Differentiation and Integration 6
Prof Richard Mitchell

Today we will start to look at more typical signals including exponentials, logarithms and hyperbolics

Some of this can be found in the recommended books

Croft 267-309; James 130-140, 514-517

Stroud 86-94,343,495-517; Singh 224-253,266,362;

Don't forget to attend the tutorials to get practice

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Exponential, Logarithm, Hyperbolic

We met exponentials in lecture 1, in the RC circuit – though they can be seen in many systems.

The current decayed smoothly – exponentially.

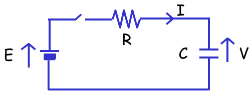
In this lecture we will look at the exponential function,
its differential and its integral.

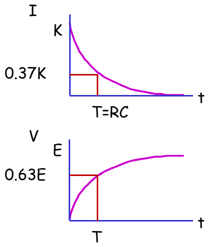
Logarithm is inverse exponential
we will consider it and some applications

Hyperbolic functions are combinations of exponentials,
so these too will be considered.

On Exponentials

In lecture 1 we argued the variation of I and V





We quoted the equation for I

$$I = \frac{E}{R} e^{-\frac{t}{RC}}$$

$E = V + I R$, so $V = E - I R$

$$V = E - E e^{-\frac{t}{RC}}$$

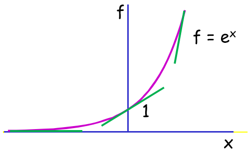
Let's look at e formally

Exponentials Formally

Function: exp(x) or e^x where e ~ 2.71828

e^0 is 1; $e^1 \sim 2.718$; $e^{-1} \sim 0.37$; $e^{\infty} = \infty$; $e^{-\infty} = 1/ e^{\infty} = 0$

Hence at $t = RC$, $I = \frac{E}{R} e^{-1} = 0.37 \frac{E}{R}$; $V = E - Ee^{-1} = 0.63E$



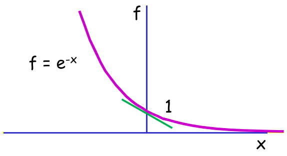
As $x \to -\infty$, $\text{Exp}(x) \to 0$
so does slope

Exp(0) = 1 so is slope

As $x \to +\infty$, $\text{Exp}(x) \to \infty$
so does slope

Suggests $\frac{d(\exp(x))}{dx} = \exp(x)$

More on Exponentials

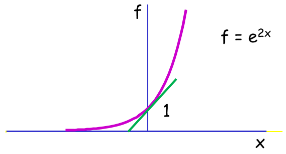


Exp(- -∞) = ∞; Slope -∞

Exp(0) = 1; Slope is -1

Exp(-∞) = 0; Slope 0

So $\frac{d(\exp(-x))}{dx} = -\exp(-x)$



Exp(-2∞) = 0; Slope 0

Exp(2*0) = 1; Slope is 2

Exp(2∞) = ∞; Slope ∞

So $\frac{d(\exp(2x))}{dx} = 2\exp(2x)$

So for Exponentials

From results, generalise: $\frac{de^{nx}}{dx} = ne^{nx}$

Therefore $\int e^{nx} dx = \frac{1}{n} e^{nx} + c$

Remember for the RC Circuit

$$I = \frac{E}{R} e^{-\frac{t}{RC}} \qquad V = E - E e^{-\frac{t}{RC}} \qquad \text{Also } I = C\frac{dV}{dt}$$

$$\frac{dV}{dt} = \frac{dE}{dt} - \frac{dEe^{-\frac{t}{RC}}}{dt} = 0 - E\left(-\frac{1}{RC}\right)e^{-\frac{t}{RC}} = \frac{E}{RC} e^{-\frac{t}{RC}}$$

$$C\frac{dV}{dt} = C\frac{E}{RC} e^{-\frac{t}{RC}} = \frac{E}{R} e^{-\frac{t}{RC}} = I$$

In Class Exercise

For the RC circuit, E = 5V, R = 100Ω and C = 0.001F

Thus $I = \frac{5}{100}e^{-\frac{t}{0.1}} = 0.05e^{-10t}$ $\qquad V = 1000 \int I\,dt$

Find an expression for V, given that V = 0 when t = 0s

$V = 1000 * \int 0.05e^{-10t}dt$

$= -\frac{1000}{10}0.05e^{-10t} + c \quad = -5e^{-10t} + c$

At $t = 0$, $0 = -5e^{0} + c$; so $c = 5$

So $V = 5 - 5e^{-10t}$

Logarithm – Inverse of Exponential

If $y = e^x$ then $x = \log_e y$ or ln(y) is log to base e

Log to base 2 $\log_2$ used in information theory, for instance

$\log_{10}$ often used – for sound : decibels $20 * \log_{10}(\text{voltage})$

If $x = \log_{10}(y)$ then $y = 10^x$

Rules $\log(a * b) = \log(a) + \log(b)$

$\log(a^n) = n * \log(a)$

So $\log(a/b) = \log(a * b^{-1}) = \log(a) - \log(b)$

$\log(1) = 0$ as for instance $10^0 = 1$

To change base : $\log_a X = \frac{\log_b(X)}{\log_b(a)}$

Differential of Logarithm

Differential of e^x is easy, but what of its inverse ln?

We find it using a neat 'trick', namely $\frac{dx}{dy} = 1\Big/\frac{dy}{dx}$

$x = \ln(y)$ So $y = \exp(x)$

$$\frac{d(\ln(y))}{dy} = 1\Big/\frac{d(\exp(x))}{dx} = \frac{1}{\exp(x)} = \frac{1}{y}$$

Note, initially the result is a function of x, but we can then change it to one of y. Consider also:

Differential of $\sin^{-1}$

The differential for $\sin^{-1}$ illustrates this further

$x = \sin^{-1}(a)$ so $a = \sin(x)$

$$\frac{d(\sin^{-1}(a))}{da} = 1\Big/\frac{d(\sin(x))}{dx} = \frac{1}{\cos(x)}$$

This is a function of x, should be a function of a.

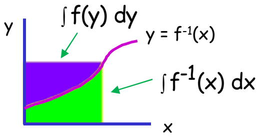
But as $\cos^2(x) + \sin^2(x) = 1$, $\cos(x) = \sqrt{(1 - \sin^2(x))}$

$$\text{So } \frac{d(\sin^{-1}(a))}{da} = \frac{1}{\sqrt{1 - \sin^2(x)}} = \frac{1}{\sqrt{1 - a^2}}$$

Similar methods can be used for $\cos^{-1}$ and $\tan^{-1}$.

Integration of $\sin^{-1}$

$\int f^{-1}(x)\,dx = xy - \int f(y)\,dy$



If $y = \sin^{-1}(x)$, $x = \sin(y)$

So $\int \sin^{-1}(x)\,dx = xy - \int \sin(y)\,dy = xy + \cos(y)$

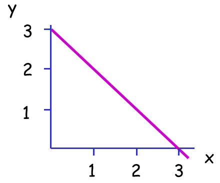
Answer should be function of x :

$y = \sin^{-1}(x)$; $\cos(y) = \sqrt{1 - \sin^2(y)} = \sqrt{1 - x^2}$

So $\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) + \sqrt{1 - x^2} + c$

Logarithmic Graphs

A normal 'linearly' scaled graph has axes like the following



The 'ticks' on the axes are linear, hence linear graph

Shows graph of : $y = 3 - x$

Sometimes better for one/both axes to be logarithmic.

Used, for instance, when variables over wide range

Application – Frequency Response

If input to RC circuit is $K_1 \sin(\omega t)$, output is $K_2 \sin(\omega t-\phi)$.

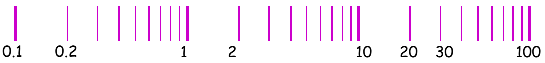
$$\text{Gain} = \frac{K_2}{K_1} = \frac{1}{\sqrt{1 + (\omega CR)^2}} \text{ and Phase } \phi = -\tan^{-1}(\omega CR)$$

Frequency Response : how Gain and Phase vary with ω

Plot Graphs : Gain vs ω and Phase versus ω

Relevant variation of Gain and Phase requires that ω is from ~ 1/30RC to 30/RC if RC=0.3 from 0.1 to 100 rad/s

The following log scale is appropriate : each power of 10 equal space

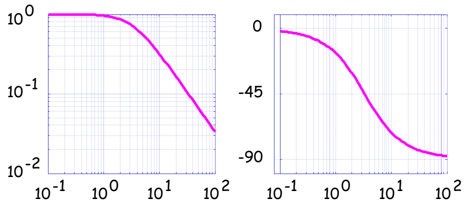


Ranges: RC = 0.3, ω 0.1 to 100 rad/s

$$\text{Gain} = \frac{1}{\sqrt{1 + (\omega CR)^2}} \text{ from } \frac{1}{\sqrt{1.0009}} \sim 1 \text{ to } \frac{1}{\sqrt{901}} \sim \frac{1}{30}$$

$$\text{Phase } \phi = -\tan^{-1}(\omega CR)\text{: } -\tan^{-1}0.03=-1.7^{O} \text{ to } -\tan^{-1}30=-88.1^{O}$$

So plot log(Gain) vs log(ω) and Phase vs log(ω)



Approximating Gain Graph

$$\text{Gain} = \frac{1}{\sqrt{1 + (\omega CR)^2}}$$

$$\text{If } \omega CR \ll 1\text{, Gain} = \frac{1}{\sqrt{1}} = 1;$$

$$\text{If } \omega CR \gg 1\text{, Gain} = \frac{1}{\sqrt{1 + (\omega CR)^2}} = \frac{1}{\omega CR};$$

