Example

Suppose we have a model

$$x_{k+1} = x_k + 3y_k$$

$$y_{k+1} = y_k + 2x_k$$

and we have observations $\tilde{x}_0, \tilde{x}_1$ of x at times t_0, t_1, with error variance σ_o^2.

Assume we have an initial estimate $\begin{pmatrix} x_0^f \\ y_0^f \end{pmatrix}$ at time t_0 with error covariance $P_f(t_0) = \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}$

We wish to assimilate the observations using a Kalman filter.

First note

$$M = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \qquad H_0 = (1 \quad 0)$$

$$H_1 = (1 \quad 0)$$

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Then we have

Kalman gain at t_0: $K(t_0) = P_f(t_0) H_0^T \left(H_0 P_f H_0^T + R \right)^{-1}$

$$= \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left[(1 \quad 0) \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sigma_o^2 \right]^{-1}$$

$$= \frac{1}{\sigma_x^2 + \sigma_o^2} \begin{pmatrix} \sigma_x^2 \\ 0 \end{pmatrix}$$

Analysis at t_0

$$\begin{pmatrix} x_0^a \\ y_0^a \end{pmatrix} = \begin{pmatrix} x_0^f \\ y_0^f \end{pmatrix} + \frac{1}{\sigma_x^2+\sigma_0^2}\begin{pmatrix} \sigma_x^2 \\ 0 \end{pmatrix}(\tilde{x}_0 - x_0^f)$$

$$= \begin{pmatrix} x_0^f + \frac{\sigma_x^2}{\sigma_x^2+\sigma_0^2}(\tilde{x}_0 - x_0^f) \\ y_0^f \end{pmatrix}$$

Analysis error covariance at t_0

$$P_a(t_0) = (I - K H_0) P_f(t_0)$$

$$K(t_0) H_0 = \frac{1}{\sigma_x^2+\sigma_0^2}\begin{pmatrix} \sigma_x^2 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \frac{1}{\sigma_x^2+\sigma_0^2}\begin{pmatrix} \sigma_x^2 & 0 \\ 0 & 0 \end{pmatrix}$$

So $$P_a(t_0) = \begin{pmatrix} 1 - \frac{\sigma_x^2}{\sigma_x^2+\sigma_0^2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sigma_0^2\sigma_x^2}{\sigma_x^2+\sigma_0^2} & 0 \\ 0 & \sigma_y^2 \end{pmatrix}$$

Now we forecast to time t_1

$$\underline{x}_f(t_1) = M\,\underline{x}_a(t_0)$$

$$= \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_0^f + \frac{\sigma_x^2}{\sigma_x^2+\sigma_0^2}(\tilde{x}_0 - x_0^f) \\ y_0^f \end{pmatrix}$$

$$= \begin{pmatrix} x_0^f + \frac{\sigma_x^2}{\sigma_x^2+\sigma_0^2}(\tilde{x}_0 - x_0^f) + 3y_0^f \\ 2x_0^f + \frac{2\sigma_x^2}{\sigma_x^2+\sigma_0^2}(\tilde{x}_0 - x_0^f) + y_0^f \end{pmatrix}$$

What is the forecast error associated with this?

$$P_f(t_1) = M P_a(t_0) M^T \quad \text{(assuming perfect model)}$$

$$= \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sigma_0^2 \sigma_x^2}{\sigma_x^2 + \sigma_0^2} & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sigma_0^2 \sigma_x^2}{\sigma_x^2 + \sigma_0^2} + 9\sigma_y^2 & \frac{2\sigma_0^2 \sigma_x^2}{\sigma_x^2 + \sigma_0^2} + 3\sigma_y^2 \\ \frac{2\sigma_0^2 \sigma_x^2}{\sigma_x^2 + \sigma_0^2} + 3\sigma_y^2 & \frac{4\sigma_0^2 \sigma_x^2}{\sigma_x^2 + \sigma_0^2} + \sigma_y^2 \end{pmatrix}$$

<u>NB.</u> It is symmetric.

We can then assimilate $\tilde{x}_1$ at time t_1 starting from $\underline{x}_f(t_1)$ and $P_f(t_1)$