SE1CY15 – Cybernetics - Calculus
Prof Richard Mitchell

One half of SE1CY15 is Calculus

Differentiation, Integration, Differential Equations

Aim - to provide students with relevant mathematical skills to formalise and analyse design problems in engineering / cybernetics

My Emphasis : covering mathematical techniques used in engineering – I mention examples, but concentrate on the associated maths

Some you may have met at A level, BUT NOT ALL

There is also some maths in SE1MA15

algebra, complex numbers, vectors, matrices, statistics

Arrangements and Assessment

One lecture and one tutorial a week on Maths – **you must attend**.

Tutorials – as best way to learn Maths is to practice.

Try all questions most in tutorial, any left in own time.

My lecture notes have questions

Take notes, books and calculators to tutorials.

Assessment (where some formulae may be provided) is by

a) Tutorials + Assessment 30%

b) One three hour exam in Summer Term 70%

Do monitor associated Blackboard course AND email regularly

Bring notes, pens, paper and calculator to Lectures and Tutorials

Books and Support

Stroud – Engineering Mathematics - Palgrave

Singh – Engineering Mathematics through Applications - Palgrave

James – Modern Engineering Mathematics - Addison-Wesley

Croft & Davison – Mathematics for Engineers - Pearson

On-line help - http://www.mathtutor.ac.uk

Also – if buy book may be able to access on-line systems

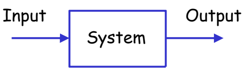
Also – support available from University's Maths Support Centre

NOTE : 75% of failures in 2008/9 did not visit:
www.reading.ac.uk/mathssupport

My notes on Blackboard and at
http://www.personal.reading.ac.uk/~shsmchlr/teach.htm

On Dynamic Systems

Engineers, Cyberneticists, Computer Scientists often study dynamic systems – which change (often with time)



Music System – In (changing) from CD, Out to Loudspeaker

Car – driver press accelerator, velocity change gradually

Aircraft: speed affected by many changing inputs

To model : use differentiation/integration/diff equations

Different frequency signals: analyse using complex numbers

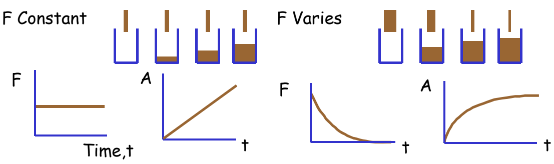
Multiple inputs/outputs – use matrices

For rest of lecture – introduce these by example

Dynamics and Maths

Consider brown liquid flowing into container

Input is Flow in, F; Output is Amount in container, A



F constant –
A change at constant rate

F variable, +ve but decr;
A incr but extra decreases

Formally : F differential of A; A is integral of F

Similarly, in electronics current flows which changes voltages …

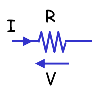
Introductory Electronic System

We will consider a simple complete electronic system to analyse

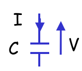
Need some basic concepts to fully understand

Battery; power supply – provides voltage, E

If applied to a complete circuit, current I flows



Resistor, resistance R. By Ohm's law, if current I flows through it, voltage across it is V = I * R



Capacitor, capacitance C. If current I *flows* into it, voltage V across it rises by amount due to I.

V is (1/C) * Integral of I

Signals and Units – must note

Voltages and currents are 'signals' that vary with time

V(t) is voltage V at time t

I(t) is current I at time t

On Units

Voltages measured in Volts(V); eg 9V battery

Currents measured in Amps(A), eg 3A current

Resistance measured in Ohms (Ω), eg 10Ω

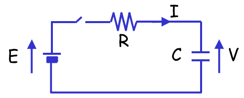
Capacitance measured in Farads (F), eg 3F

R*C is 'time constant', measured in seconds (s)

Note also, 10kΩ is 10 * 1000 Ω; 5μF is 5 * 10^{-6}F

Associated R*C is $10*10^3*5*10^{-6}$ = $50*10^{-3}$s or 50 ms

Illustrative System – RC Circuit



Voltages across

Resistor = I * R

Capacitor = V

At t = 0 capacitor uncharged, V = 0; switch then closed, I flows

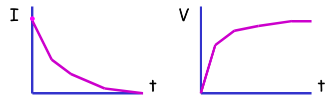
Voltage from battery equals that across R + that across C

E = I * R + V so E – V = I * R or I = (E – V) / R

Start: I = E/R; V rises

Now I less; V ↑ by less

Now I even less,

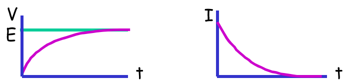


Eventually V reaches E, I becomes 0 and V unchanged

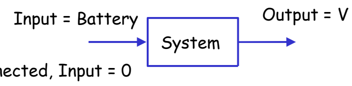
Response and Think of as System

Actual signals are smoother.

c.f. variable flow



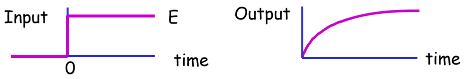
Think of as system one input one output.



For t < 0, battery not connected, Input = 0

At t = 0, a 'step' change of the Input to E; then remains const

V then changes – gradually (exponential shape)



Step Input and 'Exponential' Output occur in many systems

Exponential – a common signal/function

Exponential : exp(x) or e^x : e ~ 2.71828

e^0 is 1; $e^\infty = \infty$ $e^{-\infty} = 1/e^\infty = 0$ e^{-1} ~ 0.37 e^{-5} ~ 0.0067

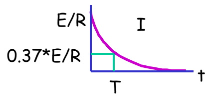
RC Circuit :

I = E/R * exp(-t/T) :

T is R*C = time constant

t = 0, I = E/R;

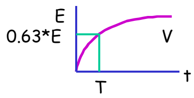
When t = T, I = 0.37 E/R (37% of initial value)



V = E - E *exp(-t/T)

when t = 0, V = 0

when t = T, V is 63% of final value E.



Differentiation

We now formally introduce differentiation:

Change in V caused by I into capacitor, capacitance C.

Change in V (with time) is expressed mathematically as

$$\frac{dV}{dt} = \frac{1}{C} I$$ Differential of V = flow I times 1/C

t is independent variable, V is dependent (on t) variable

We differentiate with respect to t.