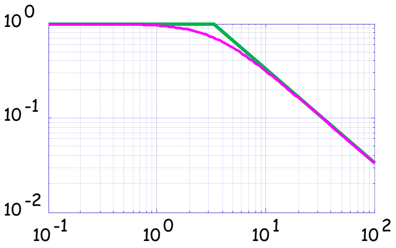
For former, graph is horizontal line

For latter, Plot $\log\frac{1}{\omega CR}$ vs $\log(\omega)$;

ie Plot $\log(\omega CR)^{-1} = -\log(\omega) - \log(CR)$ vs $\log(\omega)$

i.e Plot straight line : gradient is -1, offset -log(CR)

Gain graph with 'asymptotes'



The 2 straight lines, asymptotes, meet at ω = 1/RC

Gain then is

$$\frac{1}{\sqrt{1+1}} = 0.707$$

Gain graph starts on first asymptote and moves away eventually touching second asymptote

Approximating Freq Response like this is a powerful tool

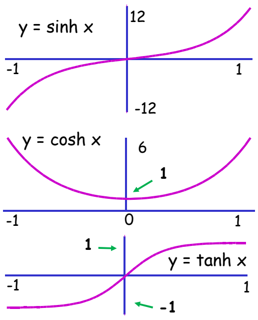
Hyperbolic Functions

The hyperbolic sine, cosine and tangent functions are:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



Applications of Hyperbolics

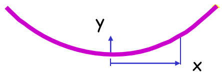
In some neural networks, the output of a neuron is a summation passed to a differentiable activation function

$$\text{If the output is to be 0..1, } f(z) = \frac{1}{1+e^{-z}} = \text{sigmoid}(z)$$

$$\text{If the output is to be -1..1, } f(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \tanh(z)$$

A catenary is a chain hanging under its own weight.

Later we derive an expression for it, in terms of unit weight w and tension T



$$y = \frac{T}{w}\left(\cosh\left(\frac{wx}{T}\right) - 1\right)$$

Differentials / Integrals

$\sinh ax = \frac{e^{ax} - e^{-ax}}{2}$ $\cosh ax = \frac{e^{ax} + e^{-ax}}{2}$

$$\frac{d\sinh ax}{dx} = a\frac{e^{ax}}{2} - -a\frac{e^{-ax}}{2} = a\frac{e^{ax} + e^{-ax}}{2} = a\cosh(ax)$$

Can show that $\frac{d\cosh ax}{dx} = a\sinh(ax)$

Thus $\int \cosh ax \ dx = \frac{1}{a}\sinh(ax) + c$

Thus $\int \sinh ax \ dx = \frac{1}{a}\cosh(ax) + c$

Hyperbolic Identities

$\cosh ax = \frac{e^{ax} + e^{-ax}}{2}$ $\sinh ax = \frac{e^{ax} - e^{-ax}}{2}$

$$\cosh^2 ax = \frac{e^{2ax} + 2e^{ax}e^{-ax} + e^{-2ax}}{4} = \frac{e^{2ax} + 2 + e^{-2ax}}{4}$$

$$\sinh^2 ax = \frac{e^{2ax} - 2e^{ax}e^{-ax} + e^{-2ax}}{4} = \frac{e^{2ax} - 2 + e^{-2ax}}{4}$$

$$\cosh^2 ax - \sinh^2 ax = \frac{2 - -2}{4} = 1$$

$$\cosh^2 x = \frac{e^{2x} + e^{-2x}}{4} + \frac{2}{4} = \frac{\cosh(2x)}{2} + \frac{1}{2}$$

Summary

Today we have looked at exponentials, logarithms and hyperbolics functions often used in systems

We have argued graphically their differential

That of exp(3x) is

That of ln(x) is

That of cosh(4x) is

Again the inverse means

$\int$ exp(-2x) dx is

Next week : chain rule and integration by substitution.

Tutorial – Week 6 – Qs 1, 2 and 3

6.1 Find the following differentials,

a) $\frac{d(5 - 5\exp(-4t))}{dt}$ b) $\frac{d\, 0.25*\ln(2x)}{dx}$ c) $\frac{d\cosh(5x)}{dx}$

6.2 Find the following integrals,

a) $\int 10\exp(-0.1x)\,dx$ b) $\int_0^1 t - 5\exp(-2t)\,dt$ c) $\int 3\cosh(4t)\,dt$

6.3 a) Find the equation of the tangent to the graph given by y = 2x - 5 exp(-3x) at x = 1.

b) Find the mean of $f(t) = e^{3t}$ from t= -2 to 1;

Tutorial – Week 6 – Qs 4 and 5

6.4 The voltage V in a RC circuit is given by the following when the input voltage is 0.1 t; $V = 0.1e^{-t} + 0.1t - 0.1$

a) Find $\frac{dV}{dt}$

b) Show that V is a solution of the equation $\frac{dV}{dt} = 0.1t - V$

6.5 The phase shift of an RC circuit is $P = -\tan^{-1}0.5\omega$.

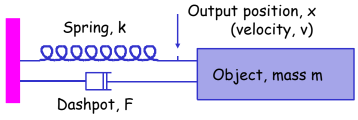
Find $\frac{dP}{d\omega}$.

Tutorial – Week 6 – Q 6

6.6 For the mass spring system below, the accelaration of the mass, $\frac{d^2x}{dt^2}$, is given by $-4e^{-t} + 9e^{-3t}$. At $t = 0$, its velocity is 1 m/s and its position is 2m.

a) Find an expression for position x.

b) Show that $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 15$



Tutorial - Week 6 - Hints

6.1 Apply standard rules

6.2 Ditto

6.3 a) $m = dy/dx$ at $x = 1$; $c = y(1) - m$

b) Use standard integral for means

6.4 a) straightforward

b) show both sides of equation equal.

6.5 Use inverse function – note diff of tan(x) is $\sec^2(x)$.

6.6 a) Integrate, find constant, integrate, find constant

b) evaluate left hand side of equation – show is 15.

Differentiation and Integration 7
Prof Richard Mitchell

Today : Chain Rule and Integration by substitution

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Stroud 384-387, 827-829; Singh 269-280, 368-370.

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Chain Rule – Why Need

From last two weeks we established graphically that

$$\frac{de^x}{dx} = e^x, \quad \frac{de^{-x}}{dx} = -e^{-x} \text{ and generalised: } \frac{de^{nx}}{dx} =$$

$$\text{Therefore } \int e^{nx}dx = \frac{1}{n}e^{nx} + c$$

$$\frac{d(\sin(a))}{da} = \cos(a) \text{ and } \frac{d(\sin(2a))}{da} = 2\cos(2a)$$

$$\text{In general } \frac{d(\sin(\omega a))}{da} =$$

We will next consider the chain rule which can formally confirm these, and then the equivalent for integration.

The Chain Rule – for sin(ωt)

sin(ωt) simple eg of a function defined by two functions:

$x = f(z)$ $\qquad$ $z = g(t)$

For $x = \sin(\omega t)$, $z = \omega t$, and $x = \sin(z)$.

For differential of x with respect to t use chain rule:

$$\frac{dx}{dt} = \frac{dx}{dz}\frac{dz}{dt}$$

So, for sin(ωt), $\frac{dx}{dz} = \frac{d(\sin(z))}{dz} =$ $\qquad$ $\frac{dz}{dt} = \frac{d(\omega t)}{dt} =$

$$\text{Hence } \frac{d\sin(\omega t)}{dt} = \frac{dx}{dt} = \cos(z)\,\omega =$$

Example – for RC Circuit

$V = 5 - 5\exp(-t/10)$: find $I = 2\,dV/dt$

To differentiate $\exp(-t/10)$, let $z = -t/10$ and $x = \exp(z)$

$$\frac{dx}{dt} = \frac{dx}{dz}\frac{dz}{dt} = \frac{d\exp(z)}{dz}\frac{d(-t/10)}{dt} = \exp(z) \;*\; -1/10 = -0.1*\exp(-t/10)$$

$$\text{Thus } 2\frac{dV}{dt} = 2\left(\frac{d5}{dt} - 5 * -0.1*\exp(-t/10)\right) = \exp(-t/10)$$

If you feel confident (comes with practice), just write

$$2\frac{dV}{dt} = 2\left(\frac{d5}{dt} - \frac{d\,5\exp(-t/10)}{dt}\right) = -10*-\frac{1}{10}\exp(-t/10) = \exp(-t/10)$$

Sigmoid in Neural Networks

Cyberneticists and Computer Scientists use neural nets. In some networks a neuron's output is given by sigmoid

$$O = \frac{1}{1+e^{-z}} \quad \{\text{z is weighted sum of neuron inputs}\}$$

For learning, we need to know the differential of O wrt z

$$\text{Let } x = 1 + e^{-z}, \text{ so } \frac{dx}{dz} = -e^{-z}$$

$$O = x^{-1}, \text{ so } \frac{dO}{dx} = -x^{-2} = -\frac{1}{\left(1+e^{-z}\right)^2}$$

$$\text{So } \frac{dO}{dz} = \frac{dO}{dx}\frac{dx}{dz} = -\frac{-e^{-z}}{\left(1+e^{-z}\right)^2} = \frac{e^{-z}}{\left(1+e^{-z}\right)^2}$$

Another Example

The gain of a system, G, as a function of angular frequency ω, is

$$G = \frac{1}{\sqrt{(1 - 0.1\omega^2)^2 + 0.04\omega^2}}$$

Find the three values of ω such that $\frac{dG}{d\omega} = 0$

Let $G = \frac{1}{\sqrt{z}}$ so $z = (1 - 0.1\omega^2)^2 + 0.04\omega^2$

Simplify : $z = 1 - 0.2\omega^2 + 0.01\omega^4 + 0.04\omega^2$

$= 1 - 0.16\omega^2 + 0.01\omega^4$

$$\frac{dz}{d\omega} = -0.32\omega + 0.04\omega^3$$

Continued

$z = 1 - 0.16\omega^2 + 0.01\omega^4$ $\qquad \frac{dz}{d\omega} = -0.32\omega + 0.04\omega^3$

$G = z^{-1/2}$ so $\frac{dG}{dz} = -\frac{1}{2}z^{-3/2}$ $\qquad \frac{dG}{d\omega} = \frac{dG}{dz}\frac{dz}{d\omega}$

$$\frac{dG}{d\omega} = -\frac{1}{2}z^{-3/2}\left(-0.32\omega + 0.04\omega^3\right) = \frac{-0.32\omega + 0.04\omega^3}{-2z^{3/2}}$$

Want ω where this is zero – need numerator only

i.e. $-0.32\omega + 0.04\omega^3 = 0$

or $\omega(-0.32 + 0.04\omega^2) = 0$

or $\omega = 0$ or $0.32 = 0.04\omega^2$ i.e. $\omega^2 = 8$

So required result is $\omega = 0$ or $\omega = \pm\sqrt{8}$ rad/s

A Related Technique

Find gradient at point x,y on circle defined by $y^2 = r^2 - x^2$

Want dy/dx; could take square root + carry on as normal.

