Extra Phase Asymptotes and e^{π}

Adapted from R.J.Mitchell "Using MATLAB GUIs to improve the learning of frequency response methods", http://ieeexplore.ieee.org/xpl/mostRecentIssue.jsp?punumber=6327840, Proc UKACC Control 2012.

Bode plots show how the gain and phase of a system vary with angular frequency, ω. These can be approximated by asymptotes. Whilst these asymptotes allow a user to sketch the gain curve easily, that is less true for the phase curve due to the step nature of the asymptotes. As such, books like Dorf and Bishop[1] or Nise[2] suggest that these phase asymptotes should be joined by 'diagonal' asymptotes between one tenth of and ten times the associated corner frequency, that is over a range of 100 rad/s, as shown in Figure 1.

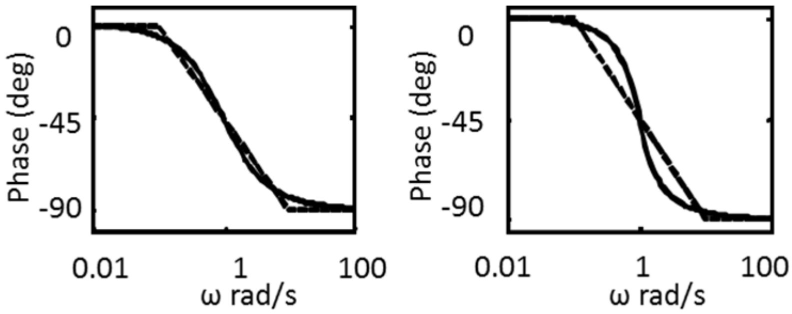

Figure 1. Phase and Asymptotes for Single and Quadratic Pole

For a single pole, this approach is quite good, as the area of the region on one side of the asymptote is approximately equal to that on the other. However for a quadratic pole the areas are quite different. A better approach, advocated here, is for the slope of the diagonal asymptote to equal that of the slope of a pole or zero at the corner frequency (ignoring the effects of other poles and zeros).

For a single pole or zero, it is shown below that this range is $e^{\pi} \approx 23$.

For a quadratic pole or zero with damping ratio ζ, the range is $e^{\zeta\pi} = 4.8$ if $\zeta = 0.5$ or 2.2 if $\zeta = 0.25$.

As regards, finding the diagonal asymptote, let r be the range over which the diagonal asymptote spans, centred on the corner frequency, ω_{CF}, of a single or a quadratic pole or zero, which changes the phase by nπ rad. The asymptote thus is from $\omega_{CF}/\sqrt{r}$ to $\omega_{CF} * \sqrt{r}$. If ϕ is the phase of such an element, plotted against log(ω), the slope of the asymptote is set by the slope of the element evaluated at ω_{CF}, so

$$\frac{n\pi}{\log(r)} = \frac{d\phi}{d\log(\omega)}\bigg|_{\omega_{CF}} = \frac{\omega}{\log(e)}\frac{d\phi}{d\omega}\bigg|_{\omega_{CF}}$$

This can be rearranged to give

$$r = e^{n\pi \Big/ \omega \frac{d\phi}{d\omega}\big|_{\omega_{CF}}}$$

For a single pole or zero, with time constant T, $n = \pm 0.5$ and $\phi = \pm\tan^{-1}(\omega T)$ and $\omega_{CF} = 1/T$, so

$$\omega\frac{d\phi}{d\omega}\bigg|_{\omega_{CF}} = \pm\frac{\omega T}{1+\omega^2T^2}\bigg|_{\frac{1}{T}} = \pm\frac{1}{2}$$

And so

$$r = e^{\pm 0.5\pi / \pm 0.5} = e^{\pi}$$

For a quadratic pole or zero with corner frequency ω_n and damping ratio ζ, for which $n = \pm 1$:

$$\omega\frac{d\phi}{d\omega}\bigg|_{\omega_{CF}} = \pm\frac{\omega\, 2\zeta\omega_n\left(\omega_n^2+\omega^2\right)}{\left(\omega_n^2-\omega^2\right)^2+4\zeta^2\omega_n^2\omega^2}\bigg|_{\omega_n} = \pm\frac{1}{\zeta}$$

And so

$$r = e^{\pi\zeta}$$

Fig 2 shows the phase and these asymptotes for a single and quadratic pole, using these values of r. The phase is easier to sketch than the conventional approach. Instead of crossing the diagonal asymptote and then intersecting with it at the corner frequency, the actual phase moves smoothly between the horizontal asymptote before or after the corner frequency (at some integer * ½ π) and the diagonal asymptote. The gain curve moves between asymptotes in a similar manner.