Note can differentiate with respect to other variables:

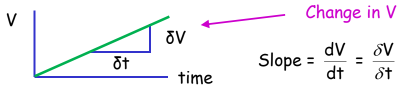
$\frac{dy}{dx}$ is differential of y with respect to (wrt) x

When plot graphs, independent variable on horizontal axis

Graphical Interpretation

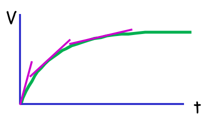
Can understand differentiation graphically : slope of graph

If function changing at constant rate – slope is constant



Slope = $\frac{dV}{dt} = \frac{\delta V}{\delta t}$ (constant)

For curve, slope changes, draw tangents, find their slopes



For differential of V at time t:

find δV = V(t+δt) – V(t) (δt small)

Slope $\frac{dV}{dt} = \frac{V(t+\delta t)-V(t)}{\delta t} = \frac{\delta V}{\delta t}$ (varies)

Clearly differential is function of time

Integration

Voltage across capacitor increases as current flows in

Mathematically integration represents this process

$\int I\,dt$ is integral of I over time (t in dt is time)

For capacitor, voltage V is proportional to integral of I

$$V = \frac{1}{C}\int I\,dt \qquad (C = \text{capacitance})$$

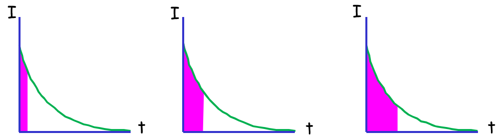
Integration also not only done with respect to time

eg $\int f(x)\,dx$

NOTE integration is inverse process to differentiation

Graphical Interpretation

Integration interpreted graphically : area under a curve.



At t = 0 area under curve is 0; after a time area increased

Over next time period, area increased, but by less, etc

So area under I curve is changing like V changes:

Starts at 0, rises rapidly, thereafter rises more slowly.

Another Topic: Differential Equations

A differential equation involves the differential of a variable.

It is solved to find how that variable changes.

For the RC circuit, we combine two equations:

$\frac{dV}{dt} = \frac{1}{C} I$ which we have already met

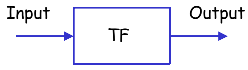
and $I = \frac{E - V}{R} = \frac{\text{voltage across resistor}}{\text{resistance}}$

So $\frac{dV}{dt} = \frac{E - V}{RC}$ Change in V is voltage across resistor divided by RC and RC = T time const

Often, we form a differential equation for a system and solve it so as (in this case) to find how V varies with t

Systems, Transfer Functions and s

We often encounter systems with an input and an output



Output = TF * Input

TF is system 'transfer function' : how Input transferred to Output

For 'Dynamic' Systems, we use 's' as shorthand : Laplace variable

In simplest case 's' means differentiation wrt time

For RC circuit, Input is E, Output is V and $\frac{dV}{dt}$ written as s V

Thus differential equation becomes

$$\frac{dV}{dt} = s\,V = \frac{E - V}{RC}$$

Laplace Operator and RC Circuit

$s\,V = \frac{E - V}{RC}$ or $RC\,s\,V = E - V$

Gathering terms in V on one side plus other basic algebra:

$V + RC\,s\,V = E$

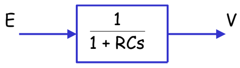
or

$V(1 + RCs) = E$

Dividing both sides by 1 + RCs $V = \frac{1}{1 + RCs} E$

Thus the transfer function is $\frac{1}{1 + RCs}$

We model the circuit as



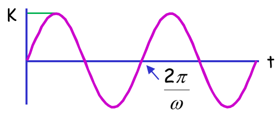
Sinusoid – another common signal

A signal that changes continuously and smoothly

Time for one complete cycle is its Period.

How often it completes cycle is its frequency, f (units Hz).

Engineers often use angular frequency, $\omega = 2\pi f$ (units rad/s)



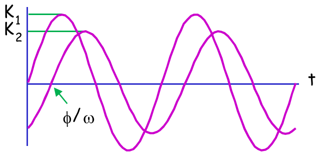
Sinusoid : K sin (ωt)

K is amplitude

ωt is an angle in radians

Example use: checking amplifier ok at audio freqs 20..20kHz

When Input to Circuit is Sinusoid



Input, $K_1 \sin(\omega t)$.

Output $K_2 \sin(\omega t+\phi)$.

Same angular frequency (ω) but different amplitude and phase shifted (delayed, $\phi<0$)

To analyse, system transfer function must involve ω

We replace s by $j\omega$: $j = \sqrt{-1}$; so $\frac{V}{E} = \frac{1}{1+j\omega CR}$

Complex numbers have size (modulus, | |) and angle (argument, $\angle$)

$\frac{K_2}{K_1} = \left|\frac{1}{1+j\omega CR}\right|$ and $\phi = \angle\frac{1}{1+j\omega CR}$ Shows Complex Numbers are used

Summary

Today aimed at setting the scene for some of the module

We have looked at dynamic systems – liquids/electronics

flow is the differential of the amount/voltag

amount/voltage is integral of the flow

We have seen common signals : step, exp, sin;

the graphical interpretation of integration and differentiation;

differential equations & transfer functions;

that complex numbers can be used

All are covered (in much more detail) in this module and elsewhere

Next week we start formally looking at differentiation and integration of some simple signals.

Tutorials

Attendance is required at all tutorials BECAUSE the best way to learn maths is to practice, and this you do, with help, in tutorials.

Help is provided – do ask if you are stuck

Complete questions in own time.

Marks contribute to the end of year module mark

Tutorial questions are with lecture notes – look in advance

Bring notes, books + calculator (paper + pen) to tutorials

Write your name clearly at the top of the paper.

Then attempt the questions

Note, there are 'hints' for each Q if you are stuck.

Please feel free to go to Maths Support Centre as well

Tutorial Week 1

1) Suppose O = (I – O) * C * P, derive an expression for O as a function of I, C and P only.

2) RC Circuit: suppose E = 10V; R = 2Ω and C = 0.05F, so T = 0.1s.

Calculate at t = 0, 0.1, 0.2, 0.3, 0.4, 0.5 :

I = E/R * exp(-t/T) and V = E - E *exp(-t/T).

Plot points on graphs of I vs t and V vs t.

Draw smooth curves through these points.

Add tangent to V at t = 0, 0.1 and 0.3 – what is its slope?

Is there a relationship between slope of V and I?

Continued

3) For t = 0, π/6, π/3, π/2, 2π/3, 5π/6 and π:

calculate sin(t) and cos(t) [remember to use radians].

Plot points on graph of sin(t) vs t and cos(t) vs t.

Sketch smooth curve through the points.

Add tangent to sin(t) at t = 0, π/3, π/2, estimate their slopes.

Compare the slopes with cos(t).

4) Repeat for sin(2t) and cos(2t).

Hints. Q1 – see slide on Laplace Operator and RC Circuit.