Easier way : differentiate both sides with respect to x.

RHS: $\frac{d(r^2-x^2)}{dx} = 0 - 2x = -2x$

LHS: $\frac{d(y^2)}{dx} = \frac{d(y^2)}{dy}\frac{dy}{dx} = 2y\frac{dy}{dx}$

So $\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{\sqrt{r^2 - x^2}}$

Note need to have result with x not y

Exercise

You are to find how the volume of a sphere ($V = 4/3\ \pi\ r^3$) changes with time if r is $5 - 5e^{-t/20}$.

Find $\frac{dV}{dr}$, $\frac{dr}{dt}$ and hence $\frac{dV}{dt}$

Answer

$$\frac{dV}{dr} = \frac{4}{3}3\pi r^2 = 4\pi r^2$$

$$\frac{dr}{dt} = \frac{-5}{-20}e^{-t/20} = \frac{1}{4}e^{-t/20}$$

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2\frac{1}{4}e^{-t/20} = \pi\left(5 - 5e^{-t/20}\right)^2 e^{-t/20}$$

Integration Using Substitution

This is related to the chain rule used in differentiation.

Let's apply to $\int \sin(\omega t)dt$

Simplify by substituting x = ω t, and then integrate

$\int \sin(\omega t)\, dt = \int \sin(x)\, dx$

But must integrate f(x) wrt x

As $x = \omega t$, $\frac{dx}{dt} = \omega$, so $dt = \frac{1}{\omega}dx$

Thus $\int \sin(\omega t)dt = \int \sin(x)\frac{1}{\omega}dx = -\cos(x) * \frac{1}{\omega} + c$

But we introduced x to help us, answer should have t

$$\int \sin(\omega t)dt = -\frac{1}{\omega}\cos(\omega t) + c$$

Application - Root Mean Square

Mean of one cycle of $\sin(\omega t)$

$$\frac{1}{\frac{2\pi}{\omega} - 0}\int_{t=0}^{t=\frac{2\pi}{\omega}} \sin(\omega t)\, dt = \frac{\omega}{2\pi}\left[-\frac{1}{\omega}\cos(\omega t)\right]_0^{\frac{2\pi}{\omega}} = -\frac{1}{2\pi}(1 - 1) = 0$$

Not useful measure, so often use root mean square or rms value of a signal.

$$\text{rms} = \sqrt{\frac{\int_a^b f^2(t)\, dt}{b-a}}$$

$$\text{rms}(\sin(\omega t)) = \sqrt{\frac{1}{2\pi/\omega}\int_0^{2\pi/\omega} \sin^2(\omega t)\, dt}$$

Looks awkward : use trig identity $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

RMS of Sinusoid

$$\text{rms} = \sqrt{\frac{1}{2\pi/\omega}\int_0^{2\pi/\omega} \sin^2(\omega t)\,dt} \qquad \sin^2(x) = \tfrac{1}{2}\,(1 - \cos(2x))$$

$$\text{rms} = \sqrt{\frac{\omega}{4\pi}\int_0^{2\pi/\omega} 1\text{-}\cos(2\omega t)\,dt}$$

$$= \sqrt{\frac{\omega}{4\pi}\Big[\, t - \tfrac{1}{2\omega}\sin(2\omega t)\,\Big]_0^{2\pi/\omega}}$$

$$= \sqrt{\frac{\omega}{4\pi}\left(\frac{2\pi}{\omega} - \frac{1}{2\omega}\sin(4\pi)\ \text{-}0 - -\frac{1}{2\omega}\sin(0)\right)} = \sqrt{\frac{1}{2}}$$

Some more examples of substitution

$\int \sin(\omega t + \phi)\,dt$: Let $x = \omega t + \phi$; $dx/dt = \omega$

$$\int \sin(\omega t + \phi)dt = \int \sin(x)\tfrac{1}{\omega}dx \quad = \frac{-\cos(x)}{\omega} + c = \frac{-\cos(\omega t + \phi)}{\omega} + c$$

$\int \frac{1}{1-t}dt$; $x = 1 - t$; $dx = -dt$;

$$\text{so} \int = \int -\tfrac{1}{x}dx = -\ln(x) + c = -\ln(1 - t) + c$$

Definite Integrals and Substitution

Beware of limits : remember

$\int_1^5 f(t)\,dt$ means integrate f(t) and evaluate t = 1 to t = 5

Can use substitution but must change limits accordingly

$\int_0^2 (5t+7)^3 dt$ we use x = 5t+7, so $\frac{dx}{dt} = 5$ or $\frac{dx}{5} = dt$

limits x = 5*2+7=17 and 5*0+ 7=7

$$\int_0^2 (5t+7)^3 dt = \int_7^{17} x^3 \frac{dx}{5} = \left[\ \frac{1}{20}\, x^4 \right]_7^{17} = \frac{1}{20}\left(17^4 - 7^4\right) =$$

Integration by Partial Fractions

A technique to simplify integrals to ones we can solve.

The velocity v of a falling object at time t found using

$t = \int \frac{1}{4 - v^2}\,dv$. Find t as function of v, then v(t).

$4 - v^2$ factorises as (2 – v) and (2 + v)

We solve the above by reorganising as follows

$t = \int \frac{A}{2 - v} + \frac{B}{2 + v}\,dv$ A and B constants

Hence t = -Aln(2 - v) + Bln(2 + v) + c

What are A and B?

Finding A and B and then t

$$\frac{1}{4 - v^2} = \frac{A}{2 - v} + \frac{B}{2 + v}$$

$$\frac{1}{4 - v^2} = \frac{A(2 + v)}{(2 - v)(2 + v)} + \frac{B(2 - v)}{(2 - v)(2 + v)} = \frac{A(2 + v) + B(2 - v)}{4 - v^2}$$

Numerators must be equal

$1 = A(2 + v) + B(2 - v) = 2(A + B) + v(A - B)$

Equating coefficients:

$1 = 2(A + B)$ $0 = (A - B)$ so A = B and 1 = 4A

$$t = \int \frac{0.25}{2 - v} + \frac{0.25}{2 + v}\,dv = -0.25\ln(2 - v) + 0.25\ln(2 + v) + c$$

Completing Problem

$$t = -0.25\ln(2 - v) + 0.25\ln(2 + v) + c = 0.25\ln\left(\frac{2 + v}{2 - v}\right) + c$$

Ok, but want v(t) not t(v)

$$t - c = 0.25\ln\left(\frac{2 + v}{2 - v}\right) \text{ or } 4(t - c) = \ln\left(\frac{2 + v}{2 - v}\right)$$

Take exp of both sides $e^{4(t - c)} = \frac{2 + v}{2 - v}$

Gather v terms $(2 - v)e^{4(t - c)} = 2 + v$

$$2e^{4(t - c)} - 2 = v(1 + e^{4(t - c)})$$

$$v = \frac{2e^{4(t - c)} - 2}{1 + e^{4(t - c)}}$$

Summary

Today we introduced the Chain rule for differentiation and some simple Integration by substitution.

The Chain Rule allows us to show

$$\frac{d\sin(5t-7)}{dt} = \frac{1}{5}\cos(5t-7) + c$$

Substitution allows us to show that

$$\int \frac{1}{9-4x}dx = -\frac{1}{4}\ln(9-4x) + c$$

There are many more ways of substituting – some relevant to Engineering Maths are given next week

Tutorial – Week 7 – Q1, 2 and 3

7.1 The current i through a diode in terms of voltage V across it is

$i = 0.05(e^{200V} - 1)$

Find the a.c. resistance r which is defined as $r = 1 \Big/ \frac{di}{dV}$.

7.2 In reliability engineering, the distribution function, F,

for a set of components is given by $F = 5 - \frac{30}{6 + 0.25\,t}$

Find the density function = $\frac{dF}{dt}$.

7.3. The gain of a mass-spring, as a function of ω, is

$G = \frac{1}{\sqrt{\omega^4 - 9\omega^2 + 25}}$. Find values of ω where $\frac{dG}{d\omega} = 0$.

Calculate G at all these values to find where G max.

Tutorial – Week 7 – Q4, 5 and 6

7.4 The position, s, of an object moving in a straight line is

$s = \int \frac{8}{\sqrt{40t+1}}\,dt.$

Find s given that s = 2 when t = 0.

7.5 The sinusoidal output of a system is given by

$O = 5\sin(4t - 0.2)$

Find its mean value between t = 0 and t = 0.5.

7.6. Find its rms value between t = 0 and t = 0.5.

Tutorial - Week 7 - Hints

7.1 Use Chain rule to find di/dV…

7.2 Again use the chain rule

7.3 Again the chain rule – look for where numerator is 0 – remember can have negative frequencies.

7.4 Use substitution

7.5 and 6

Use same substitution for both mean and rms – remember to change limits – and to use radians.

