6 : Frequency Response Part B

In the first part have looked at finding and plotting Freq Resp

- Dynamic systems modelled in terms of jω
- Complex Transfer function determined at various ω
- Plotted on Nyquist and on Bode diagrams
  - Asymptotic approximations of Bode also shown
  - Here, use system's corner frequencies CF
  - Then starting at low frequency, plot response til next CF
    - CFs can be of first or second order poles/zeros
- Will now reverse situation, have Bode Plot ..
  - Starting at low frequency, deduce each CF …
  - This is System Identification

System Identification

Suppose have measured values of m, p and ω

Can we determine what the system is.

This is System Identification.

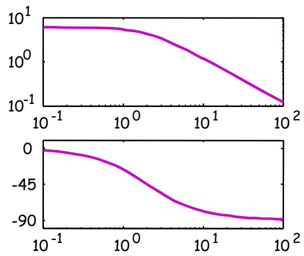
- Initially guess at structure – eg Pole, 2 Pole, Lead + Poles, etc
- But can also estimate actual corner freqs.
  - How ?

Computer can assist this.

Start with simple examples

- Have another GUI which helps
- Note is more accurate if CFs are further apart.

Simple System Identification



w	0.1	0.4	2.0	11.6	100.0
m	6.0	5.9	4.2	1.0	0.1
p	-2.9	-11.8	-45.0	-80.2	-88.9

What is system ?

$$\frac{K}{1 + sT}$$

What is K?

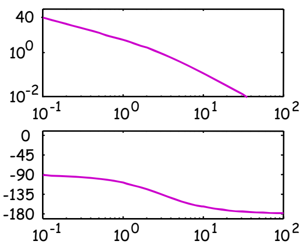
Low freq gain = 6

How find other para?

1 / freq where gain = 6/√2 or phase = -45°

T = 1 / 2

Second Simple Example



w	0.1	0.4	3.0	11.8	100.0
m	40.0	9.3	0.9	0.1	0
p	-91.9	-98.0	-135.0	-165.7	-178.3

What is system ?

$$\frac{K}{s(1 + sT)}$$

What is K?

Gain $\frac{K}{\omega}$ at 10^{-1} = 40

So K = 40*0.1 = 4

How find other para?

Where phase = -135°

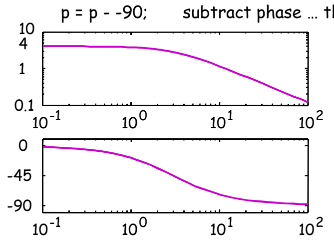
T = 1 / 3

Iterative Search

Assume: m, p, w in vectors

As system is pure integrator at low freqs, can 'remove' it :

- m = m * .w; divide all in m by gain there (= 1/ω)
- p = p - -90; subtract phase … then replot and analyse



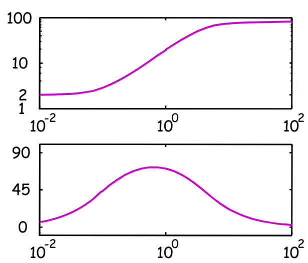
These graphs →

$$\frac{4}{1 + s/3}$$

Hence System is

$$\frac{4}{s(1 + s/3)}$$

Lead-Lag elements



Has form $K\frac{1+sTe}{1+sTa}$: K = 2

Te > Ta, as first corner freq is 1/Te

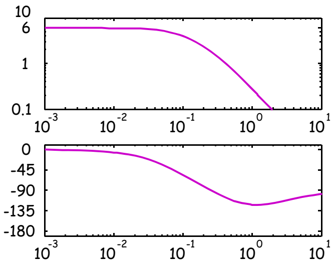
If p max = a at ω_a

$$afac = \sqrt{\frac{1 - \sin(a)}{1 + \sin(a)}}$$

$$Te = \frac{1}{\omega_a * afac}; \quad Ta = \frac{afac}{\omega_a}$$

a = 72 at ω_a = 0.633; afac = 0.7959; Te = 1.987; Ta = 1.258

Next Consider



Here $\frac{K}{1+sT} * \frac{1+sTe}{1+sTa}$

K is 6,

T found where m=6/√2:

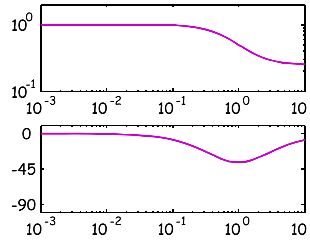
1/interp1(p,w,-45) ~ 11

So $\frac{K}{1+sT} \approx \frac{6}{1+s11}$

[ma, pa] = bode(6, [11 1], w); % find m,p for this element

m = m ./ ma; p = p – pa; % so can analyse lead-lag

Then process new m, p …



If p min = a at ω_a

$afac = \sqrt{\frac{1 - \sin(a)}{1 + \sin(a)}}$

$Ta = \frac{1}{\omega_a * afac}$ $Te = \frac{afac}{\omega_a}$

a = min(p) = -116-90

at ω_a = 1 rad/s

So afac = 2

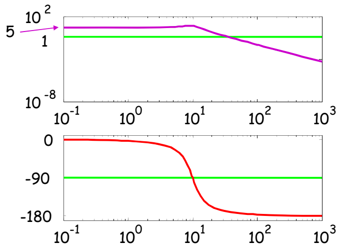
Ta = 2

Te = 1/2 = 0.5

So Sys = $\frac{6}{1+s11} * \frac{1+s0.5}{1+s2}$

Identification of Second Order

Suppose m,p,w have data for:



$P(s) = \frac{K}{\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1}$

At ω_n,

$P(j\omega) = \frac{K}{\frac{-\omega_n^2}{\omega_n^2} + j\frac{2\zeta\omega_n}{\omega_n} + 1}$

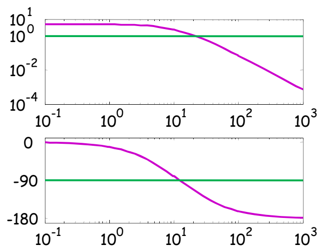
$|P| = \frac{K}{2\zeta}$; $\angle P = -90^O$

K = m(1)

w_n = 1/w(P = -90)

ζ= K/2*m(P = -90)

Next Consider



Although has form

$\frac{K}{(1+sT_1)(1+sT_2)}$

Better (as poles interact) to find K, ζ and ω_n as before, then

$T_1, T_2 = \frac{\zeta \pm \sqrt{\zeta^2 - 1}}{\omega_n}$

m(1) = 4.9983; p=-90 at 12.25 to give ω_n; ζ = 1.265;

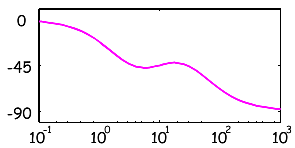
So T_1 = 0.1666; T_2 = 0.040

On interp1 and monotonicity

[m,p,w]=bode(5*[1/10 1], conv([1/2 1], [1/50 1]))

Can find ω where phase = -45 by interp1(p, w, -45, 'spline')

But, only works if first vector is monotonic – here get answer 10



Need to search where p always decreasing

Find index where p rises:

ndx = min(find(diff(p)>0))

interp1(p(1:ndx), w(1:ndx), -45, 'spline')

ans = 4.3771

Identifying Multiple Elements

Algorithm given below … implemented in another GUI

Given [m, p, w] for system

Do following until identified all elements:

Look at low freq response, decide element structure, find para(s)

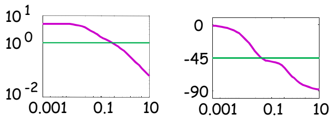
(may need to search lower freq m's and p's rather than all)

do [ma, pa] = bode (this element, w)

m := m ./ ma; p := p – pa ie whole system / this element

Early errors propagate, but can work well

Example



Looks like zero & two poles

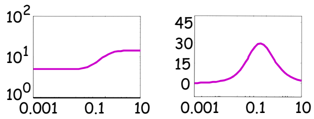
Est T where m = dcgain/√2;

Remove pole, analyse rest

T=1/interp1(m(p>-45), w(p>-45), m(1)/sqrt(2));

% search m etc where p>-45. Returns T = 23.6

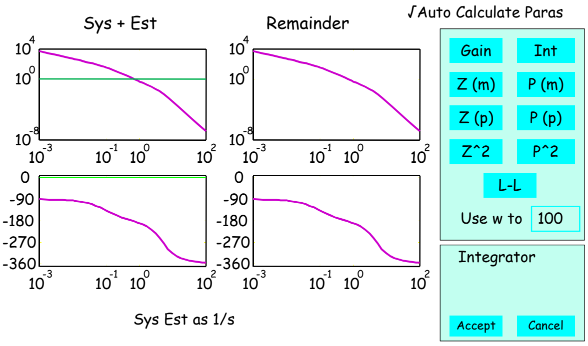
[ma,pa] = bode(1, [T 1], w); m := m ./ ma; p := p – pa;



doLeadLag(m, p, w)