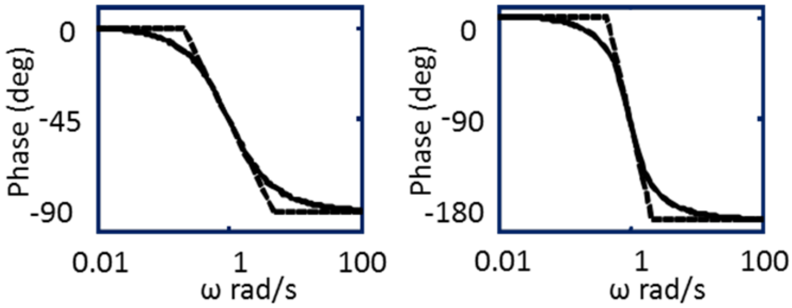

Figure 2 Better Phase Asymptotes for Single and Quadratic Pole

These plots show the asymptotes for a single pole, but multiple poles and zeros will interact. To assess the success of the approach consider the following system

$$\frac{50(1+s/2)}{(1+s/0.07)(1+s/0.4)(s^2/20^2+0.6s/20+1)}$$

By inspection the corner frequencies are 0.07, 0.4, 2 and 20 rad/s. The gain asymptote has value 50 and slope 0 until 0.07 rad/s, the slope is -1 until 0.4 rad/s, -2 until 2 rad/s, -1 until 20 rad/s and -3 thereafter. In these ranges, the phase asymptotes have values 0, -90, -180, -90 and -270 degrees. The range of the extra phase asymptotes around each corner frequency is 23 except at 20 rad/s where, as $\zeta = 0.3$, it is 2.6. These can be seen in Figure 3, where the actual Bode plot is also superimposed. The gain plot is easy to sketch from the asymptotes. The extra phase asymptotes make it easier to sketch the phase plot

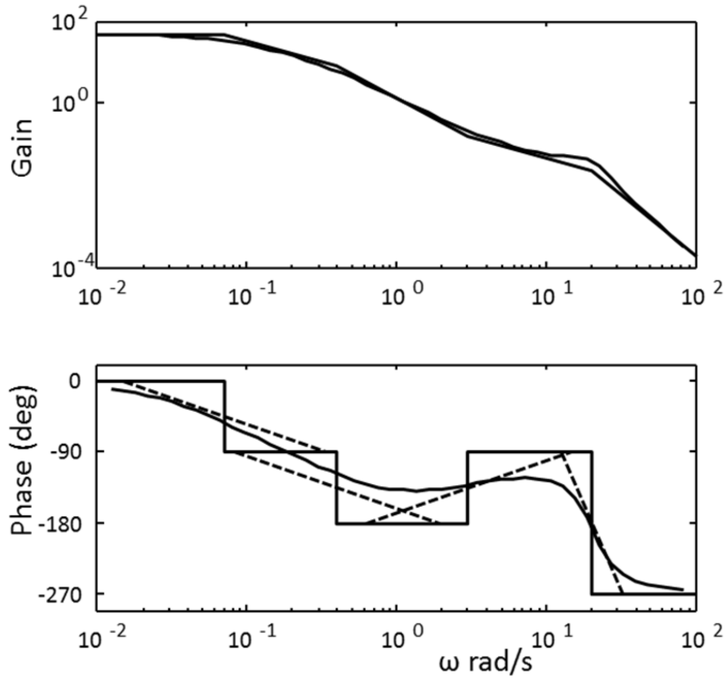

Figure 3 Bode plot of example system including asymptotes

[1] R.C. Dorf and R. H. Bishop: 'Modern Control Systems', (Pearson, 11th edn. 2008)
[2] N.S. Nise.: 'Control Systems Engineering', (John Wiley & Sons, 5th edn. 2008).

Dr Richard Mitchell, June 2014