Differentiation and Integration 8
Prof Richard Mitchell

Today : Volumes/Surfaces; More Substitution.

Some of this can be found in the recommended books

Croft 827-832, 851-890; James 559-568

Stroud 829-833,863-871,922-939; Singh 371-384,410-424

Don't forget to attend the tutorials to get practice

Also, extra support is available from

http://www.reading.ac.uk/mathssupport centre

and http://www.mathtutor.ac.uk

Integration – More Substitution

We have looked at using simple substitution for indefinite and definite integrals

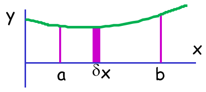
Revision $\frac{1}{\pi}\int_0^{\pi}\cos\frac{t}{2}\,dt = \frac{1}{\pi}\left[2\sin\frac{t}{2}\right]_0^{\pi} = \frac{2}{\pi}$

Now we shall look at more techniques and applications related to using substitution in integration

First we will see some geometric applications, which will provide some useful examples later

Starting with volumes of revolution which are extensions of areas under curves …

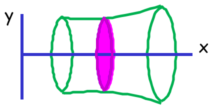
Volumes of Solids of Revolution



y = f(x): $\int f(x)\,dx$ = area under curve:

sum areas of strips: $\sum_{x=a}^{b} f(x) * \delta x$

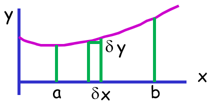
As $\delta x \to 0$, becomes $\int_a^b f(x)\,dx$

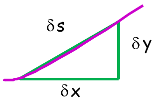


If f rotated around x axis have solid of revolution: volume found by summing volumes of cylinders, radius y & width δx

$$V = \sum_{x=a}^{b} \pi y^2 \delta x \qquad \text{As } \delta x \to 0 : V = \int_a^b \pi y^2\,dx = \int_a^b \pi (f(x))^2\,dx$$

Lengths of Curves and Surface Areas





Length of part, assume straight line, so $\delta s^2 = \delta x^2 + \delta y^2$

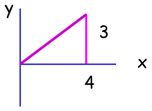
$$\text{Div by} (\delta x)^2 : \left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2 \text{ as } \delta x \to 0 \; \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

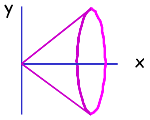
Hence length from a to b is: $s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

Surface area, when curve rotated, found similarly as $S = \int_a^b 2\pi\, y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

Example – Line Rotated to Cone

$y = \frac{3}{4}x = f(x)$





$$V = \int_0^4 \pi \left(\frac{3}{4}x \right)^2 dx = \left[\pi \frac{9}{16} \frac{x^3}{3} \right]_0^4 = \pi \frac{9}{16} \frac{4^3}{3} = 12\pi$$

$$\frac{dy}{dx} = \frac{3}{4}; \quad s = \int_0^4 \sqrt{1 + \left(\frac{3}{4}\right)^2} dx = \left[\sqrt{\frac{25}{16}} x \right]_0^4 = \frac{5}{4} 4 = 5$$

$$S = \int_0^4 2\pi \frac{3}{4} x \sqrt{1 + \frac{9}{16}} dx = \left[2\pi \frac{3}{4} \frac{x^2}{2} \frac{5}{4} \right]_0^4 = 15\pi$$

Integration of function * derivative

Find surface area when y = x^3 rotated about axis x = 0..0.5

$$\frac{dy}{dx} = 3x^2; \qquad S = \int_0^{0.5} 2\pi x^3 \sqrt{1 + 9x^4}\,dx;$$

Here the 'main' function is $\sqrt{1 + 9x^4}$, a function of x^4, multiplied by $k*x^3$ being a multiple of the derivative of x^4

The technique is to make the substitution $t = x^4$

$$\text{So } \frac{dt}{dx} = 4x^3 \text{ or } dx = \frac{dt}{4x^3} \qquad \text{so } S = \int 2\pi\, x^3 \sqrt{1 + 9t} \frac{dt}{4x^3}$$

Note x^3 terms (the main function's derivative) cancel

Completing the Example

Want $S = \int_0^{0.5} 2\pi x^3 \sqrt{1 + 9x^4}\,dx;$

Let $t = x^4$, $\frac{dt}{dx} = 4x^3$; For x = 0 to 0.5, t = 0 to $\frac{1}{16}$

$$S = \int_0^{\frac{1}{16}} 2\pi x^3 \sqrt{1 + 9t} \frac{dt}{4x^3} \qquad = \frac{\pi}{2} \int_0^{\frac{1}{16}} \sqrt{1 + 9t}\,dt \qquad = \left[\frac{\pi}{18} \frac{(1 + 9t)^{3/2}}{3/2} \right]_0^{\frac{1}{16}}$$

$$= \frac{\pi \left(1 + 9/16\right)^{3/2}}{27} - \frac{\pi}{27} \qquad = \frac{\pi}{27} \left(\left(25/16\right)^{3/2} - 1 \right) \qquad = \frac{\pi}{27} \frac{5^3 - 4^3}{4^3} = \frac{61\pi}{1728}$$

NB could have changed to f(x) & used x limits: above easier

Integrals of f'(t) / f(t)

Note, the same substitution also works when integrating the differential of a function DIVIDED by the function

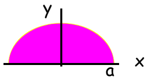
$$\text{eg } \int \tan(\omega t)\,dt = \int \frac{\sin(\omega t)}{\cos(\omega t)}\,dt$$

$$\text{Let } z = \cos(\omega t); \quad \frac{dz}{dt} = -\omega \sin(\omega t)$$

$$\int \tan(\omega t)\,dt = \int \frac{\sin(\omega t)}{-\omega \sin(\omega t) z}\,dz = \int \frac{1}{-\omega\, z}\,dz$$

$$= \frac{1}{-\omega} \ln(z) + c = -\frac{1}{\omega} \ln\left(\cos(\omega t)\right) + c$$

Exercise



Find the centre of mass, C, of the semi-circular lamina radius a :

$C = \int_0^a 2my\sqrt{a^2 - y^2}dy$

m is mass per unit area

Use $t = a^2 - y^2$

$\frac{dt}{dy} = -2y$ or $dy = \frac{dt}{-2y}$ When y = 0, $t = a^2$; when y = a, t = 0

C =

Substitution Using Trigonometry

Most Maths books cover integrals like $\int \frac{1}{\sqrt{(a^2 - t^2)}}dt$

Which does not have Engineering application

Here the √ is awkward and we substitute to remove it.

$a^2 \cos^2 x = a^2 - a^2 \sin^2 x$. Let t = a sin(x), so dt = a cos(x) dx

$$\int \frac{1}{\sqrt{a^2 - a^2\sin^2(x)}} a\cos(x)dx = \int \frac{1}{a\cos(x)} a\cos(x)dx$$

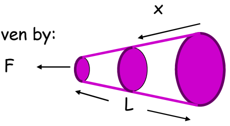
$$= \int dx = x + c = \sin^{-1}\left(\frac{t}{a}\right) + c$$

Some Useful Substitutions (others may work)

If an integral has	√(a²-t²)	a²+t²	√(t²-a²)	√(a²+t²)
Try using t =	a sin(x)	a tan (x)	a sec(x)	a sinh(x)

Example

Force F applied to bar, extension l given by:



$l = \int_0^L \frac{F}{4 + x^2} dx$

Let x = 2 tan(a) so $\frac{dx}{da} = 2\sec^2(a)$

$$l = \int_{x=0}^{x=L} \frac{F}{4 + 4\tan^2(a)} 2\sec^2(a)\,da = \int_{x=0}^{x=L} \frac{F}{4\sec^2(a)} 2\sec^2(a)\,da$$

$$l = \left[\frac{F}{2}a\right]_{x=0}^{x=L} = \left[\frac{F}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^L = \frac{F}{2}\tan^{-1}\left(\frac{L}{2}\right)$$

Integration – solving Diff Eqns

This is one application of integration – sets up an example

eg object, position x, accelerating from initial velocity

$$\frac{dx}{dt} = 5 + 3t$$

As RHS is function of t, can integrate both sides wrt t

$$\int \frac{dx}{dt} dt = \int 5 + 3t\,dt$$

Hence $\int dx = x = \int 5 + 3t\,dt = 5t + \frac{3}{2}t^2 + c$

Sometimes we need to invert the diff equation…

Application – solving Diff Eqn

The differential equation for the RC circuit: $\frac{dV}{dt} = \frac{E - V}{RC}$

But RHS is function of V not t:

So we invert the equation and get $\frac{dt}{dV} = \frac{RC}{E - V}$

Integrating both sides wrt V $\int \frac{dt}{dV} dV = \int \frac{RC}{E - V} dV$

Hence $\int dt = t = \int \frac{RC}{E - V} dV$

Recalling $\int \frac{1}{1 - t} dt = -\ln(1 - t)$ So $t = -RC\ln(E-V) + c$

Continued :

But want V as function of t, so rearrange So $t = -RC\ln(E-V) + c$

$\frac{t - c}{-RC} = \ln(E - V)$ Raise both sides to power of e

so $e^{\frac{t-c}{-RC}} = e^{\ln(E - V)} = E - V$

But $e^{\frac{t-c}{-RC}} = e^{-\frac{t}{RC}} * e^{-\frac{c}{RC}}$ ← $e^{\frac{c}{RC}}$ is constant, call it k

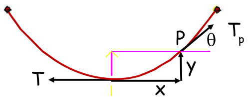
so $k e^{-\frac{t}{RC}} = E - V$ or $V = E - ke^{-\frac{t}{RC}}$

If V = 0 at t = 0, $0 = E - ke^0$; k = E. Thus $V = E - Ee^{-\frac{t}{RC}}$

Catenary: Cable Hanging Own Weight

You are not expected to remember this, just sit back and appreciate the application of various techniques.

We will use geometric techniques and others to deduce differential equations for this & then solve by integration.