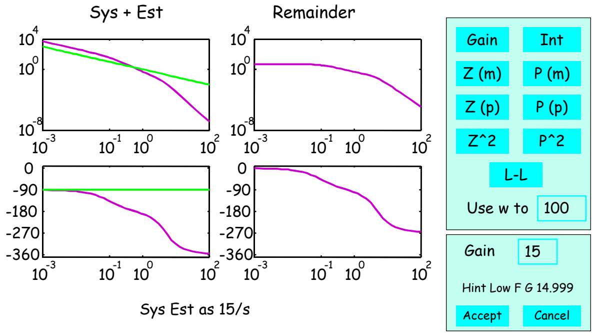
Get Te = 5.6, Ta = 1.9

MatLab GUI



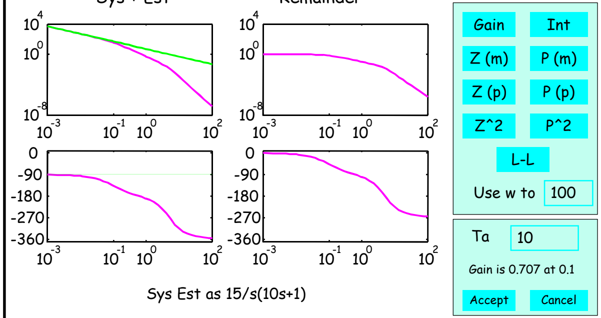
Sys Est as 1/s

After Integrator



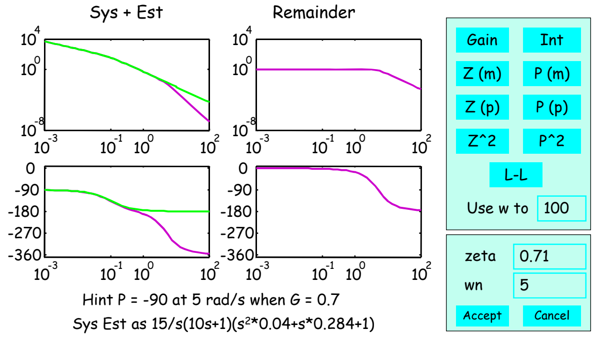
Sys Est as 15/s

After Gain



Sys Est as 15/s(10s+1)

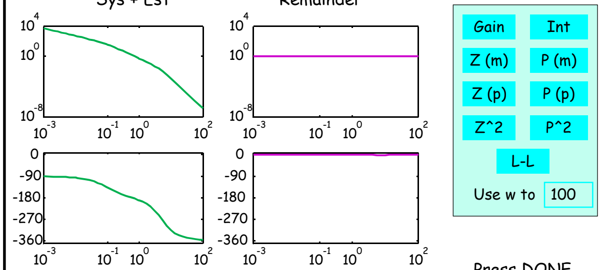
After Pole



Hint P = -90 at 5 rad/s when G = 0.7

Sys Est as 15/s(10s+1)(s²*0.04+s*0.284+1)

Add Quadratic Pole



Press DONE

Sys Est as 15/s(10s+1)(s2*0.04+s*0.284+1)

Three Pole Systems

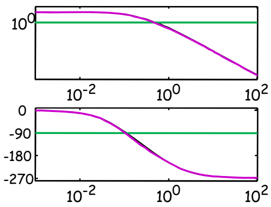
Approach here : find first pole, where gain = dcgain/√2

Then use two pole/quadratic system on system/pole

Here K = 20; Ts are 1, 4 and 15

Get gain: 19.9976 T1: 5.9411

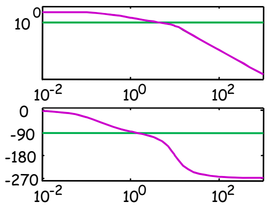
T2: 13.1470 T3: 0.8769



Here K = 20 T = 5 ζ = 0.5 ω_n = 10

Get gain: 19.9996 zeta: 0.5041

wn: 9.9992 T3: 4.9584



Summary

In this lecture we have

Looked at the identification of systems with single and quadratic poles and zeros

Next week, we will consider

Stability Margins

M-circles and then Sensitivity Circles

Then closed loop and disturbance frequency responses.

Exercise – Lecture 6 (for 2 weeks)

Go to MatLab and run sysidgui

Enter your student number and you should see system 1.

Identify the system and copy the complete system to the ClipBrd and thence to your word document. Comment on the result.

Repeat for systems 2, 3.

Then untick the 'Auto Calculate Parameters' option

Identify systems 5 then 4.

Note the system assumes parameters are 1 – you change them based on the hint you get.

Note hint gives angular freqs but you may be asked for time constants.

7 : Margins and Closed Loop FR

We have seen how we can plot the frequency response

As Bode or Nyquist diagram

We can also identify the system from the Bode Plot

Generally we look at the FR of the loop transfer function

In this lecture we start to relate loop and closed loop FR

We start with Stability Margins

Then we look at M-Circles : closed loop FR on Nyquist

We then add Sensitivity Circles and relate them to Margins

Then we will consider the closed loop FR

Next week we formally relate Loop FR to closed loop Step Resp.

Gain and Phase Margin

System oscillates when gain = 1 and phase lag = 180°

System stable if gain < 1 when phase lag = 180°

Or if phase lag < 180° when gain = 1

If system near to oscillator, then very underdamped

Hence define margins : how close to oscillating

Phase Margin : gain = 1 how much more phase lag before oscillates

Gain Margin : phase = 180°, by what multiply gain before oscillate

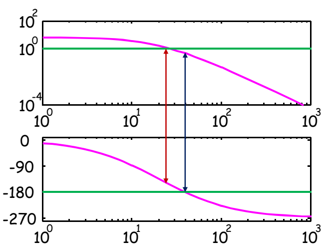
If Phase Lag is 135° when gain = 1, PM = 180-135 = 45°

Often this implies ~20% on step response

If gain = 0.25 when Phase = -180° , GM = 1/0.25 = 4

In decibels, GM = -20log(0.25) = 12dB

On Bode Plots



When phase -180, gain = 0.47,

GM = 20 log(0.47) = 6.54dB

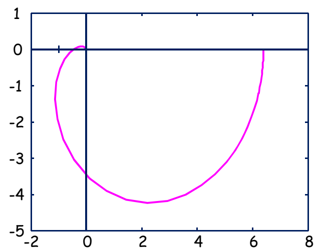
When gain = 1, phase = -154.5

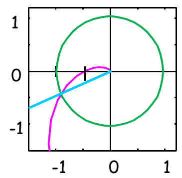
PM = 25.5

In MatLab

```
[m, p, w] = bode(num, den);
GM = -20*log10(interp1(p, m, -180, 'spline'));
PM = 180+interp1(m, p,1, 'spline');
```

On Nyquist





GM: use where locus cross -ve real axis;

PM, use where locus meet unit circle

PM related to overshoot of step response, as we shall see next week

M and N circles

Other circles are drawn on the Nyquist plane

These two are for relating loop TF to closed loop

- M-circles — locus of constant closed loop gain
- N-circles — locus of constant closed loop phase

Superimposed on a Nyquist plot : called Nichols chart.

Both circles defined by their origin and radius

M circles Origin: $-\frac{M^2}{M^2-1}, 0$ Radius $\frac{M}{M^2-1}$

N circles Origin: $-\frac{1}{2}, \frac{1}{2N}$ Radius $\frac{\sqrt{N^2+1}}{2N}$

Use of M-Circles

M is closed loop gain : for system with A and β (and – in summer)

$$M = \left|\frac{A}{1+A\beta}\right|$$

M circle defines positions on Aβ plane where M constant

Define M_{pf} = Max(M) which occurs at $\omega = \omega_{rf}$

One use of M and M-circle

(strictly for systems with very high d.c. loop gain)

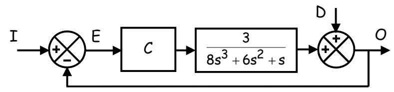
If Nyquist locus touches M = 1.3 circle (like PM = 45^o)

often implies ~ 20% overshoot to step input

ω where M = M_{pf} (ie ω_{rf}) related to freq of transient resp

This should become clearer in next lecture

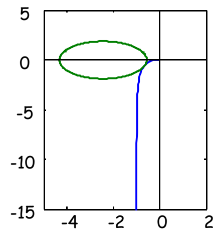
Example – from Lecture 3

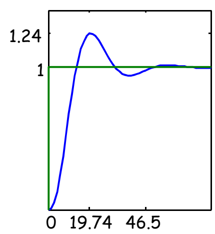


Design PM=45^o
C = 1/17.9373
= 0.0557

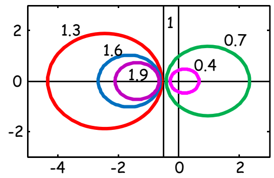
Nyq Plot of C*P

Just touch M=1.3 circle





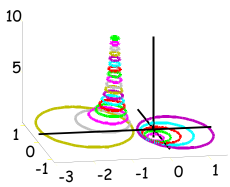
Various M-circles



$M = \left|\frac{A}{1+A\beta}\right|$

NB 1.6 circle inside 1.3

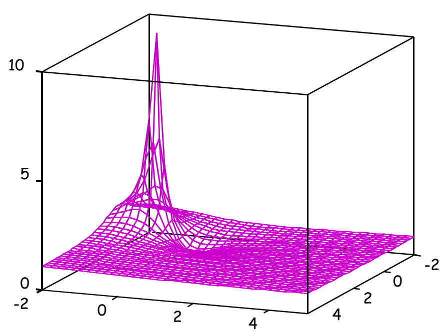
So inside 1.3 circle, M > 1.3.



