Block diagram of a self tuning regulator
Real Time Self Tuning

1. Sample plant output – at appropriate speed for plant.

2. Update parameter estimator.

3. Update controller terms.

4. Calculate new control signal

5. Update regressor, \( y(t) \rightarrow y(t-1), u(t-1) \rightarrow u(t-2) \) ....

6. Apply new control signal to the plant, etc.

7. Wait for clock pulse, then return to 1.
Minimum Variance Control

Objective – to minimize how much the system output varies, with respect to a set point, in the presence of disturbance.

This is an optimizing controller, i.e. choose $u(t)$ to minimize

$$J = E\{y^2(t + d)\}$$

(1)

Example: Suppose we have a process

$$y(t) = ay(t-1) + bu(t-1) + e(t) + ce(t-1)$$

(2)

which has a time delay $d = 1$, and $e(t)$ is a zero-mean white noise with variance $\sigma_e^2$. The next sample is

$$y(t + 1) = ay(t) + bu(t) + e(t + 1) + ce(t)$$

(3)
At time $t$, the best possible prediction for $y(t+1)$ is

$$\hat{y}(t+1|t) = ay(t) + bu(t) + ce(t) \quad (4)$$

The objective function, variance of $y(t+1)$, is

$$J = E\{y^2(t+1)\} = E\{[\hat{y}(t+1|t) + e(t+1)]^2\}$$

$$= E\{\hat{y}^2(t+1|t)\} + E\{e^2(t+1)\}$$

$$= E\{\hat{y}^2(t+1|t)\} + \sigma^2_e \quad (5)$$

Minimizing $J$ means that we can choose $u(t)$ such that

$$\hat{y}(t+1|t) = ay(t) + bu(t) + ce(t) = 0 \quad (6)$$

Writing the process as

$$e(t) = y(t) - ay(t-1) - bu(t-1) - ce(t-1) \quad (7)$$

or in transfer function term

$$e(t) = \frac{(1 - aq^{-1})y(t) - bq^{-1}u(t)}{1 + cq^{-1}} \quad (8)$$
Substitute this into the control law

\[ \hat{y}(t + 1|t) = \frac{(a + c)y(t) + bu(t)}{1 + cq^{-1}} = 0 \]  \hspace{1cm} (9)

Solving for \( u(t) \) gives

\[ u(t) = -\frac{a + c}{b}y(t) \]  \hspace{1cm} (10)

So the closed loop behavior is

\[ y(t + 1) = \hat{y}(t + 1|t) + e(t + 1) \]
\[ = \frac{(a + c)y(t) - b\frac{a + c}{b}y(t)}{1 + cq^{-1}} + e(t + 1) \]
\[ = e(t + 1) \]  \hspace{1cm} (11)

which is the best it can get.
Example: Suppose we have a process

\[ A(q^{-1})y(t) = B(q^{-1})u(t) + e(t) \]  \(12\)

where

\[ A(q^{-1}) = a_0 + a_1q^{-1} + ... a_nq^{-na}, \quad a_0 = 1. \]  \(13\)

\[ B(q^{-1}) = (b_0 + b_1q^{-1} + ... b_nbq^{-nb})q^{-1} \]  \(14\)

So

\[ y(t) = -a_1y(t-1) - a_2y(t-2) - ... - a_nay(t-na) + b_0u(t-1) + b_1u(t-2) + ... + b_nbu(t-nb-1) + e(t) \]

\[ = \theta^T \phi(t) + e(t) \]  \(15\)

where

\[ \theta = [-a_1, -a_2, ..., b_0, ... b_nb]^T \]  \(16\)

\[ \phi(t) = [y(t-1), ... y(t-na), u(t-1), ..., u(t-nb)]^T \]  \(17\)
• NB: The parameter vector $\hat{\theta}$ can be obtained from Recursive Least Squares (RLS) algorithm (System identification). $\hat{\theta} \to \theta$, $\hat{a}_1 \to a_1$, $\hat{a}_2 \to a_2$... $\hat{b}_0 \to b_0$, $\hat{b}_1 \to b_1$, ...,$\hat{b}_{nb} \to b_{nb}$.

Suggest

$$u(t) = -\frac{q^{-1}G(q^{-1})}{B(q^{-1})F(q^{-1})}y(t) \quad (18)$$

So

$$A(q^{-1})y(t) = -\frac{q^{-1}B(q^{-1})G(q^{-1})}{B(q^{-1})F(q^{-1})}y(t) + e(t) \quad (19)$$

or

$$[A(q^{-1})F(q^{-1}) + q^{-1}G(q^{-1})]y(t) = F(q^{-1})e(t) \quad (20)$$

We can simply set

$$A(q^{-1})F(q^{-1}) + q^{-1}G(q^{-1}) = 1 \quad (21)$$

and

$$F(q^{-1}) = 1 \quad (22)$$
Then

\[ y(t) = e(t) \]  \hspace{1cm} (23)

\[ G(q^{-1}) = g_0 + g_1q^{-1} + \ldots, \]  \hspace{1cm} (24)

or

\[ G(q^{-1}) = -a_1 - a_2q^{-1} - \ldots - a_{na}q^{-na+1} \]  \hspace{1cm} (25)

\[ u(t) = -\frac{1 - A(q^{-1})}{B(q^{-1})}y(t) \]  \hspace{1cm} (26)

Example: Assume the system is

\[ y(t) = -a_1y(t-1) + b_0u(t-1) + e(t) \]  \hspace{1cm} (27)

Parameter estimation:

\[ y(t) = \hat{\theta}^T \phi(t) + e(t) \]  \hspace{1cm} (28)

where

\[ \hat{\theta} = [-\hat{a}_1, \hat{b}_0]^T \]  \hspace{1cm} (29)

\[ \phi(t) = [y(t-1), u(t-1)]^T \]  \hspace{1cm} (30)

(Use RLS algorithm for parameter estimates).
\( \hat{a}_1 \rightarrow a_1, \hat{b}_0 \rightarrow b_0 \). The controller is

\[
    u(t) = -\frac{1 - A(q^{-1})}{B(q^{-1})} y(t)
\]  

(31)

or

\[
    u(t) = \frac{\hat{a}_1}{\hat{b}_0} y(t)
\]  

(32)

e.g.

\[
    A(q^{-1}) = 1 + a_1 q^{-1} = 1 + 0.5q^{-1}, \quad b_0 = 2,
\]  

(33)

Then if

\[
    \hat{a}_1 = 0.5, \hat{b}_0 = 2,
\]  

(34)

So

\[
    u(t) = 0.25y(t).
\]  

(35)
Minimum variance control algorithm (a summary)

1. Measure current output $y(t)$.

2. Recall past $y$'s and $u$'s and form data $\phi(t)$.

3. Predict estimated output $\hat{y}(t)$ from $\phi(t)$, and $\hat{\theta}$ (model).

4. Recalculate new parameter $\hat{\theta}$ from estimator, which gives new $\hat{A}$, $\hat{B}$.

5. Calculate $u(t)$ from control law.

6. Step 1-5 are repeated at each sampling period.