Horizontal tension = T

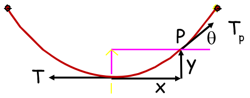
Weight of cable per unit length = w

Consider what happens at P

At P, Horizontal position x, Vertical position y;
x = 0 when y = 0

At P tension in cable is T_p, at angle θ

Catenary Continued



Length of cable 0,0 to P :

$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

At P force down = w * s

If cable not moving this force equals vertical part of T_p

So $w * s = T_p \sin(\theta)$

At P also, horiz component is $T_p \cos(\theta)$ which must equal T

Thus $\tan(\theta) = \dfrac{T_p\sin(\theta)}{T_p\cos(\theta)} = \dfrac{w*s}{T}$ But $\dfrac{dy}{dx}$ at P is slope = $\tan(\theta)$

$$\frac{dy}{dx} = \frac{w*s}{T} = \frac{w}{T} * \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{or} \quad \frac{d^2y}{dx^2} = \frac{w}{T}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Catenary Continued

$\dfrac{d^2y}{dx^2} = \dfrac{w}{T}\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$ uggh! But let $z = \dfrac{dy}{dx}$ so $\dfrac{dz}{dx} = \dfrac{w}{T}\sqrt{1+z^2}$

We can solve by $\dfrac{dx}{dz} = \dfrac{T}{w}\dfrac{1}{\sqrt{1+z^2}}$ or $x = \dfrac{T}{w}\displaystyle\int \dfrac{1}{\sqrt{1+z^2}}\,dz$

Let $z = \sinh(t)$; $\dfrac{dz}{dt} = \cosh(t)$, so $x = \dfrac{T}{w}\displaystyle\int \dfrac{1}{\sqrt{1+\sinh^2(t)}}\cosh(t)dt$

$$x = \frac{T}{w}\int \frac{1}{\cosh(t)}\cosh(t)dt = \frac{T}{w}t + c = \frac{T}{w}\sinh^{-1}(z) + c$$

But at x = 0, $z = \dfrac{dy}{dx} = 0$, so c = 0, hence $x = \dfrac{T}{w}\sinh^{-1}\left(\dfrac{dy}{dx}\right)$

Catenary Concluded

Now $x = \dfrac{T}{w}\sinh^{-1}\left(\dfrac{dy}{dx}\right)$, so $\dfrac{dy}{dx} = \sinh\left(\dfrac{wx}{T}\right)$

$$y = \int \sinh\left(\frac{wx}{T}\right)dx = \frac{T}{w}\cosh\left(\frac{wx}{T}\right) + c$$

But y = 0 when x = 0

$$0 = \frac{T}{w}\cosh\left(\frac{w*0}{T}\right) + c = \frac{T}{w} + c; \qquad c = -\frac{T}{w}$$

$$y = \frac{T}{w}\left(\cosh\left(\frac{wx}{T}\right) - 1\right)$$

Next week: differentiation and integration of a product

Tutorial – Week 8 – Q1, 2 and 3

8.1 a) Find the volume of revolution of the solid formed when $y^2 = 4ax$ is rotated about the x-axis from x = 0 to1.

b) A parabolic reflector is formed by rotating $y = 2\sqrt{x}$ about the x-axis, from x = 0 to 1. Find its surface area.

8.2 An aerofoil is defined by $y^2 = 1 - \dfrac{x^2}{4}$ for $-2 \le x \le 2$

Find the area below this function and above the x-axis.

8.3 The velocity v of a falling object is given by $\dfrac{dv}{dt} = g - 0.5v$

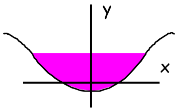
Show that $v = 2g(1-e^{-0.5t})$ if v = 0 at t = 0

Tutorial – Week 8 – Q4 and 5

8.4 The velocity v of a falling object is given by $v = 2g(1-e^{-0.5t})$. Find the distance dropped at time t.

8.5 The figure shows water in a prismatic channel, for which the water height y is defined in terms of horizontal position x by the equation

$$\frac{dy}{dx} = \frac{1}{\sqrt{16x^2 - 1}}$$



Find an expression for y if y = 0 when x = 0.25:

Tutorial – Week 8 – Q6 Extra

8.6 The logistic equation for population P is $\frac{dP}{dt} = P(10 - P)$

a) Show that $t = \int \frac{0.1}{P} + \frac{0.1}{10-P}\, dP$

b) Find P(t) if P = 20 at time t = 0 in the form $P = \frac{A}{1-Be^{-ct}}$

Tutorial - Week 8 - Hints

8.1 Use relevant formulae and integrate.
8.2 Is integral of a squareroot – use trig substitution.
8.3 Invert eqn: find t(v) -> v(t). 8.4 Integrate.
8.5 Use relevant hyperbolic substitution.
8.6 Invert eqn, use Partial Fractions and proceed

Differentiation and Integration 9
Prof Richard Mitchell

Today : Diff and Int of Products.
Some of this can be found in the recommended books
Croft 719-724, 815-821 ; James 496-504, 551-553
Stroud 379-383, 834-836; Singh 280-285, 388-396;
Don't forget to attend the tutorials to get practice
Also, extra support is available from
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and http://www.mathtutor.ac.uk

Differentiation of a Product

We can differentiate the sum of two (or more) terms; and we know the chain rule for more complicated terms. Next we will consider the product of two functions.

If the two functions are u and v, then we use the rule

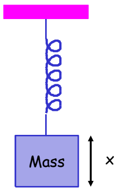
$$\frac{d(uv)}{dt} = u\frac{dv}{dt} + v\frac{du}{dt}$$

e.g. Find velocity of an object with position $x = 5 - 3te^{-2t}$.

For $3te^{-2t}$: $u = 3t$ so $\frac{du}{dt} = 3$; $v = e^{-2t}$ so $\frac{dv}{dt} = -2e^{-2t}$

Thus velocity is $0 - (3t * -2e^{-2t} + e^{-2t} * 3) = (6t - 3)e^{-2t}$

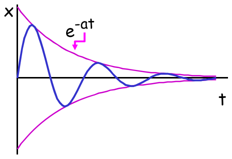
Differentiating Damped Sinusoid



If drop mass, it oscillates but oscillations die out

$x = e^{-at} \sin(bt)$

How does x change?



$u = e^{-at}$; so $\frac{du}{dt} = -a\,e^{-at}$ $v = \sin(bt)$; so $\frac{dv}{dt} = b\cos(bt)$

$$\frac{d(e^{-at}\ \sin(bt))}{dt} = e^{-at}\, b\cos(bt) + \sin(bt)\,(-a\,e^{-at})$$
$$= e^{-at}\,(b\cos(bt) - a\sin(bt))$$

Exercise

If there is considerable friction, the mass does not oscillate

Then its position varies by $x = e^{-at}\, t$.

Find $\frac{dx}{dt}$

Answer

Differentiation of a Quotient

Consider a machine with gears, used to drive its output, defined in terms of n, the ratio of one set to another.

Here the machine acceleration is

$$a = \frac{n^2 + 4}{8 - n}$$

Suppose we want to find how a changes with n: da/dn.
Here we use the quotient rule, again in terms of u and v

$$\frac{d\left(\frac{u}{v}\right)}{dt} = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$$

(This can be derived from that of $u * v^{-1}$)

Using the Quotient Rule

$$a = \frac{n^2 + 4}{8 - n} \qquad \frac{d\left(\frac{u}{v}\right)}{dt} = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$$

$$u = n^2 + 4; \text{ so } \frac{du}{dn} = 2n \qquad v = 8 - n; \text{ so } \frac{dv}{dn} = -1$$

$$\text{Thus } \frac{d\left(\frac{n^2+4}{8-n}\right)}{dn} = \frac{(8 - n)2n - (n^2 + 4)(-1)}{(8 - n)^2}$$

$$= \frac{16n - 2n^2 + n^2 + 4}{(8 - n)^2} = \frac{4 + 16n - n^2}{(8 - n)^2}$$

Quotient Rule for tan(ωt)

We know $\frac{d(\sin(\omega t))}{dt} = \omega\cos(\omega t)$ & so $\frac{d(\cos(\omega t))}{dt} = -\omega\sin(\omega t)$;

what of $\frac{d(\tan(\omega t))}{dt}$? $\qquad \tan(\omega t) = \frac{\sin(\omega t)}{\cos(\omega t)}$

Let $u = \sin(\omega t)$; $\quad v = \cos(\omega t)$

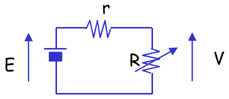
$$\text{So } \frac{d(\tan(\omega t))}{dt} = \frac{\cos(\omega t)*(\omega\cos(\omega t)) - \sin(\omega t)*(-\omega\sin(\omega t))}{\cos^2(\omega t)}$$

$$= \frac{\omega\left(\cos^2(\omega t) + \sin^2(\omega t)\right)}{\cos^2(\omega t)} = \omega\sec^2(\omega t)$$

NB sec(x)=1/cos(x)

Application : Min/Max - Optimisation

A battery, internal resistance r,
with variable load R.
Find R to maximise power.



$$P = \frac{V^2}{R} = \frac{E^2R}{(r + R)^2}$$ Find dP/dR and then R so dP/dR = 0

$$u = E^2R \text{ so } \frac{du}{dR} = E^2; \qquad v = (r + R)^2 \text{ so } \frac{dv}{dR} = 2(r + R)$$

$$\frac{dP}{dR} = \frac{(r + R)^2E^2 - E^2R*2(r + R)}{(r + R)^4}$$

$$= \frac{E^2(r + R)(r + R - 2R)}{(r + R)^4} = \frac{E^2(r - R)}{(r + R)^3}$$

Clearly, this is zero when R = r.

Integration By Parts

This is method for integrating product of two terms.