Gives no feel for loop gain or closed loop gain

So plot in 3D, Z axis is M

Or Better – a 3D plot (β = 1)



Clipped at -1,j0

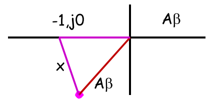
Much of plot gain = 1 (O=I)

Not so near 0,0 and -1,j0

Want Nyquist locus mainly where gain 1

But, design so locus touches suitable M circle …

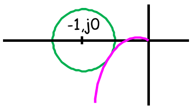
Sensitivity Function / Circle



Vector from -1,j0 to Blob = x
But going via 0,0 = 1 + Aβ
So x = 1 + Aβ
|x| = distance of point from -1,j0

The Sensitivity Function is defined as $S(j\omega) = \frac{1}{1+A\beta} = \frac{O}{D}$

M_s = Max (| S(jω) |) is
minimum distance of locus to -1,j0



Sensitivity circle: origin -1,j0, radius M_s

Can design for specific M_s value :
locus just touch it

At ω's where S(jω) > 1, Disturbances are amplified !

Finding M_s

Can find gain and phase for loop transfer function num / den.

But how to find M_s?

$$\text{If } A\beta = \frac{n}{d} \text{ then } 1 + A\beta = 1 + \frac{n}{d} = \frac{d+n}{d}$$

$$\text{Thus } M_S = \text{Max}\left(\left|\frac{1}{1+A\beta(j\omega)}\right|\right) = \text{Max}\left(\left|\frac{d(j\omega)}{d(j\omega)+n(j\omega)}\right|\right)$$

$$\text{Recall } C = 0.0557 \text{ and } P = \frac{3}{8s^3 + 6s^2 + s}$$

So have c, and num, den are polys for p:
>> [m,p,w] = bode(den, polyadd(den, c*num));
>> ms = max(m)
Ms = 1.7726

RJM function adds two polys – pads smaller with 0s til both same size

Complementary Sensitivity Function

We have met the sensitivity function $S(j\omega) = \frac{1}{1+A\beta}$

(which relates to circle, origin -1,j0)

Control Engineers also use the complementary sensitivity function $T(j\omega) = \frac{A\beta}{1+A\beta}$

Name comes from fact the S(jω) + T(jω) = 1

Control engineers assume β = 1, so T(jω) = M(jω)

It is suggested that one could do a design with relative stability specified by M_s = Max(|S(jω)|) = value 1.4 … 2

Can also specify M_t = Max(|T(jω)|) and $M_s = M_t$

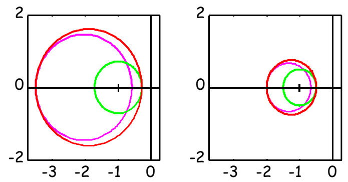
These can be considered alternative to GM and PM

M_x circles on Nyquist

M_s circles Origin : -1,0 Radius : $1/M_s$

M_t circles Origin: $-\frac{M_t^2}{M_t^2-1}$,0 Radius : $\frac{M_t}{M_t^2-1}$

M = Max(M_s, M_t) Origin: $-\frac{2M^2-2M+1}{2M(M-1)}$,0 Radius : $\frac{2M-1}{2M(M-1)}$



M = 1.4 and 2

Design so Nyquist locus kept outside circles

Closed Loop Frequency Response

Bode and Nyquist are plots of loop TF.

Can also plot how |closed loop gain| vary with frequency

Suppose $A = \frac{n(s)}{d(s)}$ and $\beta = 1$ ('-' in summer)

$$\text{Closed Loop Transfer Function is } \frac{\frac{n(s)}{d(s)}}{1 - -\frac{n(s)}{d(s)}} = \frac{n(s)}{d(s)+n(s)}$$

So use MatLab's Bode function (using RJMs polyadd function)

```
>> [m p w] = bode (n, polyadd(n,d)); % calc data
>> plot(w,m);    % plots variation of gain with angular freq
>> Mpf = max(m); wrf = w(m==Mpf);
```


Closed Loop Gain – for 5 systems

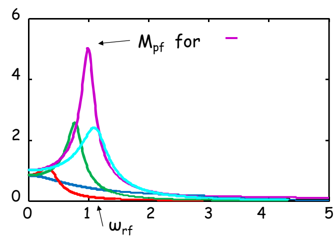
a) $\frac{5}{10s+1}$

b) $\frac{5}{(8s+1)(5s+1)}$

c) $\frac{5}{(3s+1)(2s+1)(s+1)}$

d) $\frac{5}{s(5s+1)}$

e) $\frac{5}{s(s+1)(2s+1)}$



M_{pf} is max closed loop gain, occurring at ω_{rf}

If graph peaks, ω_{rf} related to freq of damped sinusoid in step response.

We will use this later

Estimating ω_{rf}

A good estimate for ω_{rf} is where loop gain is 1. For the examples:

$\frac{5}{(8s+1)(5s+1)}$ Gain = 1 at ω = 0.3127; CLGain Max at ω = 0.3117

$\frac{5}{(3s+1)(2s+1)(s+1)}$ Gain = 1 at ω = 0.7091; CLGain Max at ω = 0.7733

$\frac{5}{s(5s+1)}$ Gain = 1 at ω = 0.9901; CLGain Max at ω = 0.9899

$\frac{5}{s(s+1)(2s+1)}$ Gain = 1 at ω = 1.212; CLGain Max at ω = 1.103

Why

ClGain = $\frac{A}{1+A\beta}$ When $|A\beta|$ = 1, $\angle A\beta \to$ -180, $|1 + A\beta| \to$ 0, CLGain max

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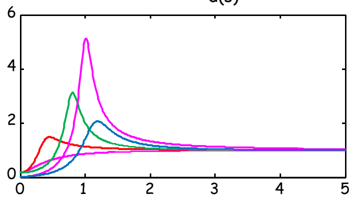
Closed Loop Disturbance Response

Closed Loop Dist Transfer Function is $\frac{O}{D} = \frac{1}{1-\frac{n(s)}{d(s)}} = \frac{d(s)}{d(s)+n(s)}$



NB This is sensitivity function – relates to Max (S) circle

At low freq, graph < 1 :reducing effect of D on O

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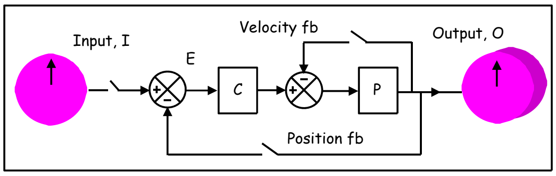
Experiment – Old Educational Servo



Closed Loop : diff to move Output slowly – til max torque

Step resp; note freq of osc : easy to move O at that freq

Open loop – very easy to move Output slowly

Diff to move O at speed of closed loop response

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Using Simple Linear Model of Servo

Open Loop: $\frac{O}{I} = \frac{O}{E} = \frac{K}{s(1+sT)}$ $\frac{O}{D} = 1$

Open loop : no feedback so I = E

Closed Loop:

$$\frac{O}{I} = \frac{\frac{K}{s(1+sT)}}{1+\frac{K}{s(1+sT)}} = \frac{K}{s^2T+s+K} \qquad \frac{O}{D} = \frac{1}{1+\frac{K}{s(1+sT)}} = \frac{s^2T+s}{s^2T+s+K}$$

Low freq, s (ie jω) negligible

Low Freq OL $\frac{O}{I} = \infty$, $\frac{O}{D} = 1$; CL $\frac{O}{I} = 1$, $\frac{O}{D} = 0$

Confirms easy to move output when OL, diff CL

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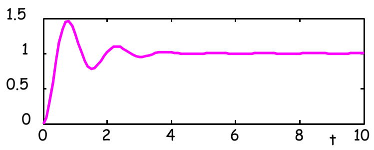
Why easy to add high freq D?