Q2-4 Calculate values using your calculator, plot some points on graphs, and sketch curves through them. Try to be reasonably accurate, but you don't have to be perfect!

Differentiation and Integration - 2
Prof Richard Mitchell

Today we will start to look formally at differentiation and integration – looking at polynomials

Some of this can be found in the recommended books

Croft 694-702, 774-776; James 480-487, 532-533

Stroud 361-375, 824; Singh 257-260, 359-361;

Don't forget to attend the tutorials to get practice

Also, extra support is available from

http://www.reading.ac.uk/mathssupport centre

and http://www.mathtutor.ac.uk

Differentiation - Introduction

Differentiation – mathematical technique to show change

That and associated process of integration are 'calculus'

Can apply to electronic, mechanical, cybernetic, computing systems…

Why need?

A manufacturing system inevitable changes, small changes ok, large ones can be very problematic.

Want to be able to find best solution – use differentiation

'optimisation' an important concept in Computer Science etc

Can use it to help sketch curves

Can use it in kinematics – movement of objects

Can use it to find roots of equations etc

Differentiation

Differential of a function (its derivative) = how it changes.

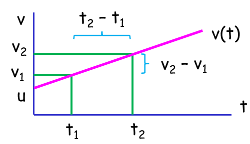
Suppose v is function for velocity of an object

Diff of v shows change of velocity or acceleration

If initial velocity u, constant acceleration a, velocity v at time t is

$v = u + a * t$

The graph of v versus t is a straight line, gradient a

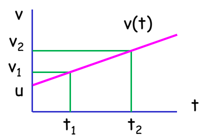


Gradient formally $\frac{v_2 - v_1}{t_2 - t_1}$

$= \frac{(u + at_2) - (u + at_1)}{t_2 - t_1}$

$= \frac{at_2 - at_1}{t_2 - t_1} = \frac{a(t_2 - t_1)}{t_2 - t_1} = a$

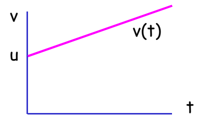
Points to Note



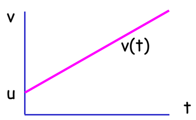
Formally for $v = u + at$,

$\frac{dv}{dt} = a$

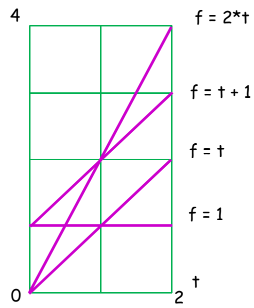
If change u, line moves but gradient unchanged



If change a, gradient of line changes



Some Examples



$f = 1$ no change : slope 0

$f = t$ slope = 1

$f = 2*t$ slope = 2

$f = t + 1$ slope = 1

So for constant

its derivative is zero;

for $f = n*t$,

derivative is constant = n.

For $f = n*t + c$ (constant)

derivative is n

What if not straight line

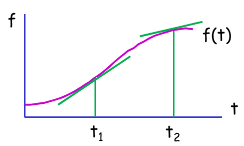
A straight line has a constant gradient

For a curve, the gradient changes.

At any time you draw a line which just touches the curve,

this is the tangent

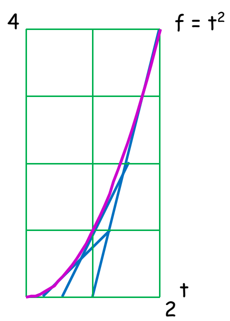
Its gradient gives derivative of curve at that time



Let f(t) show function at different points in time.

Change of f(t) at t_1 is larger than that at t_2

eg Quadratic Function $f = t^2$



Tangents added at three times

Slope is 1 at t = 0.5

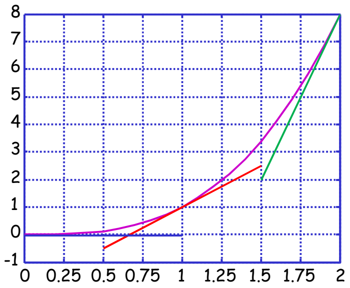
Slope is 2 at t = 1

Slope is 4 at t = 2

This suggests that

the differential of $f = t^2$ is $2*t$

In Lecture Exercise - Cubic



Add tangents at t = 0, 1 and 2

Slope at t = 0

Slope at t = 1

Slope at t = 2

Hence

Differential Notation

Let f(t) represent a function which varies with time.

Its (first) differential is represented by $f'(t)$ or $\frac{df}{dt}$

Note dt means function f (which depends on t) differentiated with respect to (wrt) the independent variable t.

f(x) is function depends on x: differentiate wrt x = $\frac{df}{dx}$

For change of circle area with radius; $A = \pi r^2$, $\frac{dA}{dr} = 2\pi r$

The differential of the differential of f is $\frac{d^2f}{dt^2}$ or $f''(t)$

If x(t) is position, velocity v = $\frac{dx}{dt}$; acceleration = $\frac{dv}{dt} = \frac{d^2x}{dt^2}$

Equation of a Tangent

The equation of a tangent to a function f(x) at x_1

$y = m x + c$

Gradient m is differential of f(x) evaluated at x_1, $f'(x_1)$

Offset c is $y - m x$ at $x = x_1$ ie $c = f(x_1) - m*x_1$

Eg for the cubic, x^3 at x = 2

Differential is $3x^2$ so $m = 3*2^2 = 12$

2^3 is 8, so c = 8 - 12 * 2 = -16

Thus equation of tangent is y = 12 x – 16

Tangent at x = 1: $m = 3*1^2 = 3$;

At x = 1, $c = 1^3 - 3*1 = -2$

Differential of Function * Constant

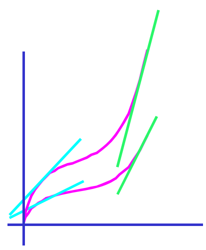
The differential of k*t is k for any constant k.

