (1.3).

**Proposition 3.1** *The solution $q(x,t)$ of (1.3) admits the following two integrals representations in terms of the given initial and boundary conditions, and of the unknown function $q_x(l(t),t)$:*
**(1)***:*

$$q(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}} \mathrm{e}^{ik_-x+ik_+t}\hat{Q}_0(k)\frac{dk}{k} - \frac{1}{4\pi}\int_{-1}^{1} \mathrm{e}^{ik_-x+ik_+t}\left[\int_0^{S(|k|)} \mathrm{e}^{-ik_-l(s)-ik_+s}\tilde{Q}_1(s,k)ds\right]\frac{dk}{k}$$
$$-\frac{1}{4\pi}\int\int_{D_3} \mathrm{e}^{ik_-(x-l(S(|k|)))+ik_+(t-S(|k|))}\frac{\partial S(|k|)}{\partial \bar{k}}\tilde{Q}_1(S(|k|),k)\frac{dk\wedge d\bar{k}}{k}. \tag{3.45}$$

*where $\hat{Q}_0$, $\tilde{Q}_1$ are given by (3.43)-(3.44), the domains $D_i$ are defined by (2.28-2.30), and $S(|k|)$ is defined by (2.25).*
**(2)***:*

$$\begin{aligned} q(x,t) &= \frac{1}{8\pi i}\left\{\int_{\mathbb{R}} \mathrm{e}^{ik_-x}[\mathrm{e}^{ik_+t}(\hat{q}_1(k_-) + ik_+\hat{q}_0(k_-)) - \mathrm{e}^{-ik_+t}(\hat{q}_1(k_-) - ik_+\hat{q}_0(k_-))]\frac{dk}{k_+}\right. \\ &- \int_{-1}^{1} \mathrm{e}^{ik_-x+ik_+t}\int_0^{S(|k|)} \mathrm{e}^{-ik_+s-ik_-l(s)}Q_1(s,k)ds\frac{dk}{k_+} \\ &+ \int_{k\in\mathbb{R},|k|>1} \mathrm{e}^{ik_-x-ik_+t}\int_0^{T(|k|)} \mathrm{e}^{ik_+s-ik_-l(s)}Q_2(s,k)ds\frac{dk}{k_+} \\ &- \int\int_{D_3} \frac{\partial S(|k|)}{\partial \bar{k}}\mathrm{e}^{ik_+(t-S(|k|))+ik_-(x-l(S(|k|)))}Q_1(S(|k|),k)ds\frac{dk\wedge d\bar{k}}{k_+} \\ &+ \left.\int\int_{B_3} \frac{\partial T(|k|)}{\partial \bar{k}}\mathrm{e}^{-ik_+(t-T(|k|))+ik_-(x-l(T(|k|)))}Q_2(T(|k|),k)ds\frac{dk\wedge d\bar{k}}{k_+}\right\}. \end{aligned} \tag{3.46}$$

*where the functions $\hat{q}_i(k)$, $Q_1(t,k)$ and $Q_2(t,k)$ are defined by (3.40)-(3.42), the domain $B_3$ is obtained from $D_3$ by the mapping $k \to -\frac{1}{k}$, and $T(|k|) = S(\frac{1}{|k|})$.*

The two approaches have distinct advantages. In this section, we sketch the derivation of both representations, and compare the resulting expressions.

**Solution representation I**

We consider the Lax pair (2.17). The solution of this pair of ODEs is a constant function of $k$ at infinity, equal to $-q(x,t)$. We first normalize it so that the solutions decay as $|k| \to \infty$. To this end we define $\tilde{\nu} = \nu + q$. This new function satisfies

$$\left\{ \begin{array}{l} \tilde{\nu}_x - ik_-\tilde{\nu} = q_t + q_x + \frac{i}{k}q, \\ \tilde{\nu}_t - ik_+\tilde{\nu} = q_t + q_x - \frac{i}{k}q. \end{array} \right. \tag{3.47}$$