Graph below shows how O responds to input step



$O = 1 - k\,e^{-at} \sin(\omega_{rt} - \phi)$

The a value – sets how quickly sinusoid decays – more later

ω_{rt} is angular freq of the sinusoid

That is freq we wish to add as disturbance

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Higher Frequency Test

Can show $\omega_{rt} = \frac{\sqrt{TK-0.25}}{T}$

Suppose K = 9, T = 0.5

Then ω_{rt} = 4.12 rad/s

Then $\left|\frac{O}{E}\right| = \frac{KT}{\sqrt{(TK-0.25)(TK+0.75)}}$ $\left|\frac{O}{E}\right| = \frac{4.5}{\sqrt{22.3}} = 0.953$

C.L. $\left|\frac{O}{I}\right| = \frac{KT}{\sqrt{TK-0.1875}}$ $\left|\frac{O}{I}\right| = \frac{4.5}{\sqrt{4.3125}} = 2.17$

C.L. $\left|\frac{O}{D}\right| = \sqrt{\frac{(TK-0.25)(TK+0.75)}{TK-0.1875}}$ $\left|\frac{O}{D}\right| = \frac{\sqrt{22.3}}{\sqrt{4.3125}} = 2.27$

At this freq, sinusoid D is amplified – confirm test

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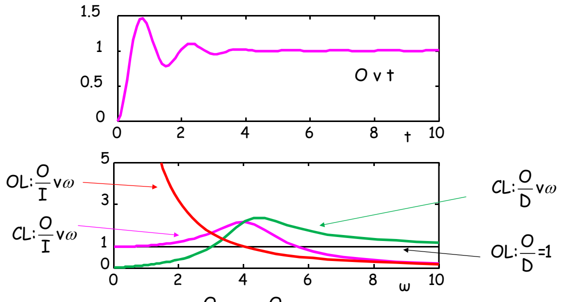
Use MATLAB: Graphs K = 9; T = 0.5



At ω_{rt}; OL: $\frac{O}{D}$=1 CL: $\frac{O}{D}$=2.23 Confirms test

Summary

This lecture has covered various related topics:

Gain and Phase Margin – we design to achieve a given PM

M-circles : where closed loop gain is constant

We can design to achieve a given max(closed loop gain)

We have looked also at sensitivity circles

We can design to achieve a given max(sensitivity)

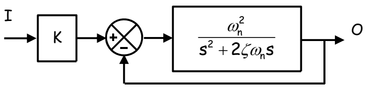
We have looked at the closed loop freq and dist responses

The design concepts relate to the relationship between loop frequency responses and closed loop step response

We will explore that relationship next week.

8 : Relating Time and Freq Domain

This week we seek to relate frequency domain with time domain
Do this by finding exact relationship for second order system
Then assess how well applies to other systems:

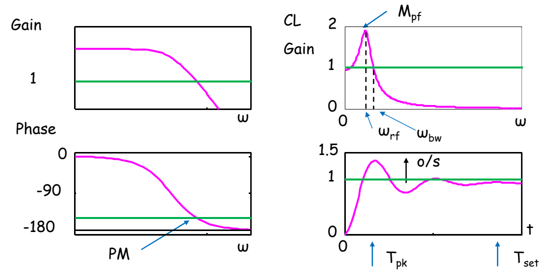


$$\text{Loop} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s};\quad \frac{O}{I} = G(s) = K * \frac{\text{Loop}}{1+\text{Loop}} = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Can find O(t) if I is a step – thence O_{ss}, Tpk %o/s, etc

Will also assess loop TF, find PM etc – then find how relate

Relationships on Bode, CLGain, Step



Show M_{pf} and PM related to ζ and o/s; ω_{rf}, ζ give T_{pk}, T_{set}

In lecture, show what do – not go through all – left to you

Time Domain Step Responses

$$O(s) = \frac{1}{s}\times\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{K}{s} - \frac{K(s+2\zeta\omega_n)\omega_n^2}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$O(t) = K - \frac{K}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\ t + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)$$

Steady State output, O_{SS} = K;

Transient O_T = -K' $e^{-\zeta\omega_n t}$ sin (ω_{rt} t + ϕ)

Sets settling time

Freq of oscillations $\omega_{rt} = \omega_n\sqrt{(1-\zeta^2)}$;

Peak Value – when dO/dt = 0

$$\dot{O}(s) = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{K\omega_n^2}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$\text{So } \dot{O} = \frac{dO}{dt} = \frac{K\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\ t\right)$$

Zero at sin (rπ), first peak when sin(π), next at sin(3π)

$$\text{So } T_{pk} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_{rt}}$$

Per unit overshoot

$$\text{when}\quad O = K - Ke^{-\frac{\zeta\omega_n\pi}{\omega_n\sqrt{1-\zeta^2}}}\cos(\pi) = K + K\,e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

O/S 20.5% if ζ 0.45

Settling Time, T_{set}

$O_T(t) = K' e^{-\zeta\omega_n t} \sin(\omega_{rt}\, t + \phi)$

O settles within 2% of O_{ss} when $K' e^{-\zeta\omega_n t}$ within 2% of K

T_{set} is t when $\frac{K}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} < 0.02\ K$

$$t = T_{set} \geq \frac{-\ln\left(0.02 * \sqrt{1-\zeta^2}\right)}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n}$$

{ For 0.3 < ζ < 0.7, 3.96 < numerator < 4.25 }

Frequency Response: Max Gain M_{pf}

$$G(j\omega) = \frac{K\omega_n^2}{\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega};\ \text{so } M = |G(j\omega)| = \frac{K\omega_n^2}{\sqrt{(\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$

Want ω where M max : when diff of square of denom is 0

Straightforward to show $\omega_{rf}^2 = (1-2\zeta^2)\ \omega_n^2$

Thus max gain, M_{pf}, = $\frac{K}{2\zeta\sqrt{1-\zeta^2}} = \frac{O_{ss}}{2\zeta\sqrt{1-\zeta^2}}$

Rearranging, gives $\zeta^2 = 0.5 - 0.5\sqrt{1 - \frac{O_{ss}^2}{M_{pf}^2}}$

ζ negative if system unstable

On T_{pk}

Can use

$$\omega_{rf} = \omega_n\sqrt{1-2\zeta^2};\qquad T_{pk} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_{rf}}\frac{\sqrt{1-2\zeta^2}}{\sqrt{1-\zeta^2}}$$

Fine if 2ζ² < 1, or ζ < 0.707. If not, use closed loop bandwidth …

$$G(j\omega) = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2};\ \text{Find } \omega \text{ where } |G(j\omega)| = \frac{1}{\sqrt{2}}$$

i.e. $\frac{\omega_n^4}{\left(\omega_n^2-\omega^2\right)^2 + (2\zeta\omega_n\omega)^2} = \frac{1}{2}$ $\qquad \omega_{bw}^2 = \omega_n^2\left(1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right)$

$$\text{So } T_{pk} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi\sqrt{1-2\zeta^2+\sqrt{4\zeta^4-4\zeta^2+2}}}{\omega_{bw}\sqrt{1-\zeta^2}}$$

Loop Transfer Function for PM

Loop TF $\quad L(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}$ $\qquad |L(j\omega)| = \frac{\omega_n^2}{\omega\sqrt{\omega^2 + (2\zeta\omega_n)^2}}$

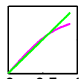
$\angle L(j\omega) = -90 - \tan^{-1}\frac{\omega}{2\zeta\omega_n}$

Want ω where |L(jω)| = 1

Form quadratic in ω^2 from $|L|^2$=1. Solve: $\omega^2 = \omega_n^2\left(-2\zeta^2 + \sqrt{4\zeta^4+1}\right)$

$$PM = 180 + \text{Phase} = 180 - 90 - \tan^{-1}\frac{\omega}{2\zeta\omega_n} = 90 - \tan^{-1}\frac{\omega}{2\zeta\omega_n}$$

$$= \tan^{-1}\frac{2\zeta\omega_n}{\omega} = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{4\zeta^4+1}}}$$



Ugg – but for ζ < 0.6 good approx PM = 100 **ζ**

Summary

ζ ≈ PM / 100 : from PM can estimate ζ.

Or if know peak closed loop freq: using M_{pf} and ω_{rf}

$$\zeta^2 = 0.5 - 0.5\sqrt{1-\frac{O_{ss}^2}{M_{pf}^2}} \qquad \omega_n = \frac{\omega_{rf}}{\sqrt{1-2\zeta^2}}$$

Settling time, $T_{settle} \approx \frac{4}{\zeta\omega_n}$ $\qquad$ overshoot = $e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$

$$\text{Time to Peak, } T_{pk} = \frac{\pi}{\omega_{rt}} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi\sqrt{1-2\zeta^2+\sqrt{4\zeta^4-4\zeta^2+2}}}{\omega_{bw}\sqrt{1-\zeta^2}}$$

We can use these to numerically estimate step resp from bode data

Predicting Response from Bode data

Suppose have [m, p, w] that gives Bode plot of loop TF

At each w, have m cos(p) + j m sin(p) : But need closed loop gain

$$\left|\frac{m\cos(p) + jm\sin(p)}{1 + m\cos(p) + jm\sin(p)}\right| = \frac{\sqrt{m^2\cos^2(p) + m^2\sin^2(p)}}{\sqrt{1 + 2m\cos(p) + m^2\cos^2(p) + m^2\sin^2(p)}}$$

$$= \frac{m}{\sqrt{1 + m^2 + 2m\cos(p)}}$$

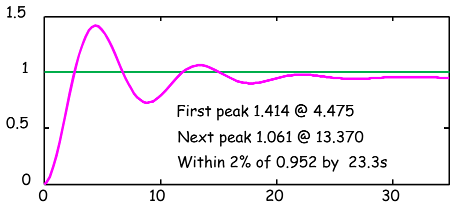
```
mc = m ./ sqrt(1 + m.^2 + 2*m.*cosd(p));  % closed loop gain
mpf = max(mc);                             % find Mpf  : max cl gain
wrf = w(mc == mpf);                        % w where max cl gain
yss = mc(1);                               % steady state = dc gain
zeta = sqrt(0.5 - 0.5 * sqrt(1-yss^2/mpf^2));  % and ζ
```