More generally, for any function f(t), if k constant:

diff of k*f(t) = k*diff of f(t)

Makes sense: if plot graph k * f(t), each value of f(t) is multiplied by k, so tangent slopes multiplied by k.

$$\text{e.g.} \, \text{diff}\left(5t^3\right) = 5 * \text{diff}\left(t^3\right) = 5*3*t^2$$



Differential of Sum of Functions

This is sum of differentials of each function

Consider distance travelled by an object under constant acceleration a from initial velocity u:

$s = u * t + \frac{1}{2} * a * t^2$

Has two functions	$u * t$	and	$\frac{1}{2} * a * t^2$
Their differentials :	u	and	$\frac{1}{2} * a * 2 * t$
So differential of s is	$u + a * t$		

So velocity, $v = u + a * t$

The differential of v, is acceleration, which = 0 + a = a

So diff(v) = diff(u) + diff(at) = 0 + a = a

Example – Capacitor of Capacitance C

If v is voltage across a capacitor, current i through it is

$i = C \frac{dv}{dt}$ where C is capacitance of capacitor

Suppose v is $3 + 5 * t - 2 * t^2$ and C = 0.1F, then

$$i = 0.1 \frac{d(3 + 5t - 2*t^2)}{dt}$$

$$= 0.1*(0 + 5 - 4*t)$$

$$= 0.5 - 0.4*t$$

In Class Exercise

If i is current through an inductor, whose inductance is L, the voltage across it is

$$v = L \frac{di}{dt}$$

Find v if L = 2H and $i = 5t^2 - 3t - 2$

Answer

Integration

Here introduced as reverse of differentiation

Has geometric interpretation – area under a curve

We shall cover indefinite and definite integrals

Numerous techniques, we concentrate on those for which engineering applications exist

Eg for : mean and rms of signals; volumes and surface areas of solids of revolution, lengths of curve; solution of differential equations; numerical methods.

As acceleration is change (ie differential) of velocity,

velocity is integral of acceleration

similarly, position is integral of velocity …

As Reverse of Differentiation

So, for $f(t) = t^3$, its differential is $3t^2$

Therefore the integral of $3t^2$ must be t^3

Formally we write integration as follows

$\int f(t)\, dt$

$\int$ indicates integration, f(t) is function being integrated,

$\int dt$ means that integration performed with respect to t

$\int 3t^2\, dt = t^3$

Again can integrate wrt other variables , so for instance

$\int 2x\, dx = x^2$

Two Simple Rules of Integration

The differential of k * f(t) is k * differential of f(t)

So $\int k * f(t)\, dt = k * \int f(t)\, dt$

so $\int 10 * t\, dt = 10 * \int t\, dt = 10 * \frac{1}{2} t^2 = 5t^2$

Also, as differential of t^3 is $3t^2$

so $\int t^2\, dt$ is $t^3 / 3$

so $\int 2\pi r\, dr$ is πr^2

Given that $\frac{d(f+g)}{dt} = \frac{df}{dt} + \frac{dg}{dt}$ it is not suprising that

$\int f(t) + g(t)\, dt = \int f(t)\, dt + \int g(t)\, dt$

So $\int u + at\, dt = \int u\, dt + \int at\, dt = ut + \frac{1}{2} a t^2$

Constant of Integration

Straightforward, but

$\frac{dt^3}{dt} = 3t^2$ But, for any constant c, $\frac{d(t^3 + c)}{dt} = 3t^2$

So integral of a function is a function PLUS a constant, so

$\int 3t^2\, dt = t^3 + c$

$\int x\, dx = \frac{1}{2} x^2 + c$

$\int \pi r^2 + 2\pi r\, dr = 1/3\, \pi r^3 + \pi r^2 + c$

only one c!

These are indefinite integrals : *function* plus a *constant*

Later we meet definite integrals.

Constants Can Mean Something

Object accelerating: $a = 4\ ms^{-2}$: initial velocity $5ms^{-1}$

Integrate to find velocity: $v = \int 4\, dt = 4t + c$

c is initial velocity (at t = 0) so v = 5 + 4t

If object at position 3m at t= 0, where is it at t = 10s?

Its position, p, is the integral of v, $\int v\, dt$,

$p = \int 5 + 4t\, dt = 5t + 4 * t^2/2 + c = 5t + 2t^2 + c$

At t = 0, p = 0 + 0 + c = 3, so c = 3

At time 10, position is 5 * 10 + 2 * 100 + 3 = 253 m.

Often information given to allow constants to be found.

Summary

Today we have introduced differentiation and integration

Graphically we have shown

- differentiation gives gradients of lines
- differential of a curve is found by gradient of tangent

We have looked at simple functions like

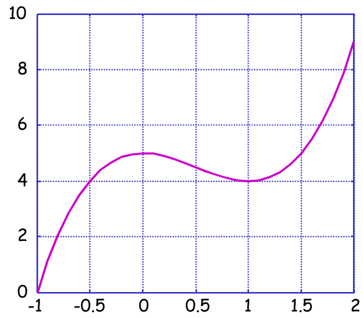
- const, x, x^2, x^3; sums of functions and const*functions

Integration has been introduced as the reverse process

- We have seen the need for 'constant' of integration

Next week we define differentiation and consider the graphical interpretation of integration which leads to definite integrals.

Tutorial – Week 2 – Q1



2.1 $y = 2x^3 - 3x^2 + 5$

a) What is $\frac{dy}{dx}$?

b) Draw tangents at x = -1, 0, 1 and 2 and evaluate their slope.

c) Show answers are consistent.

Tutorial – Week 2 – Q2 and 3

2.2. For the following,

- write down the differential of the function
- find the equation of the tangent at the given point

a)	function:	$y = 3x^4$	tangent at x = 2
b)	function:	$y = 12x^5 - 2x^2 + 5$	tangent at x = 1
c)	function:	$s = 3t + \frac{1}{2}7t^2$	tangent at t = 4

2.3 Find the following integrals,

a) $\int 10t^4\, dt$ b) $\int 5t^3 - 6t\, dt$

c) $\int 2\pi r\, dr$ d) $\int 4\pi r^2\, dr$ e) $\int 7 + 5t\, dt$

Tutorial – Week 2 – Q4 and 5

2.4 a) An object is accelerating at $3ms^{-2}$ and at time t = 1 s its velocity is $2ms^{-1}$, and at t = 4 s its position is 25m. What are its velocity and position at time t = 10s?

b) Suppose its accelation is 6 t ms^{-2} and at time t = 1 s its velocity is 1 ms^{-1} and its position is 2m. What are its velocity and position at t = 5s?

2.5. The current flowing into a 0.2F capacitor is $i = 2t - 8t^3$

The voltage across it is $\frac{1}{C}\int i\, dt$, and this is 4V at time 1s.

Find an expression for this voltage.

Tutorial – Week 2 – Q6 and Hints

2.6. The voltage across an L = 0.1H inductor is $v = 6t^2 - 3t^4 + 2$

The current through it, $\frac{1}{L}\int v\, dt$, is 3A at time 2s.

Find an expression for this current.