Is extension of method for differentiating products

$$\frac{d(uv)}{dt} = u\frac{dv}{dt} + v\frac{du}{dt}$$

Integrating both sides gives

$$\int\frac{d(uv)}{dt}dt = uv = \int u\frac{dv}{dt}dt + \int v\frac{du}{dt}dt$$

Which rearranged gives

$$\int u\frac{dv}{dt}dt = uv - \int v\frac{du}{dt}dt$$

Key to Using Integration By Parts

$$\int u\frac{dv}{dt}dt = uv - \int v\frac{du}{dt}dt$$

For integrating product of two functions, u and dv/dt.

The key is to select which function is u and which dv/dt.

The aim is to choose these suitably so that the term

$$\int v\frac{du}{dt}dt \quad \text{is easier to solve than} \quad \int u\frac{dv}{dt}dt$$

Definite version of integral: $\int_q^p u\frac{dv}{dt}dt = [uv]_q^p - \int_q^p v\frac{du}{dt}dt$

Showing Effect of Wrong Choice

Consider $\int t\cos(t)\,dt \qquad \int u\frac{dv}{dt}dt = uv - \int v\frac{du}{dt}dt$

If let $u = \cos(t)$ & $\frac{dv}{dt} = t$

$$\frac{du}{dt} = -\sin(t) \quad \text{and} \quad v = \int t\,dt = \frac{1}{2}t^2$$

So integral is $\int t\cos(t)\,dt = -\cos(t)\frac{t^2}{2} + \int\frac{t^2}{2}\sin(t)\,dt$

This has made matters harder.

But with right choice

Again $\int t\cos(t)\,dt$ $\qquad \int u\frac{dv}{dt}dt = uv - \int v\frac{du}{dt}\,dt$

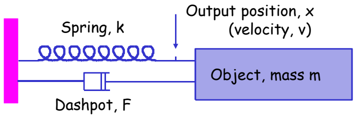
But if u = t and $\frac{dv}{dt} = \cos(t)$

$\frac{du}{dt} = 1$ and $v = \int\cos(t)\,dt = \sin(t)$

$$\text{So } \int t\cos(t)\,dt = t*\sin(t) - \int 1*\sin(t)\,dt$$
$$= t\sin(t) + \cos(t) + c$$

Shows why important to make right choice !

Example – Mass Spring System



Typical in control
Let m = 1kg,
$F = 4\text{Nsm}^{-1}$,
$k = 4\text{Nm}^{-1}$

Mass pulled, Spring extended, forced back , opposed by friction.

Then $\frac{dv}{dt} = -4*x - 4*v$

Suppose $v = 2t\,e^{-2t} - 3e^{-2t}$. Find x if, at time t = 0, x = 1m

Second term easy, Use integration by parts for first:

Integrating 2t e^{-2t}

$\int u\frac{dv}{dt}dt = uv - \int v\frac{du}{dt}\,dt$

So to integrate 2t e^{-2t} we choose u and v terms sensibly

Let u = 2t and $\frac{dv}{dt} = e^{-2t}$

Then $\frac{du}{dt} = 2$ and $v = \int e^{-2t}\,dt = -\frac{1}{2}e^{-2t}$

$$\text{So } \int 2t\,e^{-2t}dt = 2t\left(-\frac{1}{2}e^{-2t}\right) - \int -\frac{1}{2}e^{-2t}2dt$$
$$= -te^{-2t} + \frac{1}{-2}e^{-2t}$$
$$= -te^{-2t} - \frac{1}{2}e^{-2t}$$

Completing Problem

$v = 2t\,e^{-2t} - 3\,e^{-2t}$

We now know $\int 2t\,e^{-2t}dt = -t\,e^{-2t} - \frac{1}{2}e^{-2t}$

Also $\int 3\,e^{-2t}dt = -\frac{3}{2}e^{-2t}$

$$\text{So } x = \int v\,dt = -t\,e^{-2t} - \frac{1}{2}e^{-2t} + \frac{3}{2}e^{-2t} + c = e^{-2t} - t\,e^{-2t} + c$$

At t = 0, x = 1; so 1 = 1 - 0 + c; c = 0; $x = e^{-2t} - t\,e^{-2t}$

Extension: verify that x and v are solutions of

$\frac{dv}{dt} = -4*x - 4*v$ (would need to differentiate v to solve this problem)

Application – Solving Diff Eqn

Suppose we want to solve $\frac{dV}{dt} + 5V = t$

If multiply by e^{5t}, get $\quad e^{5t}\frac{dV}{dt} + 5Ve^{5t} = te^{5t}$

But $\frac{d(Ve^{5t})}{dt} = \frac{dV}{dt}e^{5t} + V5e^{5t}$

So $\frac{d(Ve^{5t})}{dt} = te^{5t}$ or $Ve^{5t} = \int te^{5t}\,dt$

We use integration by parts to evaluate $\int te^{5t}\,dt$

And then divide by e^{5t} to find V:

Completing Problem

$Ve^{5t} = \int te^{5t}\,dt$

For $\int te^{5t}\,dt$: Let u = t so $\frac{du}{dt} = 1$; $\frac{dv}{dt} = e^{5t}$ so $v = 0.2e^{5t}$

So $\int te^{5t}dt = t\,0.2e^{5t} - \int 0.2e^{5t}\,dt = t\,0.2e^{5t} - 0.04\,e^{5t} + c$

So $Ve^{5t} = t\,0.2e^{5t} - 0.04\,e^{5t} + c$

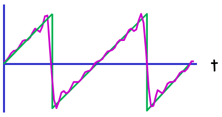
Thus $V = 0.2\,t - 0.04 + c\,e^{-5t}$

Suppose V = 0 at t = 0:

At t = 0: 0 = 0 - 0.04 + c; so c = 0.04

Hence $V = 0.2t - 0.04 + 0.04e^{-5t}$

Using Definite Version of Integral



We quoted earlier that saw(t) ~

$$\sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \frac{\sin(4t)}{4}$$

Means $a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\left(\frac{n2\pi t}{T}\right)dt$ should be 0

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\left(\frac{n\,2\pi t}{T}\right)dt$$

$$= -\frac{\cos(n\pi)}{n} = \begin{cases} +1/n & \text{if n is odd} \\ -1/n & \text{if n is even} \end{cases}$$

Let's do b_n

Finding b_n for Sawtooth

$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} t\sin(2nt)\ dt$ { remember n is constant }

Let u = t, So $\frac{du}{dt} = 1$;

and $\frac{dv}{dt} = \sin(2nt)$ and $v = -\frac{1}{2n}\cos(2nt)$

$$\text{So } b_n = \frac{2}{\pi}\left(\left[-\frac{t}{2n}\cos(2nt)\right]_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} -\frac{1}{2n}\cos(2nt)\ dt\right)$$

$$= \frac{2}{\pi}\left(-\frac{1}{2n}\left[t\ \cos(2nt)\right]_{-\pi/2}^{\pi/2} + \frac{1}{4n^2}\left[\sin(2nt)\right]_{-\pi/2}^{\pi/2}\right)$$

Example Continued

$$b_n = \frac{2}{\pi}\left(-\frac{1}{2n}\left[t\ \cos(2nt)\right]_{-\pi/2}^{\pi/2} + \frac{1}{4n^2}\left[\sin(2nt)\right]_{-\pi/2}^{\pi/2}\right)$$

But, cos(x) = cos(-x);

So $\left[t\ \cos(2nt)\right]_{-\pi/2}^{\pi/2}$ is $\frac{\pi}{2}\cos(n\pi) - -\frac{\pi}{2}\cos(-n\pi) = \pi\cos(n\pi)$

Also sin(x) = –sin(-x); sin(nπ) = 0 (n integer)

So $\left[\sin(2nt)\right]_{-\pi/2}^{\pi/2}$ is $\sin(n\pi) - \sin(-n\pi) = 2\pi\sin(n\pi) = 0$

So $b_n = \frac{2}{\pi}*\left(-\frac{1}{2n}*\pi*\cos(n\pi)\right) = -\frac{\cos(n\pi)}{n}$ (as stated)

You find a_n term in tutorial!

Summary

Here we have looked at the differential and integral of product (or quotient terms)

$$\frac{d(uv)}{dt} = u\frac{dv}{dt} + v\frac{du}{dt} \qquad \frac{d\left(\frac{u}{v}\right)}{dt} = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$$

$$\int u\frac{dv}{dt}dt = uv - \int v\frac{du}{dt}\ dt$$

The key to integration is to choose u so second integral is easier.

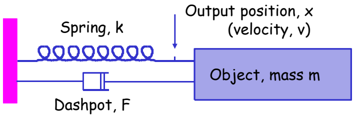
For $\int e^{2t}\,5t\ dt$, what is u and what is v?

Tutorial – Week 9 – Q1

9.1 In the mass-spring system below, the position of the mass is given by $x = 2 + e^{-2t}(3\sin(t) + 2\cos(t))$

a) Show that at time t = 0, the velocity v is -1 m/s.

b) Find $\frac{d^2x}{dt^2}$ and hence show that $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 10$



m = 1 kg;
F = 4 Ns/m;
k = 5 N/m

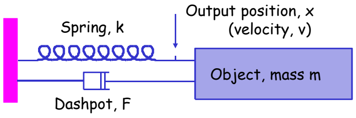
Tutorial – Week 9 – Q2

9.2 For the mass-spring system, v, the mass' velocity, is given by $v = e^{-4t} - 20\,t\,e^{-4t}$

a) Find $\int 20te^{-4t}\ dt$

b) Hence find x, the position of the mass, if at t = 0, x = 3m.

c) Find $\frac{d^2x}{dt^2}$ and so show that $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 32$



m = 1 kg;
F = 8 Ns/m;
k = 16 N/m

Tutorial – Week 9 – Q3

9.3 Differentiate the following using the quotient rule

a) $f(x) = \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ [used in information theory]

b) The displacement of a damped vertical pendulum

$$x = \frac{\cos(5t)}{\exp(2t)}$$

c) The voltage in an electronic circuit

$$V = \frac{t+2}{\exp(0.1t)}$$

Tutorial – Week 9 – Q4,5 and 6

9.4 The gain of an electronic circuit, in terms of angular frequency ω is given by $G = \frac{\sqrt{1+\omega^2}}{3+\omega^2}$

Find ω such that G is maximised.

