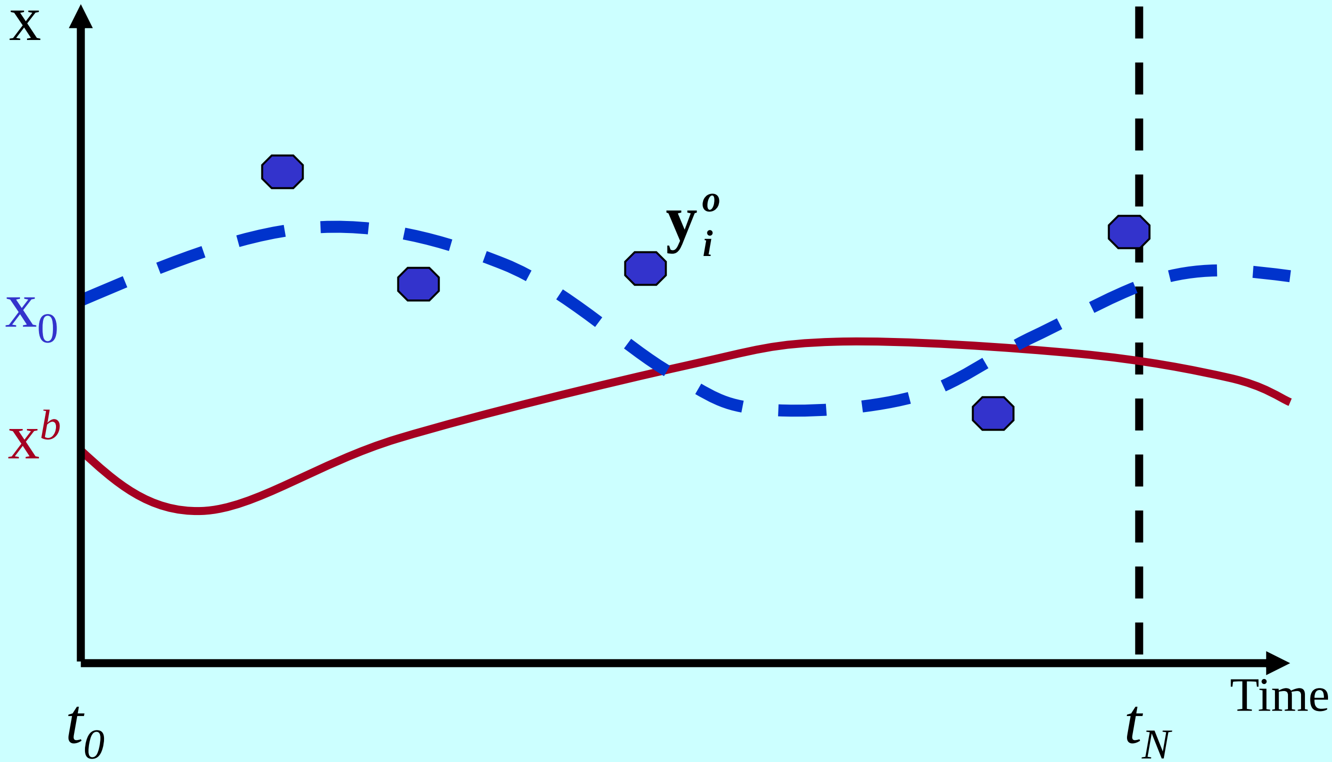
Variational data assimilation

Amos S. Lawless

a.s.lawless@reading.ac.uk

Four-dimensional variational assimilation (4D-Var)

Aim: Find the best estimate of the true state of the system (*analysis*), consistent with both observations distributed in time and the system dynamics.



Nonlinear least squares problem

Minimize

$$\mathcal{J}(\mathbf{x}_0) = \frac{1}{2}(\mathbf{x}_0 - \mathbf{x}^b)^{\mathrm{T}}\mathbf{B}^{-1}(\mathbf{x}_0 - \mathbf{x}^b) + \frac{1}{2}\sum_{i=0}^{N}(\mathcal{H}_i(\mathbf{x}_i) - \mathbf{y}_i)^{\mathrm{T}}\mathbf{R}_i^{-1}(\mathcal{H}_i(\mathbf{x}_i) - \mathbf{y}_i)$$

with respect to x_0, subject to

$$\mathbf{x}_{i+1} = \mathcal{M}_i(\mathbf{x}_i),$$

x^b - *a priori* (background) state – Size of order 10^8 - 10^9

y_i - Observations – Size of order 10^6 - 10^7

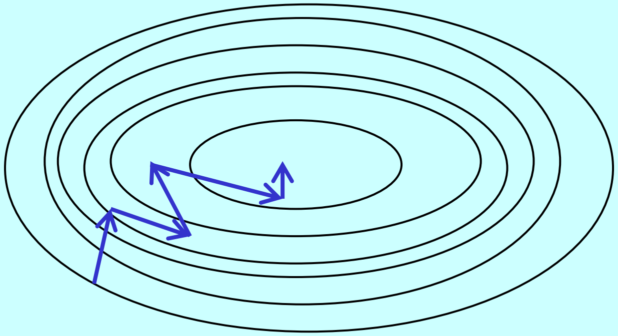
H_i - Observation operator

B - Background error covariance matrix

R_i - Observation error covariance matrix

Numerical minimization - Gradient descent methods

Iterative methods, where each successive iteration is based on the value of the function and its gradient at the current iteration.



$$\mathbf{x}_0^{(k+1)} = \mathbf{x}_0^{(k)} - \alpha\, \varphi(\mathbf{x}_0^{(k)})$$

where α is a step length and φ is a direction that depends on $J(\mathbf{x}_0^{(k)})$ and its gradient.