Hints

2.1 : a) use formula, b) draw lines over t = 0.5 touch curve

2.2 : can write down function. See slide on finding tangent

2.3 : Use formulae

2.4 : see slide on 'constants can mean something'

2.5 : Do indefinite integral, use initial value to find const

2.6 : Ditto

Differentiation and Integration - 3
Prof Richard Mitchell

Today we continue differentiation and integration formalising polynomials, looking at areas under curves

Some of this can be found in the recommended books

Croft 697-707, 781-801; James 481-487, 532-549

Stroud 366-375, 824; Singh 260-269, 375-379;

Don't forget to attend the tutorials to get practice

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http://www.reading.ac.uk/mathssupport centre

and http://www.mathtutor.ac.uk

Differentiation of Polynomials

We established graphically:

diff of kt is k; of t^2 is 2t; of t^3 is

Also diff of k*f(t) is k*diff of f(t)

And diff of f(t) + g(t) is diff (f(t)) + diff(g(t))

So diff of 5t + $6t^2$ is

Integration is the reverse process

So initially we said Integ of $3t^2$ is t^3, etc.

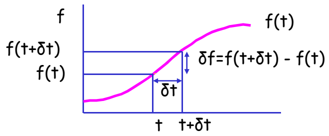
But as Diff (t^3 + c) is $3t^2$, Integ ($3t^2$) is

Today we formalise and generalise for Diff(t^n), look at the graphical interpretation of integration and so definite integrals

Finding Derivatives Analytically

Evaluate function at t and at t + δt

δt is small value



Gradient is change in f over that in t = $\frac{\delta f}{\delta t} = \frac{f(t+\delta t)-f(t)}{\delta t}$

For f(t) = 1; $\frac{\delta f}{\delta t} = \frac{1-1}{\delta t} = 0$

For f(t) = t: $\frac{\delta f}{\delta t} = \frac{(t+\delta t) - t}{\delta t} = \frac{\delta t}{\delta t} = 1$

For f(t) = 2t: $\frac{\delta f}{\delta t} = \frac{2(t+\delta t) - 2(t)}{\delta t} = \frac{2\delta t}{\delta t} = 2$

These agree with earlier estimates

Continued

For f(t) = t^2: $\frac{\delta f}{\delta t} = \frac{(t+\delta t)^2 - (t)^2}{\delta t} = \frac{t^2 + 2t\,\delta t + \delta t^2 - t^2}{\delta t}$

$= \frac{2t\delta t + \delta t^2}{\delta t} = 2t + \delta t$

Not the 2t we expected.

We should do the above and then see what happens when δt → 0.

tends to zero

Strictly $\frac{df}{dt} = \lim_{\delta t \to 0} \frac{\delta f}{\delta t}$

For f = t^2, when δt → 0, 2t + δt → 2t,

So differential of t^2 is 2t

Cubic

$\frac{d(t^3)}{dt} = \lim_{\delta t \to 0} \frac{(t+\delta t)^3 - (t)^3}{\delta t}$

$= \lim_{\delta t \to 0} \frac{t^3 + 3t^2\delta t + 3t\,\delta t^2 + \delta t^3 - t^3}{\delta t}$

$= \lim_{\delta t \to 0} \frac{3t^2\delta t + 3t\,\delta t^2 + \delta t^3}{\delta t}$

$= \lim_{\delta t \to 0} 3t^2 + 3t\,\delta t + \delta t^2$

$= 3t^2$

Arrange so don't divide at end by δt ie 0

Similarly can shown diff(t^4) is $4t^3$, and diff(t^5) is $5t^4$

Summary and Suggestion

f(t)	f'(t)
$1=t^0$	0
$t=t^1$	1
t^2	2 t
t^3	3 t^2
t^4	4 t^3
t^5	5 t^4

Hence, differential of t^n is

REMEMBER

Which suggests

differential of t^6 is

It also for negative powers:

differential of t^{-3} is

And for fractional powers

$\frac{d\left(t^{5/3}\right)}{dt} =$

If want to, can use analysis given earlier to show this.

Extending Integration

Given that differential of t^n is n * t $^{n-1}$

$\int t^n\,dt = \frac{t^{n+1}}{n+1} + c$

And so

integral of t^6 is

Also (for negative powers)

integral of t^{-3} is

And for fractional powers

$\int t^{5/3} dt =$

Integration Graphically

Differentiation has a graphical interpretation - slope

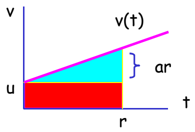
For integration it is area under a curve

We will start with a line:

$v = u + a\,t$

At time $t = r$, $v = u + a\,r$

Area from $t = 0$ to r is



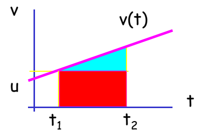
Area of rectangle : $u * r$

+ Area of triangle: $\frac{1}{2}\, r * a\, r = \frac{1}{2}\, a\, r^2$

So at any time r, area since $t = 0$ is $u * r + \frac{1}{2}\, a\, r^2$

Consistent with $\int u + at\; dt = ut + \frac{1}{2}\, a\, t^2 + c$

What if want area starting at t ≠ 0



Now want area under line from t_1 to t_2

At t_1, $v = u + a\,t_1$;

at t_2, $v = u + a\,t_2$

Area of rectangle = $(u + a\,t_1) * (t_2 - t_1)$

Height of triangle = $(u + a\,t_2) - (u + a\,t_1) = a\,(t_2 - t_1)$

So its area = $\frac{1}{2} * a\,(t_2 - t_1) * (t_2 - t_1)$

Total area = $(\,(u + a\,t_1) + \frac{1}{2}\, a\,(t_2 - t_1)\,) * (t_2 - t_1)$

$= (u + \frac{1}{2}\, a\,(t_2 + t_1)) * (t_2 - t_1)$

$= u * (t_2 - t_1) + \frac{1}{2}\, a * (t_2^2 - t_1^2)$ $(x+y)(x-y)=x^2-y^2$

Definite Integrals

Area found is $u * (t_2 - t_1) + \frac{1}{2}\, a * (t_2^2 - t_1^2)$

Which can be written $u\,t_2 + \frac{1}{2}\, a\, t_2^2 - (u\,t_1 + \frac{1}{2}\, a\, t_1^2)$

Ignoring constant, $\int u + at\; dt = u\,t + \frac{1}{2}\, a\, t^2$ call it $s(t)$

So area under line is $s(t_2) - s(t_1)$

ie, we find the function $s(t)$ the integral of $u + at$

Then area is $s(t)$ evaluated at t_2 minus $s(t)$ at t_1

An indefinite integral is where

$\int f(t)\; dt = g(t) + c$

A definite integral is   same g()

$\int_a^b f(t)dt = [\, g(t)\,]_a^b = g\,(b) - g(a)$ Note notation: g(t) evaluated at b & a