9.5 Find the a_n term for the Fourier Series of a sawtooth.

ie. find $a_n = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} t \cos(2nt)\, dt$ { recall n is constant }

9.6. Find area A under of Kte^{-at} for t>0 : K and a constant

i.e. solve $A = \int_0^{\infty} K\, t\, e^{-at} dt$; Nb as $t \to \infty$, so $t\, e^{-at} \to 0$

Extra!

Tutorial – Week 9 – Hints

9.1 a) Find dx/dt, and put t = 0. Use diff of product

b) Use diff of product and then show LHS = RHS

9.2 a) Choose u and dv/dt as per examples

b) Straightforward

c) Diff of a product than show LHS = RHS

9.3 Straightforward.

9.4 Use diff of quotient to find dG/dω, find ω where this is zero and evaluate G at these values.

9.5 Use Integration by Parts

9.6 Ditto

Differentiation and Integration 10

Prof Richard Mitchell

Numerical Differentiation and Integration

Some of this can be found in the recommended books

Croft 809-813; James 611-13, 709-11;

Stroud 673-682; Singh 426-439;

Don't forget to attend the tutorials to get practice

Also, extra support is available from

http://www.reading.ac.uk/mathssupport centre

and http://www.mathtutor.ac.uk

Numerical Differentiation

If have a function, can almost always differentiate it.

But if signal is just numbers, can only differentiate by estimating: called numerical differentiation.

In control systems, a common controller is PID, it takes a signal x & returns P * x + I * integral(x) + D * diff(x)

Suppose data values at times h apart (eg at f(t), f(t+h))

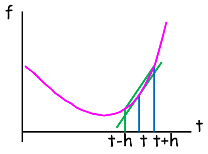
Given rule for differentiation $f'(t) = \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t}$

Might expect estimate of diff $f'(t) \approx \frac{f(t+h) - f(t)}{h}$

However

As want differential at t, don't just use gradient before t, rather use gradient before and after t. Use f(t+h), f(t-h)

$$f'(t) \approx \frac{f(t+h) - f(t-h)}{2*h}$$



e.g. estimating f'(t) for

$f(t) = t^2$ at t = 2 : h = 0.1.

f(1.9) = 3.61

f(2) = 4.00

f(2.1) = 4.41

Method 1: (4.41-4.0)/0.1 = 4.1

Method 2: (4.41-3.61)/0.2 = 4.0

Why Avoid Numerical Diff

Numerical Differentiation should be avoided if possible.

This is cos it usually involves dividing by a small number

The value being divided could have errors - due to poor measurement or rounding errors in earlier calculations.

As denominator < 1, these errors are amplified.

Suppose numerator = 4.01 but should be 4: error = 0.01

Suppose also, denominator 0.1

Differential estimated as 4.01 / 0.1 = 40.1

It should be 4 / 0.1 = 40.

Thus error of 0.01 amplified so error now 0.1.

Application : Small Changes

A circular piece of metal is to be copied, but its radius is mismeasured

How do you find the error in the area of the metal?

For f(t), if δf is small change in f & δt a small change in t.

$$\frac{df}{dt} = \lim_{\delta t \to 0} \frac{\delta f}{\delta t}$$

For small changes, this can be approximated as $\frac{df}{dt} \approx \frac{\delta f}{\delta t}$

So a small change in f can be estimated by $\delta f \approx \frac{df}{dt}\delta t$

Applying to Circular Disc

The area, A, is πr^2, where r is radius.
The change in area due to a small change in r is:

$$\delta A \approx \frac{dA}{dr}\delta r = 2\pi r \delta r$$

Suppose radius was measured as 10.01, not 10, i.e. $\delta r = 0.01$
The error in the area is:

$$2\pi * 10 * 0.01 = 0.628$$

Another example: the error in volume of sphere when radius measured as 4.01 when it should have been 4.0

$V = \frac{4}{3}\pi r^3$ so $\frac{dV}{dr} = 4\pi r^2$. Thus $\delta V = 4\pi 4^2 * 0.01 = 2.01$

Why Use Numerical Integration

Often have a function to be integrated, for which an analytical function can be found

eg $\int \cos(t)\,dt = \sin(t) + c$

For some functions, a substitution is needed

However, for other functions there is no such solution

eg $\int \exp(t^2)\,dt$

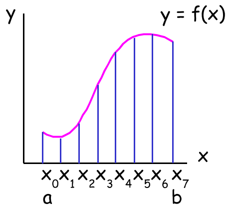
Also, sometimes engineers have just data values (from an experiment say), so no function to be integrated

The solution: approximate integration numerically.

The concept can also be extended for numerical solution of differential equations.

Basic Concept

If cant integrate f(x), or don't know f(x) analytically, divide area into strips, find area of each and sum.



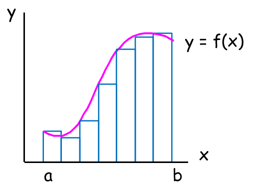
Each strip has same width - call it h

Want $\int_a^b f(x)\,dx$

Various ways of finding area of each strip ....

Reactangular Integration

Simplest (least accurate) method; assume each strip rectangle



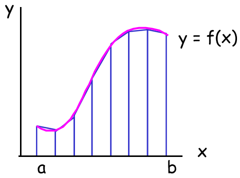
($a = x_0$ $b = x_6$)

Area of r'th rectangle
width * height
$h * f(x_r)$

Area of 7 strips: $\sum_{r=0}^{6} h * f(x_r) = h * \sum_{r=0}^{6} f(x_r)$

Trapezoidal Integration

More accurate – assume each strip is trapezium – in effect approximate curve as straight lines between points



$(a = x_0 \; b = x_6)$

Area of r'th strip is

$h * \frac{1}{2}(\, f(x_r) + f(x_{r+1})\,)$

Total area is sum of these

If do this directly count f(n) twice, so better to do

$$h * \left(\tfrac{1}{2}(f(x_0) + f(x_6)) + \sum_{r=1}^{5} f(x_r) \right)$$

Example

To test use a function which has is an analytical solution so can see how accurate the two methods are.

$$\int_1^5 \exp\left(x/10\right) dx$$

The correct answer is 10 * (exp(0.5) - exp(0.1)) = 5.435504

First, set h = 1, so use values at x = 1, 2, 3, 4 and 5.

For the rectangular method, the area is:

$1 * (e^{0.1} + e^{0.2} + e^{0.3} + e^{0.4}) = 5.168257$ poor

For the trapezoidal method, the area is:

$1 * (0.5 * (e^{0.1} + e^{0.5}) + e^{0.2} + e^{0.3} + e^{0.4}) = 5.440032$ ~ok

Making Strips Narrower

You should expect a better result if narrower strips used

Width, h	Rectangular	Trapezoidal
1	5.168257	5.440032
0.5	5.300748	5.436636
0.1	5.408371	5.435549
0.05	5.421926	5.435515
0.01	5.432786	5.435504
0.001	5.435232	5.435504
0.0001	5.435476	5.435504

Rect poor; Trap good with h = 0.01, or smaller

Simpson’s Rule

Trapezium rule assumes straight line between adjacent pts

Simpson’s rules goes one stage further ;

- It uses an even number of strips
- For first three adjacent points (and hence two strips)
  - find quadratic function going through them
  - calculate area
- For points 3, 4, 5 (i.e. the following two strips)
  - find quadratic function and area
- etc
- Sum all these areas.
  - e.g exp(x/10) for x = 1..5; use $e^{0.1}$ $e^{0.2}$ $e^{0.3}$ and $e^{0.3}$ $e^{0.4}$ $e^{0.5}$

Finding Quadratic Function

Suppose have three x values x_0, x_1, x_2 at intervals h, and are trying to integrate f(x) knowing $f(x_0)$, $f(x_1)$ and $f(x_2)$.

Use Lagrange Polynomials to find p(x) passing through pts.

Use shorthand : $f_0 = f(x_0)$, $f_1 = f(x_1)$, $f_2 = f(x_2)$.

$$p(x) = \frac{(x-x_1)\;(x-x_2)}{(x_0-x_1)\;(x_0-x_2)} f_0 + \frac{(x-x_0)\;(x-x_2)}{(x_1-x_0)\;(x_1-x_2)} f_1 + \frac{(x-x_0)\;(x-x_1)}{(x_2-x_0)\;(x_2-x_1)} f_2$$

But $x_0-x_1=-h$; $x_0-x_2=-2h$, etc., so

$$p(x) = \frac{(x-x_1)\,(x-x_2)}{2h^2} f_0 - \frac{(x-x_0)\,(x-x_2)}{h^2} f_1 + \frac{(x-x_0)\,(x-x_1)}{2h^2} f_2$$

Simpson’s Continued

For simplicity, define r such that

	$x - x_1 = r * h$	when $x = x_1$, $r = 0$
So	$x - x_0 = (r + 1) * h$	when $x = x_0$, $r = -1$
	$x - x_2 = (r - 1) * h$	when $x = x_2$, $r = +1$