Thence can estimate ω_n, ω_{rt}, Overshoot, Tpk, Tset …

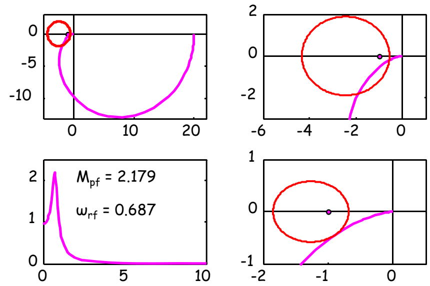
Step Response of 2nd Order System

This is first example where we try these ideas out

```
n = 20;                              % num = 20 simple controller
d = conv([5 1],[8 1]);               % denom = (1+s5)*(1+s8)
[y,x,t]=step (n, [0 0 n] + d);       % of closed loop system
plot(t,y,'k-',[min(t), max(t)], [1 1],'k-');   % plot y and i/p
```



Nyquist and CL-Freq Resp



If locus touch M=1.3 circle, often 20%o/s.

Here Mpf is 2.179,

so locus touch M=2.179 circle at ω_{rf}

```
[m p w] = bode (n, [0 0 n] + d);   % calc closed loop freq resp
plot (w,m,'k-');                   % plot magnitude vs freq
```


Applying These

M_{pf} = 2.179 and ω_{rf} = 0.687 rad/s;

So ζ = 0.2243 and ω_{rt} = 0.706 rad/s;

Actual Step response:

peak of 1.40 @4.47 s and 1.06@13.4 s;

Tset (settling time) = 23.3s

Predictions from Freq Domain

peaks at π/0.706 = 4.45; that + 2π/0.706 = 13.3 s

Peak from formula (20/21)*(1+0.486) = 1.41 s

Tset = 23.9 s

Quite close – not exact as loop tf not in form K/s(1+sT)

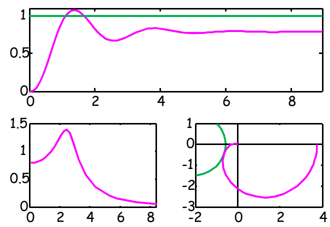
Another Example

num = 37.58;
den = (s + 1)(s + 2)(s+5)

M_{pf} 1.394 ω_{rf} = 1.682
ζ = 0.2968 ω_n = 1.853

Tpk ~ 1.776 Tpk = 1.395
%os ~ 37.67 %os = 37.01
Tset ~ 5.714 Tset = 5.23

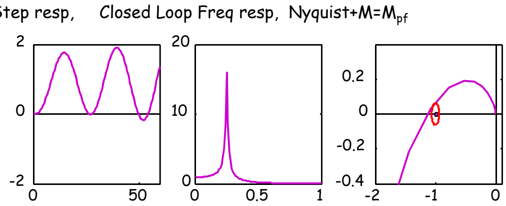
Predictions are ok



Now try on an Unstable System

n = 10; d = (1+s5)(1+s8)(1+s10);

Step resp, Closed Loop Freq resp, Nyquist+M=M_{pf}



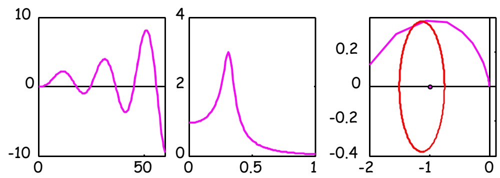
M_{pf} = 16.1; ω_{rf} = 0.251; ζ = -0.028; ω_{rt} = 0.251

Actual Peak 1.76 at 14.96s; Next at 38.9s

Predict as 1.90 at π/ω_{rt} = 12.5s; Next at 3 π/ω_{rt} = 37.6s

Increase n to 20: More Oscillatory

Step resp, Closed Loop Freq resp, Nyquist + M=2.995



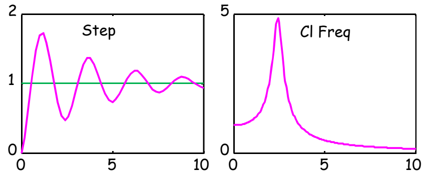
M_{pf} = 2.995; ω_{rf} = 0.311; ζ = -0.161, ω_{rt} = 0.315

Actual Peak 2.22 at 11.6s; Next at 31.3s

Predict as 2.54 at π/ω_{rt} = 10.0s; Next at 3π/ω_{rt} = 29.9s

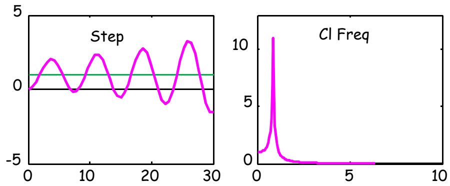
Now consider this system

Plant = $\frac{(s+2)(s+3)}{s^2(s+1)(s+24)(s+30)}$; Series Controller K = 1000



M_{pf} = 4.85; ω_{rf} = 2.43; ζ = 0.104; ω_{rt} = 2.44
First peak 1.73 @ 1.14s; Next 1.37 @ 3.70s; Settle by 14.2s
Predict as 1.72 at π/ω_{rt} = 1.29s; 3 π/ω_{rt} = 3.86; Tset: 15.7s

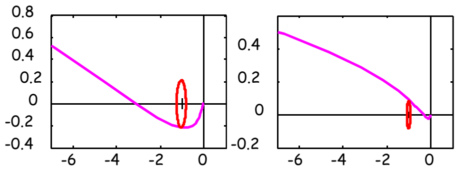
Same System, But K down To 100



M_{pf} = 11.1; ω_{rf} = 0.846; ζ = -0.045, ω_{rt} = 0.847
Peak pred as 2.15 at π/ω_{rt} = 3.71s $3\pi/\omega_{rt}$ = 11.1s
First peak 2.04 at 3.68s; Next at 11.0s

Nyquists for These Two Systems

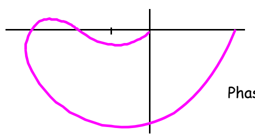
K = 1000 + M = 4.85 Circle; For K = 100 + M = 12.0

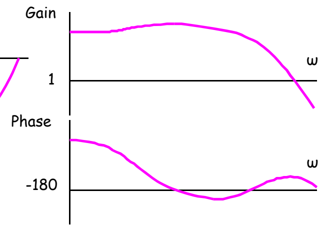


When K = 100, curve encircles the –1,j0 point – so unstable.
When K = 1000, resonant freq where phase < 180°, so stable
Conditionally stable system - increase gain to stablise it
Although gain > 1 when phase = -180° system not unstable
Because freq of oscillation of transient where CL gain max

On Conditionally Stable Systems

So system unstable if gain > 1 when phase -180° not always true
can have conditionally stable systems
In such cases reduce gain to make system unstable!





NB these are rare, but now know why stable!

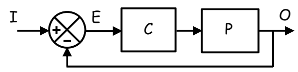
Summary

Have shown how time domain responses can be estimated from frequency responses
Ok for 2nd order and others if dominant mode appropriate
We see that PM related to overshoot
Hence we can design controllers to achieve such a specification
We have already done this for P controller
Next week we remind ourselves of this and then consider more sophisticated controllers.

9. Frequency Response Designs

We have considered the analysis of systems in the freq domain
We can determine stability, plot responses, estimate time response
We have seen how to Design P
In this lecture
- We remind ourselves of P control
- and consider Phase Lead Control, P +I and PID control
- We will also look at how these controllers change Bode plots

Proportional Design - Reminder

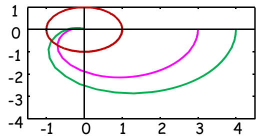


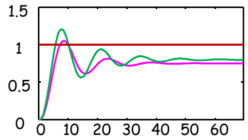
Design C so C*P has given PM

Let P = num/den

C = 1 / gain of C*P when phase is -180^O + PM

e.g. $P(s) = \frac{4}{24s^3 + 26s^2 + 9s + 1}$; PM = 45^O; C = 1/1.334 = 0.7495





Stable, but not very good

Now Another Example

$P(s) = \frac{1}{(s+5)(s+1)(s+2)}$

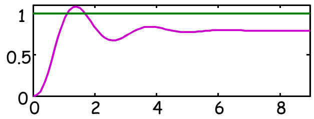
Set PM = 45^O; C = 37.58

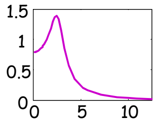
Code looks at Bode data of C * P

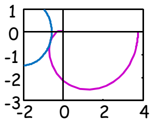
Calculates CL Freq Resp

Hence estimates ζ & ω_n

Thence T_{pk}, %os and Tset







Mpf 1.394 Wrf 2.49 zeta 0.2968 wn 2.743

Estimates: Tpk ~ 1.199 %os ~ 37.67 Tset ~ 3.859

Actual: Tpk = 1.395 %os = 37.01 Tset = 5.23 ok

More Complicated Example

$P(s) = \frac{36(s/11 + 1)}{(s/71 + 1)(s/30 + 1)(s/222 + 1)(s/448 + 1)}$

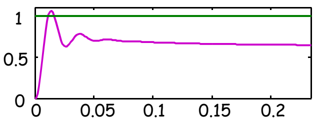
Set PM = 45^O; C = 0.0491

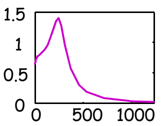
Mpf 1.396 Wrf 245.5
zeta 0.2353 wn 260.3