On Definite Integrals

So if $g(t)$ is integral of $f(t)$, then $\int_a^b f(t)dt = g(b) - g(a)$

$$\int_1^5 t^3 dt = \left[\frac{t^4}{4}\right]_1^5 = \frac{5^4}{4} - \frac{1^4}{4} = \frac{625}{4} - \frac{1}{4} = 156$$

With definite integrals no constant needed: it's cancelled

Suppose $\int f(t)dt = g(t) + c$

If include constant in definite integral:

$\int_a^b f(t)dt = g(b) + c - g(a) - c = g(b) - g(a)$

In Class Exercises

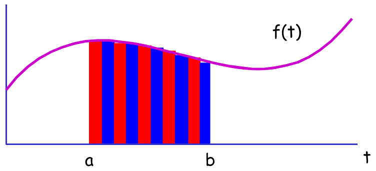
$\int_1^4 t^2 dt \qquad =$

$\int_1^5 3 + 6t\; dt \qquad =$

Integrals of Curves

Areas under lines are easy, but for curves …

Easiest is to split area into a series of narrow strips



Suppose each strip of width δt

Area of strip at t = r is

~ $f(r) * \delta t$

Thus area from t = a to t = b is $\sum_{r=a}^{b} f(r) * \delta t$

Formal Definition

Formally for the differential of f(t) we say

$$\frac{df}{dt} = \lim_{\delta t \to 0} \frac{\delta f}{\delta t} = \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t}$$

For integration, we do something similar.

We divide the area into strips, and see what happens as the widths gets smaller and number of strips increase.

Area is sum all strips' areas as $\delta t \to 0$, so formally

$$\int_a^b f(t)\,dt = \lim_{\delta t \to 0} \sum_{r=a}^{b} f(r) * \delta t$$

Example : Area under cubic graph

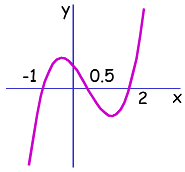
Consider $y = x^3 - 1.5x^2 - 1.5x + 1$

Factorises as $y = (x - 0.5)(x - 2)(x + 1)$

y -ve when $x < -1$, or $0.5 < x < 2$

y +ve when $-1 < x < 0.5$ or $x > 2$

Consider the area under curve from -1 to 2:



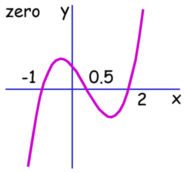
$$\int x^3 - 1.5x^2 - 1.5x + 1\,dx = \frac{x^4}{4} - \frac{x^3}{2} - \frac{3x^2}{4} + x + c$$

$$\int_{-1}^{2} x^3 - 1.5x^2 - 1.5x + 1\,dx = \left[\frac{x^4}{4} - \frac{x^3}{2} - \frac{3x^2}{4} + x\right]_{-1}^{2}$$

$$= (4 - 4 - 3 + 2) - (\frac{1}{4} + \frac{1}{2} - \frac{3}{4} - 1) = -1 - -1 = 0$$

Look at Positive and Negative Areas

Correct in one sense, but visibly areas are not zero



$$\int_{-1}^{0.5} x^3 - 1.5x^2 - 1.5x + 1\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{2} - \frac{3x^2}{4} + x\right]_{-1}^{0.5} = \frac{17}{64} - -1 = \frac{81}{64}$$

$$\int_{0.5}^{2} x^3 - 1.5x^2 - 1.5x + 1\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{2} - \frac{3x^2}{4} + x\right]_{0.5}^{2} = -1 - \frac{17}{64} = -\frac{81}{64}$$

If want actual area, use positive area plus – negative area

So 'area' is 81/64 – -81/64 = 81 / 32

Application – Mean of Signal f(t)

Mean of f(t) between t = a and t = b is $\frac{1}{b-a}\int_a^b f(t)dt$

Suppose velocity of object is $v = 5 + 3t$

Then mean velocity between t = 2 and t = 5 is

$$\frac{1}{5-2}\int_2^5 5 + 3t\,dt = \frac{1}{3}\left[5t + 1.5t^2\right]_2^5 = \frac{25+37.5-10-6}{3} = \frac{46.5}{3} = 15.5$$

Distance travelled is $5t + 1.5t^2$; so mean of this distance is

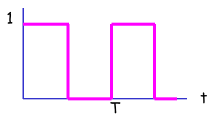
$$\frac{1}{5-2}\int_2^5 5t+1.5t^2 dt = \frac{1}{3}\left[\frac{5}{2}t^2 + \frac{1.5}{3}t^3\right]_2^5 = \frac{125+125-20-8}{3*2} = \frac{222}{6} = 37$$

Mean Value of Periodic Signal

Repeats every T secs: $f(t) = f(t+T) = f(t+2T)$

Mean is $\frac{1}{T}\int_0^T f(t)\,dt$

A square wave is a periodic signal.

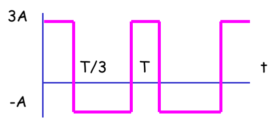


$f(t) = 1$ when $0 < t < T/2$

$= 0$ when $T/2 < t < T$

$$\text{Mean} = \frac{1}{T}\left(\int_0^{T/2} 1\,dt + \int_{T/2}^{T} 0\,dt\right) = \frac{1}{T}\left([t]_0^{T/2} + 0\right) = \frac{1}{T}\left(\frac{T}{2} - 0\right) = \frac{1}{2}$$

Uneven Mark/Space Ratio + Offset



$f(t) = 3A$ from 0 to $\frac{T}{3}$

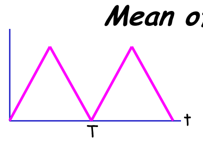
$= -A$ from $\frac{T}{3}$ to T

$$\text{Mean} = \frac{1}{T}\left(\int_0^{T/3} 3A\,dt + \int_{T/3}^{T} -A\,dt\right)$$

$$= \frac{1}{T}\left([3At]_0^{T/3} + [-At]_{T/3}^{T}\right)$$

$$= \frac{1}{T}\left(AT - 0 + -AT - -A\frac{T}{3}\right) = \frac{A}{3}$$

Mean of Triangle Wave



From 0.. $\frac{T}{2}$, $Tri(t) = t*\frac{2}{T}$

From $\frac{T}{2}$ to T, $Tri(t) = 2 - t*\frac{2}{T}$

$$Mean = \frac{1}{T}\left(\int_0^{T/2} \frac{2}{T} t\,dt + \int_{T/2}^{T} 2 - \frac{2}{T} t\,dt\right)$$

$$= \left[\frac{2}{T^2} * \frac{t^2}{2}\right]_0^{T/2} + \left[\frac{2t}{T} - \frac{2}{T^2}\frac{t^2}{2}\right]_{T/2}^{T}$$

$$= \left(\frac{T^2}{4T^2} - 0\right) + \left(\frac{2T}{T} - \frac{2T^2}{2T^2} - \frac{2T}{2T} + \frac{T^2}{4T^2}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Summary