Particular solutions of this pair of ODEs have the form

$$\tilde{\nu}(x,t,k) = \mathrm{e}^{ik_-x}\int_{x*}^{x} \mathrm{e}^{-ik_-y}(q_t + q_x + \frac{i}{k}q)(y,t)dy$$

$$+\mathrm{e}^{ik_-(x-x*)}\int_{t*}^{t} \mathrm{e}^{ik_+(t-s)+ik_-(l(t)-l(s))}\left[(1+l'(s))(q_t+q_x) + \frac{i}{k}(1-l'(s))q\right](l(s),s)ds. \tag{3.48}$$

We define three particular solutions by specifying the pair $(x*, t*)$. These solutions, and the domain where they are bounded as functions of $k$, are the following:

$$\begin{array}{l} \tilde{\nu}_1 \Rightarrow x* = l(t),\;\; t* = 0, \quad k \in D_1 \cap (\mathbb{C}^*)^+ \\ \tilde{\nu}_2 \Rightarrow x* = l(t),\;\; t* = S(|k|), \quad k \in (D_2 \cup D_3) \cap (\mathbb{C}^*)^+ \\ \tilde{\nu}_- \Rightarrow x* = \infty,\;\; t* = t, \quad k \in (\mathbb{C}^*)^-, \end{array}$$

where the domains $D_i$ are defined by (2.28-2.30), and $S(|k|)$ is defined by (2.25). It can easily be verified that these solutions decay both as $k \to \infty$ and $k \to 0$, see also [15].
As in the previous section, using the Cauchy-Green formula we can find a function $\tilde{\nu}(x,t,k)$ defined and bounded for every $k \in \mathbb{C}^*$, which concides with the three functions defined above in each respective domain. In particular, $\tilde{\nu}$ is *not* analytic for $k \in D_3$. Computing the jumps along the common boundaries, and the Dbar derivative of $\nu_2$ in $D_3$, we find for this function the following representation:

$$\begin{aligned} \tilde{\nu}(x,t,k) \;=\; & \frac{1}{2\pi i}\int_{\mathbb{R}} \mathrm{e}^{i\lambda_-x+i\lambda_+t}\hat{Q}_0(\lambda)\frac{d\lambda}{\lambda-k} \\ & -\frac{1}{2\pi i}\int_{-1}^{1} \mathrm{e}^{i\lambda_-x+i\lambda_+t}\left[\int_0^{S(|\lambda|)} \mathrm{e}^{-i\lambda_-l(s)-i\lambda_+s}\tilde{Q}_1(s,\lambda)ds\right]\frac{d\lambda}{\lambda-k} \\ & -\frac{1}{2\pi i}\int\int_{D_3} \frac{\partial S(|\lambda|)}{\partial\bar{\lambda}}\mathrm{e}^{i\lambda_-x+i\lambda_+t}\tilde{Q}_1(S(|\lambda|),\lambda)\frac{d\lambda\wedge d\bar{\lambda}}{\lambda-k} \end{aligned} \tag{3.49}$$

where $Q_0(k)$ is given by (3.43) and $\tilde{Q}_1(t,k)$ by (3.44).
Computing $\tilde{\nu}_x - ik_-\tilde{\nu}$, we find for $q(x,t)$ the representation (3.45).

**Solution representation II**

We now derive a solution $M$ of (2.16) with the property that $M$ is bounded for all values of $k \in \mathbb{C}^*$. To this end we perform the spectral analysis of (2.17) and (2.18). This

analysis is similar to the one performed in the previous section, with the only difference that this case we do not regularise the functions $\nu$ and $\mu$ at infinity.
This analysis yields for $M(x,t,k)$ the representation:

$$
\begin{aligned}
M(x,t,\lambda) &= \frac{1}{2\pi i}\left\{\int_{\mathbb{R}} \frac{\mathrm{e}^{ik_- x}}{2ik_+}[\mathrm{e}^{ik_+ t}(\hat{q}_1(k_-)+ik_+\hat{q}_0(k_-)) - \mathrm{e}^{-ik_+ t}(\hat{q}_1(k_-)-ik_+\hat{q}_0(k_-))]\frac{dk}{k-\lambda}\right.\\
&- \int_{-1}^{1}\frac{\mathrm{e}^{ik_- x+ik_+ t}}{2ik_+}\int_0^{S(|k|)}\mathrm{e}^{-ik_+ s-ik_- l(s)}Q_1(s,k)ds\frac{dk}{k-\lambda}\\
&+ \int_{k\in\mathbb{R},|k|>1}\frac{\mathrm{e}^{ik_- x-ik_+ t}}{2ik_+}\int_0^{T(|k|)}\mathrm{e}^{ik_+ s-ik_- l(s)}Q_2(s,k)ds\frac{dk}{k-\lambda}\\
&- \int\int_{D_3}\frac{\partial S(|k|)}{\partial\bar{k}}\mathrm{e}^{ik_+(t-S(|k|)+ik_-(x-l(S(|k|)))}Q_1(S(|k|),k)ds\frac{dk\wedge d\bar{k}}{2ik_+(k-\lambda)}\\
&+ \left.\int\int_{B_3}\frac{\partial T(|k|)}{\partial\bar{k}}\mathrm{e}^{-ik_+(t-T(|k|)+ik_-(x-l(T(|k|)))}Q_2(T(|k|),k)\frac{dk\wedge d\bar{k}}{2ik_+(k-\lambda)}\right\}. \quad (3.50)
\end{aligned}
$$

where the functions $Q_1(t,k)$ and $Q_2(t,k)$ are defined by (3.41) and (3.42), the domain $B_3$ is obtained from $D_3$ by the mapping $k \to -\frac{1}{k}$, and similarly $T(|k|) = S(\frac{1}{|k|})$.
Finally, using $M_x - i\lambda_- M = q(x,t)$, we find (3.46).

**Comparison of the two representations**

The integral representations (3.45) and (3.46) are equivalent, formal representations of the solution $q(x,t)$ of the boundary value problem. The equivalence can be established by verifying that the representation (3.46) is essentially the sum of two copies of (3.45), obtained by changing variable using the map $k \to -\frac{1}{k}$ and manipulating the result.
However, these representation are formal, because the integrand in both cases is expressed not only in terms of the given initial and boundary conditions, $q(x,0)$, $q_t(x,0)$ and $q(l(t),t)$, but contains also the unknown boundary value $q_x(l(t),t)$. In the next section, we discuss the characterisation of this unknown function.
We stress that although these two representations are equivalent, they present some distinct advantages. The representation (3.45) is more compact and it is preferable for symbolic manipulations. However, the second representation is more appropriate for the direct verification that indeed, the function defined by these integrals satisfies not only the PDE but also the given initial and boundary conditions. In addition, it is possible to read off immediately from (3.46) that the solutions propagate in two directions (corresponding to the exponential terms $\pm ik_+ t$), in analogy with the solutions of the wave equation. Indeed, for the case of the wave equation and of the Klein-Gordon equation posed on the half line [14, 15], we used a representation analogous to (3.46) to prove our results. In particular we used this representation to evaluate the unknown boundary value without resorting to the global relation. For the wave equation, this approach yields a representation that reduces directly to the usual one for the case of the half line. In the present case, we cannot avoid using the global relation, as the direct evalution of $q_x(l(t),t)$ from expression (3.46) leads to an identity.

### 3.2 The characterisation of the unknown boundary value $q_x(l(t),t)$

In this last section, we give the proof of Proposition 1.1.
The unknown boundary value $q_x(l(t),t)$ can be characterised using the *global relation* associated with the Lax pair (2.17), namely the relation (2.21) which we write, using the given initial and boundary conditions in (1.3), as follows:

$$\int_0^\infty \mathrm{e}^{-ik_+s-ik_-l(s)}(1-l'(s)^2)q_x(l(s),s)ds = \int_0^\infty \mathrm{e}^{-ik_-y}[q_1(y)+ik_+q_0(y)]dy- \tag{3.51}$$
$$-\int_0^\infty \mathrm{e}^{-ik_+s-ik_-l(s)}[l'(s)f_0'(s)+(ik_+l'(s)+ik_-)f_0(s)]ds.$$