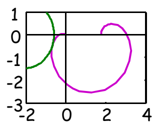
Tpk : %os : Tset

Est 0.012 : 46.7 : 0.051
Act 0.014 : 65.9 : 0.183

Pred ok.
Controller Naff







Phase Lead Control – for speed up

Speed of response set by ω_{rf}, which is near where loop phase -180

To speed response, have Phase Lead instead of P control:

$C(s) = K_p\left(\frac{1 + sTe}{1 + sTa}\right)$

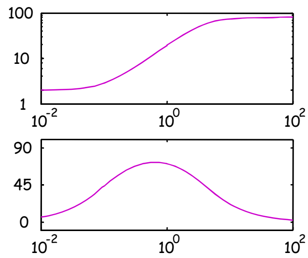
Te > Ta, phase ^ at low freqs,

So phase -180 at higher freq

Phase max α at $\omega = \frac{1}{\sqrt{Te*Ta}}$

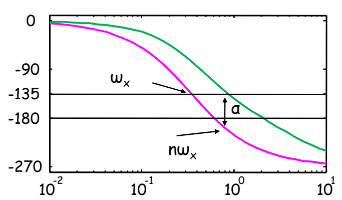
Then $\alpha = \tan^{-1}\sqrt{\frac{Te}{Ta}} - \tan^{-1}\sqrt{\frac{Ta}{Te}}$

Can show $\alpha = \frac{\pi}{2} - 2\tan^{-1}\sqrt{\frac{Ta}{Te}}$



Phase Lead Controller

To speed by fac n, if phase -180+PM at ω_x, design so phase -180+PM at $n*\omega_x$.



$C(s) = K_p\left(\frac{1 + sTe}{1 + sTa}\right)$

pn = $\angle P(n*\omega_x)$

α = -180 + PM - pn

(must be < 90)

Let $G = \tan\left(\frac{90 - \alpha}{2}\right)$

$Te = \frac{1}{n*\omega_x*G}$

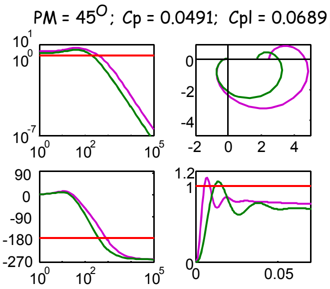
$Ta = \frac{G}{n*\omega_x}$

Then Kp to meet PM

Results – speed up by 2

$P(s) = \frac{36(s/11 + 1)}{(s/71 + 1)(s/30 + 1)(s/222 + 1)(s/448 + 1)}$

PM = 45^O; Cp = 0.0491; Cpl = $0.0689\left(\frac{1 + s0.00624}{1 + s0.000902}\right)$

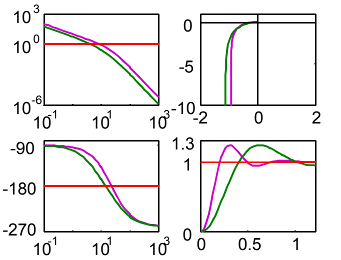


C =	P	PhL
Tpk	0.014	0.007
TSet	0.18	0.172
%os	65.9	55.9
Oss	0.639	0.713

PhL : faster, slightly better Oss … naff

Example for a Type 1 System

$P(s) = \frac{70}{s(s/6+1)(s/35+1)}$; $Cp = 0.0836$; $Cpl = 0.154\left(\frac{1+s0.176}{1+s0.067}\right)$



C =	P	PhL
Tpk	0.616	0.308
TSet	1.47	0.70
%os	23.3	23.5
Oss	1	1

PL Speeds up.
Here Oss = 1…

P + I Controller

For Type 0 system, to ensure O_{ss} is 1 C(s) includes an integrator eg a P+I controller

$$C(s) = K_p\left(1+\frac{1}{sT_i}\right) = K'_p\left(\frac{1+sT_i}{sT_i}\right)$$

The integrator adds 90° phase lag, the associated lead adds some phase lead – which can help

As we shall see, can design for this, but system slow.

Aim, choose a working freq, ω_c, and find Ti and Kp so that system has unity gain and phase -180° + PM.

Note ω_c typically near where plant has phase lag of ~ 90°

This is lower than that used for phase lead, so system slow…

Details

$C(j\omega) = K_p\frac{1+j\omega T_i}{j\omega T_i}$ $\qquad \angle C(j\omega_c) = \tan^{-1}\omega_c T_i - \frac{\pi}{2}$

Want $\angle C(j\omega_c) * P(j\omega_c) = -\pi + PM$

So $\angle C(j\omega_c) = -\pi + PM - \angle P(j\omega_c) = \phi$

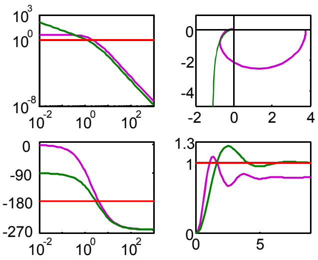
$\phi = \tan^{-1}\omega_c T_i - \frac{\pi}{2} \Rightarrow T_i = \frac{1}{\omega_c}\tan\left(\phi+\frac{\pi}{2}\right)$

$$|C(j\omega_c)| = K_p\frac{\sqrt{1+(\omega_c T_i)^2}}{\omega_c} = K_p\frac{\sqrt{1+\tan^2(\phi+\frac{\pi}{2})}}{\omega_c T_i} = \frac{K_p}{\omega_c T_i\cos(\phi+\frac{\pi}{2})}$$

As $|C(j\omega_c)||P(j\omega_c)| = 1$; $\frac{K_p}{-\omega_c T_i \sin\phi}|P(j\omega)| = 1$; so $K_p = -\frac{\omega_c T_i \sin\phi}{|P(j\omega_c)|}$

Example – on same plant

$P(s) = \frac{1}{(s+5)(s+1)(s+2)}$ PM = 45°; $C_p = 37.58$; $C_{pi} = 12.45\left(1+\frac{1}{0.894s}\right)$



$\angle P(j\omega) = -90$ at $\omega = 1.118$

$\phi = -180 + 45 - -90 = -45$

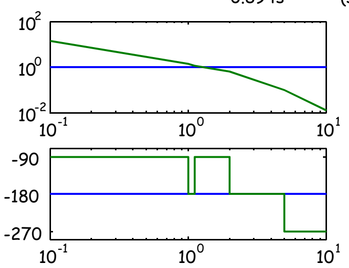
tan(45)/1.118 = 0.894

C =	P	P+I
Tpk	1.4	2.58
TSet	2.5	5.86
%os	37	23.6
Oss	0.787	1

Better: Oss 1, but slow

Estimating Time Response from Asyms

$$C*P(s) = \frac{12.45(0.894s+1)}{0.894s} * \frac{1}{(s+5)(s+1)(s+2)}$$



Clearly |CP| = 1 bit after 1 rad/s, say 1.2

Give estimate for ω_{rf}

PM design at 45, ζ = 0.45

$Tpk \sim \frac{\pi}{\omega_{rt}} = \frac{\pi}{\omega_{rf}}\sqrt{\frac{1-2\zeta^2}{1-\zeta^2}}$

Estimate is 2.3s

Actual is 2.6s

Do able in an Exam!

Confirmation : How to do in an Exam

$$C*P(s) = \frac{12.45(0.894s+1)}{0.894s} * \frac{1}{(s+5)(s+1)(s+2)}$$

Corner Freqs 1; 1/0.894 = 1.2; 2 and 5

Before 1 : TF = $\frac{12.45}{0.894j\omega} * \frac{1}{(5)(1)(2)} = \frac{1.39}{j\omega}$ Gain = 1 at ω = 1.39

1..1.2 TF = $\frac{1.39}{j\omega} * 0.894j\omega = 1.24$ Gain not equal 1 !

1.2..2 TF = $1.24 * \frac{1}{j\omega} = \frac{1.24}{j\omega}$ Gain = 1 at ω = 1.24

This is in range 1.2 ..4, so this is a good estimate of |CP| = 1

Which itself is an estimate of where CL Gain maximum

Modified Ziegler Nichols PID

To speed up, add to controller a term prop to differential of error

Results in most common type of industrial controller - PID

$$C(s) = K_p\left(1+\frac{1}{sT_i}+sT_d\right) = K_p\left(\frac{1+sT_i+s^2T_iT_d}{sT_i}\right)$$

Numerous ways to find parameters – Modified Ziegler Nichols common

Choose a partic freq, ω_c, design controller for desired PM

ie C*P has gain 1 and phase -180°+PM at ω_c

Can be done by P+I controller – K_p and T_i define operation.