We have looked formally at differentiation of t^n

And we have considered integrals and areas and signal means

REMEMBER THESE: to help, complete the following

$$\frac{d\left(5 - 3t + 7t^2 - 8t^3\right)}{dt} =$$

$$\int 10t^4 + 3t^2 + 5\,dt =$$

a) $40t^5 + 6t^3 + 5t + c$

b) $2t^5 + t^3 + 5t + c$

c) $\frac{10}{4}t^5 + \frac{3}{2}t^3 + 5t + c$

Next week: we apply these to maxima and minima and Newton Raphson

Tutorial – Week 3 – Q1, 2 and 3

3.1 Complete the following $\frac{d(t^4)}{dt} = \lim_{\delta t \to 0} \frac{(t + \delta t)^4 - (t)^4}{\delta t}$

3.2. Find the following differentials

a) $\frac{d(t^7)}{dt}$ b) $\frac{d(4t^{-3})}{dt}$ c) $\frac{d(3x^{1/2})}{dx}$ d) $\frac{d(\frac{4}{3}\pi r^3)}{dr}$

3.3. Find the following integrals

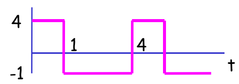
a) $\int t^7 dt$ b) $\int_{-1}^{2} 5t^{-2} dt$ c) $\int 3x^{1/2}\,dx$ d) $\int_{0.1}^{0.5} 4\pi r^2 dr$

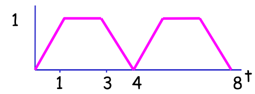
Tutorial – Week 3 – Q4, 5 and 6

3.4. Expand the right hand side of $y = (x-1)^2(x-3)$:
Find the area under the curve between x = 1 and x = 3.

3.5. Find the mean of $f(t) = t^3 - 6t^2 + 12t - 10$ between t = 1 and t = 3.

3.6. Evaluate the mean of the following periodic signals

a) 

b) 

Tutorial Week 3 Hints

3.1 Look at notes for t^3 – Remember 1 4 6 4 1

3.2 Use standard rules

3.3 Ditto

3.4 Expand out; then integrate and evaluate

3.5 Just integrate & evaluate (remember to div by 2)

3.6 Derive f(t) for each line, means will be the summation of integrals of each line.

Differentiation and Integration - 4
Prof Richard Mitchell

Today : maxima/minima and Newton Raphson

Some of this can be found in the recommended books

Croft 747-750, 755-770; James 492-494, 619-621

Stroud 388-393, 672-682; Singh 308-329, 351-356;

Don't forget to attend the tutorials to get practice

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and http://www.mathtutor.ac.uk

Two Applications of Differentiation

We know the differential and integral of polynomials

$$\frac{d\left(2t^6 - 5t^{-3}\right)}{dt} = 12*t^5 + 15*t^{-4}$$

$$\int 2t^6\, dt = \frac{2}{7}t^7 + c$$

Today two applications of differentiation

We start with maxima and minima

- useful for optimisation

Then proceed to Newton Raphson

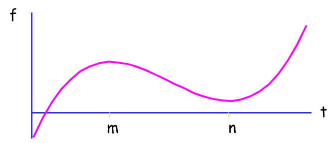
Application – Finding Max and Min

A function f(t) will have a maximum at a value t = m, if

f(t) rises before t = m, f'(t) > 0; and falls after, f'(t) < 0

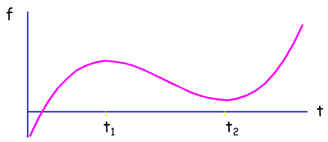
So f'(m) is 0.

If f(t) is minimum at t = n, f'(n) = 0 also



NB these are strictly local maxima and minima…

Example of finding Maxima/Minima



Here the function f is

$2t^3 - 21t^2 + 60t - 20$

Global max at t = ∞ ..

To find the local maxima and minima, solve df/dt = 0

$$\frac{df}{dt} = 6t^2 - 42t + 60 = 0$$

So solve $6(t^2 - 7t + 10) = 0$ or $6(t - 2)(t - 5) = 0$

Therefore maxima and minima occur at t = 2 and t = 5.

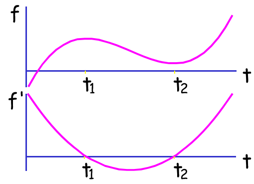
How Tell if Maximum or Minimum

Before max, slope +ve, after -ve

So at max, slope change -ve.

At min, slope change +ve

$\frac{d^2f}{dt^2} < 0$ at max, $\frac{d^2f}{dt^2} > 0$ at min



e.g. For $\frac{df}{dt} = 6t^2 - 42t + 60$ (zero at t = 2 and at t = 5)

$$\frac{d^2f}{dt^2} = \frac{d}{dt}\left(6t^2 - 42t + 60\right) = 12t - 42$$

At t = 2, the above is -18, so f is a maximum at t = 2.

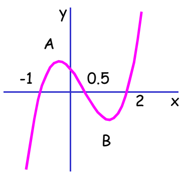
At t = 5, the above is 18, so f is a minimum at t = 5.

Another Example – last week's cubic

$y = x^3 - 1.5x^2 - 1.5x + 1$

$\frac{dy}{dx} = 3x^2 - 3x - 1.5$

$\frac{dy}{dx} = 0$ when $x = \frac{3 \pm \sqrt{9-(4*3*-1.5)}}{6}$



$x = \frac{1}{2} \pm \frac{\sqrt{27}}{6} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}$

A : x = 0.5 - 0.866 = -0.366;

B : x = 0.5 + 0.866 = 1.366

$\frac{d^2y}{dx^2} = 6x - 3$

At A: $3-3\sqrt{3} - 3 < 0$

At B: $3+3\sqrt{3} - 3 > 0$

In Class Exercise

Suppose $f = 5 + 25t + 5t^2 - t^3$

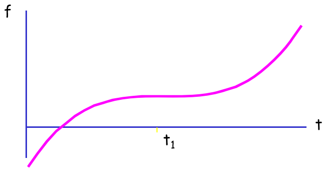
Find values of t where f is a maximum or a minimum

Answer

$\frac{df}{dt} =$

$\frac{d^2f}{dt^2} =$

Slope 0 When Neither Max Nor Min



$f(t) = t^3 - 6t^2 + 12t - 10$

$f'(t) = 3t^2 - 12t + 12$

$f''(t) = 6t - 12$

$f'(t) = 3t^2 - 12t + 12 = 3(t^2 - 4t + 4) = 3(t - 2)^2$

So $f'(t)=0$ @ $t = 2$. $f''(2) = 6*2 - 12 = 0$.