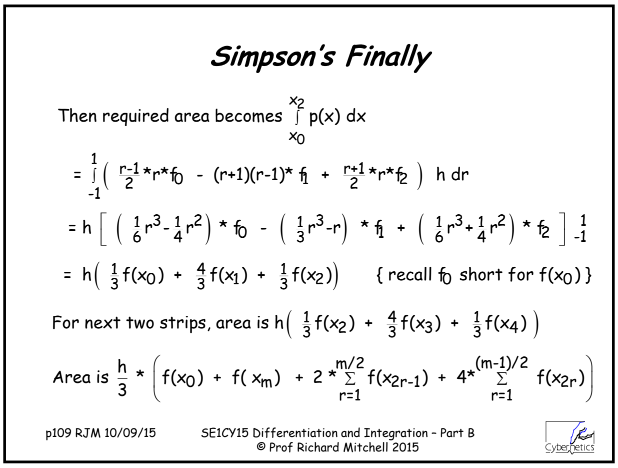
Then Polynomial

$$p(x) = \frac{(x-x_1)\,(x-x_2)}{2h^2} f_0 - \frac{(x-x_0)\,(x-x_2)}{h^2} f_1 + \frac{(x-x_0)\,(x-x_1)}{2h^2} f_2$$

Becomes

$$p(x) = \tfrac{1}{2}(r-1)*r*f_0 - (r+1)(r-1)*f_1 + \tfrac{1}{2}(r+1)*r*f_2$$

We can now integrate p(x) to get area.
We note that dx = h dr;



Simpson's Finally

Then required area becomes $\int_{x_0}^{x_2} p(x)\, dx$

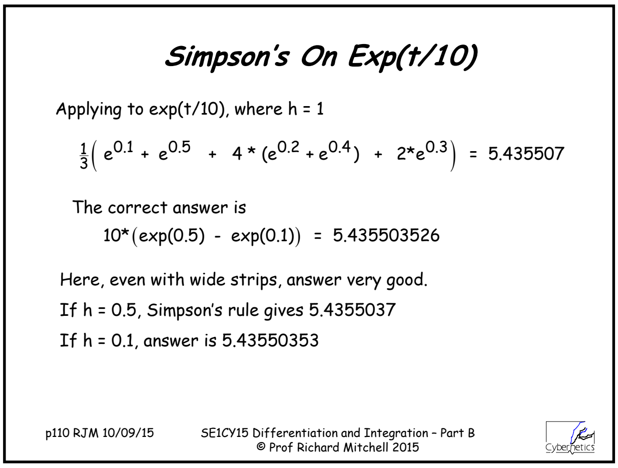
$$= \int_{-1}^{1} \left(\frac{r-1}{2} * r * f_0 \;-\; (r+1)(r-1) * f_1 \;+\; \frac{r+1}{2} * r * f_2 \right) h\, dr$$

$$= h \left[\left(\frac{1}{6}r^3 - \frac{1}{4}r^2 \right) * f_0 \;-\; \left(\frac{1}{3}r^3 - r \right) * f_1 \;+\; \left(\frac{1}{6}r^3 + \frac{1}{4}r^2 \right) * f_2 \right]_{-1}^{1}$$

$$= h\left(\frac{1}{3} f(x_0) + \frac{4}{3} f(x_1) + \frac{1}{3} f(x_2) \right) \qquad \{ \text{recall } f_0 \text{ short for } f(x_0) \}$$

For next two strips, area is $h\left(\frac{1}{3} f(x_2) + \frac{4}{3} f(x_3) + \frac{1}{3} f(x_4) \right)$

Area is $\frac{h}{3} * \left(f(x_0) + f(x_m) + 2 * \sum_{r=1}^{m/2} f(x_{2r-1}) + 4 * \sum_{r=1}^{(m-1)/2} f(x_{2r}) \right)$



Simpson's On Exp(t/10)

Applying to exp(t/10), where h = 1

$$\frac{1}{3}\left(e^{0.1} + e^{0.5} + 4 * (e^{0.2} + e^{0.4}) + 2 * e^{0.3} \right) = 5.435507$$

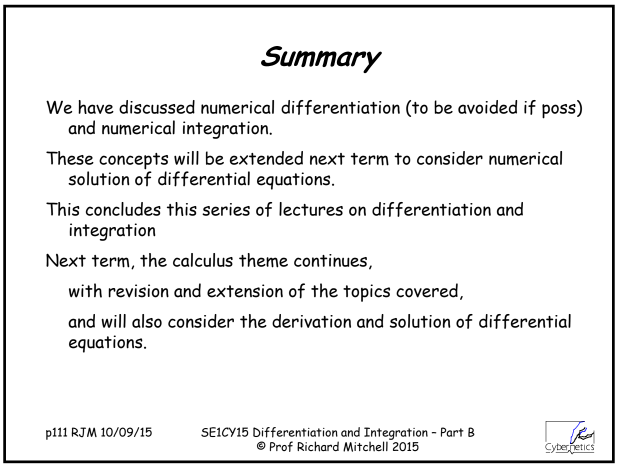
The correct answer is

$10*(\exp(0.5) - \exp(0.1)) = 5.435503526$

Here, even with wide strips, answer very good.

If h = 0.5, Simpson's rule gives 5.4355037

If h = 0.1, answer is 5.43550353



Summary

We have discussed numerical differentiation (to be avoided if poss) and numerical integration.

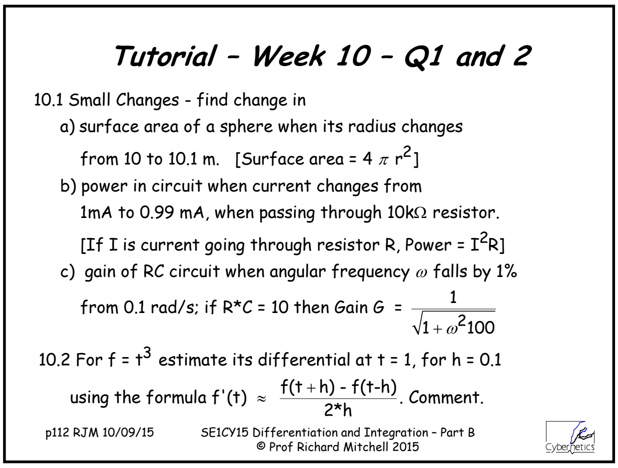
These concepts will be extended next term to consider numerical solution of differential equations.

This concludes this series of lectures on differentiation and integration

Next term, the calculus theme continues,

with revision and extension of the topics covered,

and will also consider the derivation and solution of differential equations.



Tutorial – Week 10 – Q1 and 2

10.1 Small Changes - find change in

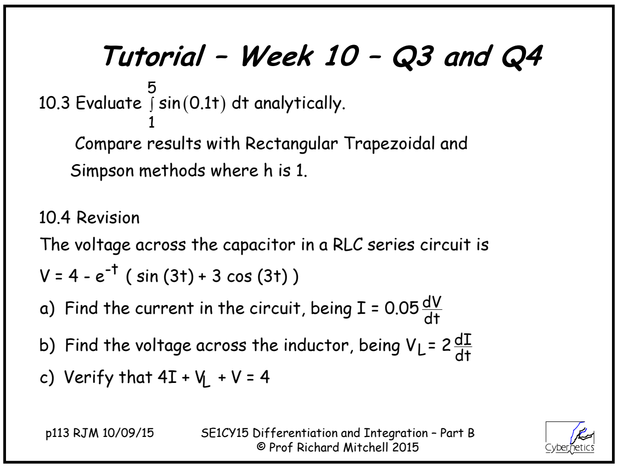
a) surface area of a sphere when its radius changes from 10 to 10.1 m. [Surface area = 4 π r^2]

b) power in circuit when current changes from 1mA to 0.99 mA, when passing through 10kΩ resistor.

[If I is current going through resistor R, Power = I^2R]

c) gain of RC circuit when angular frequency ω falls by 1% from 0.1 rad/s; if R*C = 10 then Gain $G = \frac{1}{\sqrt{1+\omega^2 100}}$

10.2 For f = t^3 estimate its differential at t = 1, for h = 0.1 using the formula $f'(t) \approx \frac{f(t+h) - f(t-h)}{2*h}$. Comment.



Tutorial – Week 10 – Q3 and Q4

10.3 Evaluate $\int_1^5 \sin(0.1t)\, dt$ analytically.

Compare results with Rectangular Trapezoidal and Simpson methods where h is 1.

10.4 Revision

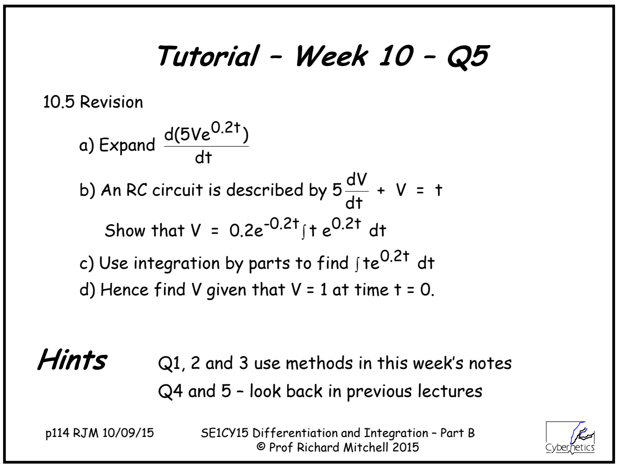
The voltage across the capacitor in a RLC series circuit is

$V = 4 - e^{-t} (\sin(3t) + 3 \cos(3t))$

a) Find the current in the circuit, being $I = 0.05\frac{dV}{dt}$

b) Find the voltage across the inductor, being $V_L = 2\frac{dI}{dt}$

c) Verify that $4I + V_L + V = 4$



Tutorial – Week 10 – Q5

10.5 Revision

a) Expand $\frac{d(5Ve^{0.2t})}{dt}$

b) An RC circuit is described by $5\frac{dV}{dt} + V = t$

Show that $V = 0.2e^{-0.2t} \int t\, e^{0.2t}\, dt$

c) Use integration by parts to find $\int te^{0.2t}\, dt$

d) Hence find V given that V = 1 at time t = 0.

Hints Q1, 2 and 3 use methods in this week's notes

Q4 and 5 – look back in previous lectures