Problem: How do we calculate the gradient of $J(\mathbf{x}_0^{(k)})$ with respect to $\mathbf{x}_0^{(k)}$?

Method of Lagrange multipliers

We introduce Lagrange multipliers $\boldsymbol{\lambda}_i$ at time t_i and define the Lagrangian

$$\mathcal{L}(\mathbf{x}_i, \boldsymbol{\lambda}_i) = \mathcal{J}(\mathbf{x}_0) + \sum_{i=0}^{N-1} \boldsymbol{\lambda}_{i+1}(\mathbf{x}_{i+1} - \mathcal{M}_i(\mathbf{x}_i))$$

Then necessary conditions for a minimum of the cost function subject to the constraint are found by taking variations with respect to $\boldsymbol{\lambda}_i$ and $\mathbf{x}_i$.

Variations with respect to $\boldsymbol{\lambda}_i$ simply give the original constraint.

$$\mathcal{L}(\mathbf{x}_i, \boldsymbol{\lambda}_i) = \mathcal{J}(\mathbf{x}_0) + \sum_{i=0}^{N-1} \boldsymbol{\lambda}_{i+1}(\mathbf{x}_{i+1} - \mathcal{M}_i(\mathbf{x}_i))$$

Variations with respect to $\mathbf{x}_i$ give the *adjoint* equations

$$\boldsymbol{\lambda}_i = \mathbf{M}_i^T \boldsymbol{\lambda}_{i+1} - \mathbf{H}_i^T \mathbf{R}_i^{-1}(\mathcal{H}_i(\mathbf{x}_i) - \mathbf{y}_i)$$

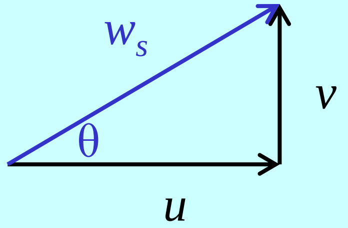
with boundary condition $\boldsymbol{\lambda}_{N+1} = 0$.

Then at initial time we have

$$\nabla \mathcal{J}(\mathbf{x}_0) = -\boldsymbol{\lambda}_0 + \mathbf{B}^{-1}(\mathbf{x}_0 - \mathbf{x}^b)$$

An aside – What are the linear operators **H** & **M**?

Let us go back to the example of the first lecture and suppose we observe only the wind speed w_s.



Then we have $\mathbf{x} = \begin{pmatrix} u \\ v \end{pmatrix}$, $\mathbf{y} = w_s$ and $\mathbf{y} = H(\mathbf{x})$

with

$$H(\mathbf{x}) = \sqrt{u^2 + v^2}$$

Then

$$\mathbf{H} = \begin{pmatrix} \frac{\partial H}{\partial u} & \frac{\partial H}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{u}{\sqrt{u^2+v^2}} & \frac{v}{\sqrt{u^2+v^2}} \end{pmatrix}$$

Back to the adjoint equation

$$\mathcal{L}(\mathbf{x}_i, \boldsymbol{\lambda}_i) = \mathcal{J}(\mathbf{x}_0) + \sum_{i=0}^{N-1} \boldsymbol{\lambda}_{i+1}(\mathbf{x}_{i+1} - \mathcal{M}_i(\mathbf{x}_i))$$

Let's consider a simple example - see separate sheet.

So where have we got to?

We wish to minimize

$$\mathcal{J}(\mathbf{x}_0) = \frac{1}{2}(\mathbf{x}_0 - \mathbf{x}^b)^{\mathrm{T}}\mathbf{B}^{-1}(\mathbf{x}_0 - \mathbf{x}^b) + \frac{1}{2}\sum_{i=0}^{N}(\mathcal{H}_i(\mathbf{x}_i) - \mathbf{y}_i)^{\mathrm{T}}\mathbf{R}_i^{-1}(\mathcal{H}_i(\mathbf{x}_i) - \mathbf{y}_i)$$

with respect to x_0, subject to

$$\mathbf{x}_{i+1} = \mathcal{M}_i(\mathbf{x}_i),$$

On each iteration we have to calculate *J* and its gradient

- To calculate *J* we need to run the nonlinear model
- To calculate the gradient of *J* we need one run of the adjoint model (backward in time)

Properties of 4D-Var

- Observations are treated at correct time.
- Use of dynamics means that more information can be obtained from observations.
- Standard formulation assumes model is perfect. Weak-constraint 4D-Var being developed to relax this assumption.
- In practice development of linear and adjoint models may be complex, but can be done at level of code.
- 4D-Var is currently used operationally at Met Office and ECMWF, among others.