The term on the left hand side is the transform of the function $(1-l'(t)^2)q_x(l(t),t)$ as defined by (1.6):

$$F(k)=\int_0^\infty \mathrm{e}^{-ik_+s-ik_-l(s)}(1-l'(s)^2)q_x(l(s),s)ds.$$

Therefore we can use the inversion formula (1.9) to derive a Volterra integral equation for this function:

$$(1-l'(t)^2)q_x(l(t),t) = -\frac{1+l'(t)}{2\pi}\int_{\partial D_2^-}\mathrm{e}^{ik_+t+ik_-l(t)}F(k)dk$$
$$-\frac{1+l'(t)}{2\pi}\int_0^t\left[\int_0^\pi i\rho(s)\mathrm{e}^{i\theta}E(\rho(s)\mathrm{e}^{i\theta},s,t)d\theta+2Re\int_0^{\rho(s)}E(k,t,s)dk\right](1-l'(s)^2)q_x(l(s),s)ds$$

with notation as for (1.9).
To use this result, we multiply the global relation (3.51) by $\mathrm{e}^{ik_+t+ik_-l(t)}$ and integrate along $\partial D_2^-$. Hence we obtain

$$\int_{\partial D_2^-}\mathrm{e}^{ik_+t+ik_-l(t)}F(k)dk = H(t), \tag{3.52}$$

$$H(t)=\int_{\partial D_2^-}\mathrm{e}^{ik_+t+ik_-l(t)}\left\{\int_0^\infty \mathrm{e}^{-ik_-y}[q_1(y)+ik_+q_0(y)]dy-\right.$$
$$\left.-\int_0^\infty \mathrm{e}^{-ik_+s-ik_-l(s)}[l'(s)f_0'(s)+(ik_+l'(s)+ik_-)f_0(s)]ds\right\}dk$$

Using (1.9), equation (3.52) yields the sought characterisation of the unknown boundary value:

$$(1-l'(t)^2)q_x(l(t),t) = -\frac{(1+l'(t))}{2\pi}H(t)$$
$$-\frac{1+l'(t)}{2\pi}\left\{\int_0^t\left[\int_0^\pi i\rho(s)\mathrm{e}^{i\theta}E(\rho(s)\mathrm{e}^{i\theta},s,t)d\theta+2Re\int_0^{\rho(s)}E(\lambda,t,s)d\lambda\right](1-l'(s)^2)q_x(l(s),s)ds\right\}. \tag{3.53}$$

Dividing through by $1-l'(t)^2$, we finally obtain (1.14).

## 4 Conclusions

We consider a boundary value problem for the Klein Gordon equation in a time-dependent domain, and solve it in terms of the unique solution of a Volterra linear integral equation. Hence the solution representation is not fully explicit, but it is characterised implicitly through the solution of an integral equation. This is an essential difference with the case of a boundary value problem posed on a time-independent domain. Analytically, this difference is reflected in the fact that the associated spectral problem cannot be solved in terms of a Riemann-Hilbert problem, as in the fixed domain case. Instead, the spectral problem is solved in terms of a Dbar problem, which can be viewed as a generalisation of a Riemann-Hilbert problem when the representation of the solution is not analytic in some domain of the spectral complex sphere. This also implies that these boundary value problems could not be solved by the classical Fourier transform.

This approach was first presented in a series of papers by Fokas and one of the authors [8, 12] where moving boundary problems for linear evolution PDEs were solved. We note that although Dbar problems have appeared in the context of boundary value problems for nonlinear integrable equations, they first appeared in association with a linear problem in the mentioned papers.

The main steps of the solution presented here follows closely the methodology for solving evolution problems. However, for the hyperbolic case presented here, this outcome is more surprising. Indeed, for these PDEs, the solution of boundary value problems posed in fixed domains can be decoupled in the solution of two first order systems, and can be solved explicitly by inverting the representation obtained, evaluated at the boundary. Hence in these cases it is not necessary to consider the global relation. However, for moving boundary value problems, this is no longer possible, as this evaluation leads to an identity.

We assume here that the time dependence of the domain is very smooth, and that the given boundary conditions are of Dirichlet type. However, both these assumptions can be relaxed, as mentioned in the introduction. In particular we can easily generalise our solution method to the case that the given boundary condition is of the more general form $q(l(t),t) + cq_x(l(t),t) = f(t)$.

Finally, we stress that the results presented here are formal, as no analysis of the kernel of the Volterra linear equation is presented. However, this analysis is similar to the one presented for the case of the kernel of the analogous integral equation derived in [8], where it is shown that this kernel is well defined, and in [2], where a numerical evaluation of the unknown boundary value is also presented. The details of this analysis, as well as numerical computations based on the formulas presented here, will be presented elsewhere.

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