For PID, suggest $T_d \le 0.25 * T_i$ so C(s) has two real zeros

Here set $T_d = 0.25 * T_i$ as it makes Maths easier

PID Controller

If $T_d = 0.25T_i$, $C(j\omega) = K_p\dfrac{(1+j\omega 0.5T_i)^2}{j\omega T_i}$

$\angle C(j\omega_c) = 2*\tan^{-1}0.5\omega_c T_i - \frac{\pi}{2}$

Want $\angle C(j\omega_c)*P(j\omega_c) = -\pi + PM$

So $\angle C(j\omega_c) = -\pi + PM - \angle P(j\omega_c) = \phi$

$$T_i = \frac{2}{\omega_c}\tan\left(\frac{\phi+\frac{\pi}{2}}{2}\right) = \frac{2}{\omega_c}\frac{\sin(\phi+\frac{\pi}{2})}{1+\cos(\phi+\frac{\pi}{2})} = \frac{2}{\omega_c}\frac{\cos\phi}{1-\sin\phi}$$

Continued

$$T_i = \frac{2}{\omega_c}\frac{\cos\phi}{1-\sin\phi} \qquad |C(j\omega_c)| = K_p\frac{1+(0.5\omega_c T_i)^2}{\omega_c T_i}$$

$$|C(j\omega_c)| = K_p\frac{1+\left(\frac{\cos\phi}{1-\sin\phi}\right)^2}{\frac{2\cos\phi}{1-\sin\phi}} = K_p\frac{(1-\sin\phi)^2+\cos^2\phi}{2\cos\phi(1-\sin\phi)}$$

$$= K_p\frac{1-2\sin\phi+\sin^2\phi+\cos^2\phi}{2\cos\phi(1-\sin\phi)} = K_p\frac{2(1-\sin\phi)}{2\cos\phi(1-\sin\phi)} = \frac{K_p}{\cos\phi}$$

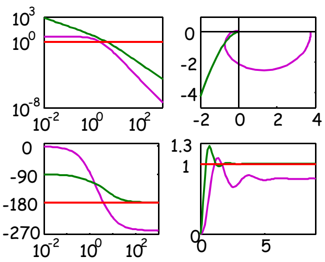
As $|C(j\omega_c)||P(j\omega_c)| = 1; \Rightarrow \dfrac{K_p}{\cos\phi}|P(j\omega_c)| = 1;$

$$\text{so } K_p = \frac{\cos\phi}{|P(j\omega_c)|}$$

On Same Plant

Plant has phase -180° at 4.12 rad/s – this can be ω_c set PM 45°

$$T_i = \frac{2}{4.12}*\frac{\cos 45}{1-\sin 45} = 1.17; \quad |P(j4.12)| = 0.0079; \; Kp = \frac{\cos(45)}{0.0079} = 89$$



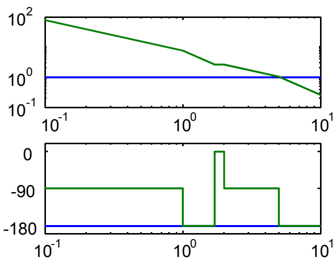
$$89.09\left(1+\frac{1}{1.171s}+0.293s\right)$$

C =	P	PID
Tpk	1.4	0.72
TSet	2.5	1.63
%os	37	25.4
Oss	0.787	1

Better: Oss 1, and faster

On Asymptotes

$$C(j\omega) = \frac{Kp}{j\omega Ti}*\left(1+j\omega\frac{Ti}{2}\right)^2 = \frac{76}{j\omega}*(1+j\omega 0.58)^2; \; P(j\omega) = \frac{1}{(j\omega+5)(j\omega+1)(j\omega+2)}$$



Clearly |CP| = 1 ~ 5

Is est for ωrf

PM design at 45, ζ = 0.45

$$Tpk \sim \frac{\pi}{\omega_{rt}} = \frac{\pi}{\omega_{rf}}\sqrt{\frac{1-2\zeta^2}{1-\zeta^2}}$$

Estimate is 0.5s

Actual is 0.7s

Again – estimate by Asymptote

$$C(j\omega)*P(j\omega) = \frac{76}{j\omega}*(1+j\omega 0.58)^2*\frac{1}{(j\omega+5)(j\omega+1)(j\omega+2)}$$

CFs 1, 1/0.58 = 1.72, 2, 5;

1.72 v close to 2, so replace 1.72 and 2 by ~1.9

$$(1+j\omega 0.58)^2*\frac{1}{(j\omega+2)} \approx 1+j\omega/1.9 = \frac{j\omega+2}{1.9}$$

$$\text{So } C(j\omega)*P(j\omega) \approx \frac{40\,(j\omega+1.9)}{j\omega(j\omega+5)(j\omega+1)}$$

Asyms	<1	1..1.9	1.9..5	>5
	$\frac{16}{j\omega}$	$\frac{16}{(j\omega)^2}$	$\frac{8}{j\omega}$	$\frac{40}{(j\omega)^2}$

Gain = 1 in last asym, at √40 = 6.3 similar to est of 5 from figure

Summary

We have seen how to design, in the frequency domain

P, Phase Lead, P+I and PID controllers

P is simplest – often not acceptable in type 0 systems

For type 1 systems, Oss = 1, so P ok, but Phase Lead speeds up

For type 0 systems, an integrator needed for Oss = 1

P+I ok, but slow, so PID often used – most common in industry

One method has been shown – there are others.

Next week we finish the course by considering

Positive and Negative Feedback

Estimating frequency response from time domain samples

Assignment

You are now in a position to complete the assignment

See the sheet for details, but essentially you will

Design P, Phase Lead, P+I and PID controllers for systems based on your student number using another GUI

Copy relevant code, results, etc., into the Word doc

Submit by deadline onto Blackboard

10 : Final Topics

In this lecture we finish the course

We look into concept of negative and positive feedback

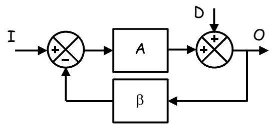
Consider definition and claims

and relate this to the frequency domain

We also look at estimating the freq resp from time samples

Positive and Negative Feedback

Various views and erroneous comments exist.
Here give sensible definition and consistent claims for effect.
Bode's colleague Black's *Change in Gain due to Feedback*
Bell System Technical Journal, Vol XIII, pp1-18, Jan 1934

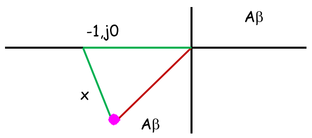


Amplification (or Gain) without feedback = $|A|$

with feedback $|G| = \dfrac{|A|}{|1 + A\beta|}$

Positive Feedback if $|G| > |A|$; that is $|1 + A\beta| < 1$
But note, $A\beta$ is a function of frequency …

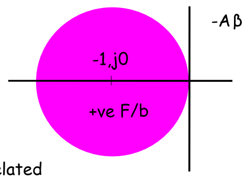
Distance from -1,j0



$x = 1 + A\beta$

So distance from -1,j0 = $|1 + A\beta|$

Any point in circle is where $|1 + A\beta| < 1$



Positive Feedback at freqs where Nyquist plot in circle

Negative Feedback at others

+ve feedback and instability not related

Consider these systems

used earlier

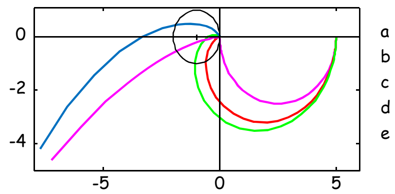
a) $\dfrac{5}{10s+1}$

b) $\dfrac{5}{(8s+1)(5s+1)}$

c) $\dfrac{5}{(3s+1)(2s+1)(s+1)}$

d) $\dfrac{5}{s(5s+1)}$

e) $\dfrac{5}{s(s+1)(2s+1)}$



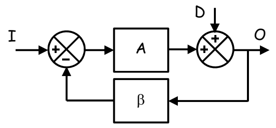
Locus	Some +ve fb	Stable
(a)		
(b)		
(c)		
(d)		
(e)		

Claims for Negative Feedback

1) It reduces error (or errors) in the system
   many books not define what error is
2) It reduces the effects of disturbances
   sometimes disturbances are not defined
3) It reduces effects of changes in the forward path gain
   this is a useful effect in electronics, for instance, as gains change, as well as in other systems
4) It reduces the magnitude of the system gain
   Black's claim – often ignored by control engineers

Let us investigate these claims – doing 2, 3 and then 1.

Reducing Effect of Disturbances



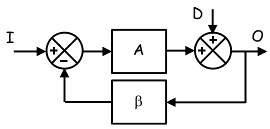
Assuming I = 0, for open loop, O = D

for closed loop, $O = \frac{1}{1+A\beta}D$

So effect of D on O reduced if

$|D| > \left|\frac{1}{1+A\beta}D\right|$ or $\left|\frac{1}{1+A\beta}\right| < 1$ i.e. if negative fb

On Changes in Forward Path Gain, A



$\frac{O}{I} = \frac{A}{1+A\beta}$

Define $G = \frac{A}{1+A\beta}$

What is effect on G of changing A, assuming β is constant?

$$\frac{dG}{dA} = \frac{(1+A\beta)*\frac{dA}{dA} - A*\frac{d(1+A\beta)}{dA}}{(1+A\beta)^2} = \frac{(1+A\beta)-A\beta}{(1+A\beta)^2} = \frac{1}{(1+A\beta)^2}$$

Investigating This

Change in G on its own not useful, the relative change in G better, ie dG/G

$$dG = \frac{dA}{(1+A\beta)^2}$$

$$\text{So } \frac{dG}{G} = \frac{\frac{dA}{(1+A\beta)^2}}{G} = \frac{\frac{dA}{(1+A\beta)^2}}{\frac{A}{(1+A\beta)}} = \frac{dA}{A}\frac{1}{1+A\beta}$$

So proportional change in G = prop. change in A * $\frac{1}{1+A\beta}$

For effect of change in A to be reduced by feedback

$\left|\frac{1}{1+A\beta}\right| < 1$ i.e. if system has negative feedback

What About Changing β?

$$\frac{dG}{d\beta} = \frac{(1+A\beta)*0-A*.A}{(1+A\beta)^2} = \frac{A^2}{(1+A\beta)^2} = G^2$$

$$\frac{dG}{G} = Gd\beta = \frac{A\beta}{1+A\beta}\frac{d\beta}{\beta}$$

So cant say, if negative feedback, effect of changing β reduced.

In fact, if have (as want) high loop gain:

$$\frac{dG}{G} \approx +\frac{d\beta}{\beta}$$

However, usually, more likely that A changes, not β.

Overall – definition pretty consistent with claims.

Negative Feedback Reduces Error

'Desired Output' = -I / β, what is actual output?

Open Loop, Actual Output = A * I

Closed Loop Actual Output = $\frac{A}{1+A\beta}$ * I

Output Error = Desired Output - Actual Output

But, size of error affected by output size: define Error Ratio :

$$ER = \frac{\text{Output Error}}{\text{Desired Output}}$$

$$ER = \frac{\text{Output Error}}{\text{Desired Output}} = \frac{\text{Desired Output - Actual Output}}{\text{Desired Output}}$$

Error Ratio

Closed Loop: ER = $\dfrac{I/\beta - AI/(1+A\beta)}{I/\beta}$ = $1 - \dfrac{A\beta}{1+A\beta}$ = $\dfrac{1}{1+A\beta}$

Open Loop : ER = $\dfrac{I/\beta - AI}{I/\beta}$ = $1 - A\beta$

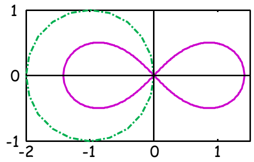
Feedback has reduced the size of the error if:

$\left|\dfrac{1}{1+A\beta}\right| < |1 - A\beta|$ or $1 < |1\text{-}A\beta||1+A\beta|$

Not negative feedback definition – but similar

Let's show the regions on the Argand Plane :

Consider these on Argand Plane



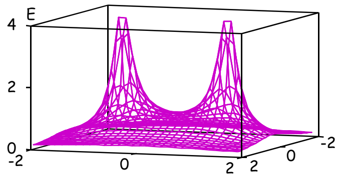
Error raised inside figure of 8

$((1+x)^2 + y^2)((1-x)^2 + y^2) = 1$

NB y = 0 when x = 0 or $\pm\sqrt{2}$

Positive feedback inside

$(1+x)^2 + y^2 = 1$



3D plot, Error vs plane

Much of plane E < 1

E > 1 near -1,j0 and +1,j0

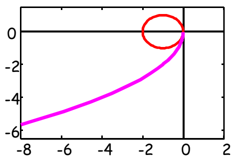
Easy to avoid +1,j0 by high low freq loop gain

Then -ve fb ~ reduce E

2nd Order System no Positive Feedback

Loop TF $\dfrac{1+2s}{s^2}$

CL: $\dfrac{\frac{1+2s}{s^2}}{1+\frac{1+2s}{s^2}} = \dfrac{1+2s}{1+2s+s^2}$



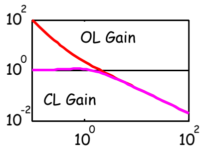
High F, LoopTF ~2/s, up imag axis

Re no +ve feedback, look at | denom |

Open Loop : ω^2

Closed Loop: $\sqrt{1\text{-}2\omega^2 + \omega^4 + 4\omega^2}$

$= \sqrt{1+2\omega^2+\omega^4} = 1+\omega^2$



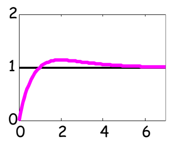
Second Order Correlations On This

M_{pf} = 1.16 ω_{rf} = 0.707 rad/s O_{ss} = 1

ζ = 0.5 ω_{rt} = 0.866 rad/s ω_n = 1 rad/s

	Estimate	Actual
T_{pk}	3.6s	1.98s
O_{pk}	1.16	1.14
T_{settle}	8s	5.4s

Why the difference?



Step Resp overshoots though no +ve fb

Comparing $\dfrac{1+2s}{1+2s+s^2}$ and $\dfrac{1}{s^2+2\zeta\omega_n s+\omega_n^2} = \dfrac{1}{s^2+s+1}$

Dominant mode of this system too different.

Estimating System Frequency Response

Fourier Transform

process a signal (a function of time),

generate its frequency response

Result a set of complex numbers at various frequencies,

This is the power spectrum

$y(t) \rightarrow Y_{PS}(j\omega)$

Could then reconstruct signal by summing sinusoids …

Note, real signals are the 'true' value, plus noise

So can only generate estimates of the spectrum.

How to get Transfer Function?

In Laplace Domain : Y(s) = G(s) * U(s)

In Freq domain : Y(jω) = G(jω) * U(jω)

$G(j\omega) = \dfrac{Y(j\omega)}{U(j\omega)}$

So an estimate of the power spectrum of G is found by

$G_{PS}(j\omega) = \dfrac{Y_{PS}(j\omega)}{U_{PS}(j\omega)}$

Frequency Resp from Time Domain

MatLab has the tools to do this. Uses structs for data

Basic algorithm

```
y = output, u = input, over time = t
dat = iddata(y, u, t(2)-t(1))  % form time domain data
datd = detrend(dat)            % remove best straight line fit
                               % removes mean value
idplot(ze)                     % plot in and out
fr = spa(datd)                 % finds spectral response
bode(fr)                       % plot it as bode plot
```


Persistently Exciting Input

Important that the system is 'persistently excited'

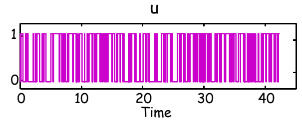
step signal not enough, as output over small freq range

'random' pulse train better

also called pseudo random binary sequence

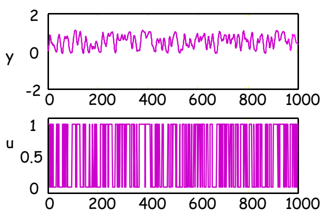
Eg to generate 1000 bit sequence, u(t) = randomly u(t-1) or 1-u(t-1)

```
r = rand(1000,1);
u = r; u(1) = 1;
for ct = 2:length(u),
  if r(ct)<0.8, u(ct)=u(ct-1);
  else      u(ct) = 1-u(ct-1);
  end;
end;
```



Example – test on 2nd order system

```
Ts = 0.04; t = [0 : Ts : Ts*999];   % 1000 element time
y = lsim(50, [1 8 50], u, t);       % sim known 2nd order sys
dat = iddata(y, u, Ts);
datd = detrend(dat);
idplot(dat)
```



Do Spectral Analysis

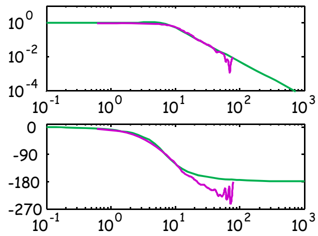
```
[ma, pa, wa] = bode(50, [1 8 50]);  % get actual bode to test
fr=spa(datd);                       % now do spa
[m,p,w]=bode(fr);
mm=squeeze(m);
pp=squeeze(p);
ww=squeeze(w);
```

mm is 1,1,n matrix

squeeze so 1,n vector

Plot m and mm; p and pp

Results from spa close to actual, until high freqs



Summary

In this lecture

- Looked at positive and negative feedback
  - Have a definition consistent with claims
  - Showed that systems often have both +ve and -ve
- Also, briefly introduced Spectral Analysis
  - For finding frequency response of plant from I/O data
  - Note, I must persistently excite system

Overall, in these lectures

- Have considered how to find and plot frequency responses
- To identify and control systems using the frequency response
- To find frequency response from time domain data