When $f'(t) = f''(t) = 0$, function has neither a max nor min.

Such a point is termed a point of inflection.

Max and Min Help to Sketch Curves

Sketch the curve $y = x^3/3 - 2x^2 + 3x$ for $0 \le x \le 4$

$\frac{dy}{dx} = x^2 - 4x + 3 = (x-1)(x-3)$ Max/Min at x = 1, x = 3

$\frac{d^2y}{dx^2} = 2x - 4$; at x = 1, $\frac{d^2y}{dx^2} < 0$, so Max; at x = 3 have Min

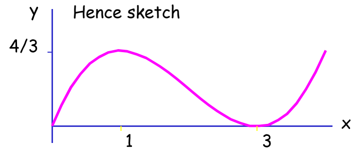
At x = 0, y = 0

And dy/dx = 3

At x = 1, y = 4/3

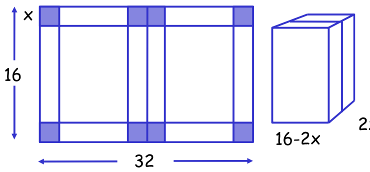
At x = 3, y = 0

Hence sketch



Max Min for Optimisation

Can often choose a value to maximise (or minimise) some function – to optimise it. e.g, maximising volume of carton made from sheet of laminated card with squares cut.



Squares (x*x) cut from card then card folded

Volume $V = (16-2x)(16-2x)*2x = 512x - 128x^2 + 8x^3$

Let's find x to maximise V

Optimisation of Carton Continued

Volume $V = 512x - 128x^2 + 8x^3$ $\frac{dV}{dx} = 512 - 256x + 24x^2$

Dividing by 8: Volume max when $64 - 32x + 3x^2 = 0$

Factorises as $(x-8)(3x-8) = 0$,

so max/min when x=8, 8/3

$\frac{d^2V}{dx^2} = -256 + 48x$ At x = 8, this is 128
At x = 8/3, this is -128

Therefore volume maximised when x = 8/3.

What is volume when x = 8 ?

Minimise Metal in Fixed Volume Can

A cylindrical metal can with volume V to be manufactured using smallest amount of metal. What is can's radius r?

Surface area = that of top πr^2 + bottom πr^2 + side $2 \pi r h$

h is height which we can remove as we know $V = \pi r^2 h$

Hence area $A = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2} = 2\pi r^2 + 2\frac{V}{r}$

So $\frac{dA}{dr} = 4\pi r - 2\frac{V}{r^2}$

This is zero when $4\pi r = 2\frac{V}{r^2}$

i.e. solution is: $r^3 = \frac{V}{2\pi}$

Newton-Raphson For Finding Roots

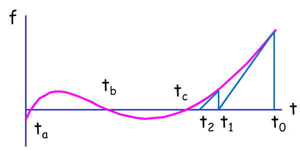
We often want to solve equations of the form f(t) = 0

If first, second order - easy; eg use $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Else: consider this,

f(t) = 0 at t_a, t_b, t_c

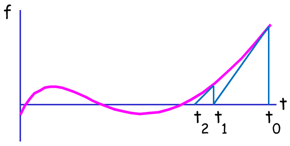
These are the three roots of the equation. How find?



Here estimate a root, by a series of guesses, t_0 t_1 & t_2.

Found using the curve's tangents & hence differentiation.

Newton-Raphson Method



Start with guess, t_0, hopefully near a root

The next guess is t_1 where tangent at t_0 meets axis

$f'(t_0) = \frac{f(t_0)-0}{t_0-t_1}$ Rearranged to $t_1 = t_0 - \frac{f(t_0)}{f'(t_0)}$

The next guess is t_2 where tangent to f(t) at t_1 meets axis

$f'(t_1) = \frac{f(t_1)-0}{t_1-t_2}$ Rearranged to $t_2 = t_1 - \frac{f(t_1)}{f'(t_1)}$

In general, the n+1th guess (using nth) $t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$

Newton-Raphson Example

$f(t) = t^3 - 4t^2 - 11t + 30$; $f'(t) = 3t^2 - 8t - 11$

Let $t_0 = 3$,

Then $t_1 = t_0 - \frac{f(t_0)}{f'(t_0)} = 3 - \frac{-12}{-8} = 1.5$

$t_2 = t_1 - \frac{f(t_1)}{f'(t_1)} = 1.5 - \frac{7.875}{-16.25} = 1.9846$

$t_3 = t_2 - \frac{f(t_2)}{f'(t_2)} = 1.9846 - \frac{0.2312}{-15.0608} = 1.99995$

$t_4 = t_3 - \frac{f(t_3)}{f'(t_3)} = 1.99995 - \frac{0.0001}{-15.001} = 2.000$

Can Be Problems If Two Roots Close

$f(t) = t^3 - t^2 - 8t + 12 = (t - 2)^2 (t + 3)$ Every 3rd Iteration:

t_n	$f(t_n)$	$f'(t_n)$	t_{n+1}
2.000000E+0	6.000000E+0	1.30000E+1	2.538461E+0
2.144514E+0	1.074405E-1	1.507800E+0	2.073258E+0
2.018513E+0	1.720025E-3	1.861595E-1	2.009273E+0
2.002321E+0	2.696214E-5	2.323238E-2	2.001161E+0
2.000290E+0	2.214546E-7	2.903458E-3	2.000145E+0
2.000036E+0	6.585563E-9	2.629231E-4	2.000018E+0
2.000004E+0	1.029001E-10	2.536525E-5	2.000002E+0

Here does not actually reach 2.0 though gets close.

Polynomial Example

Sketch $y = 2x^3-21x^2+60x-25$, x = 0 to 6, having found its maxima and minima. Hence you will find one (repeated) root of y. Use Newton-Raphson to find the other root.

$\frac{dy}{dx} = 6x^2 - 42x + 60 = 6(x^2-7x+10) = 6(x-5)(x-2)$

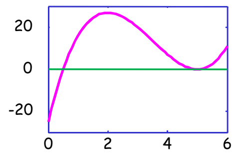
Clearly $\frac{dy}{dx}$ is 0 at x = 2 and x = 5; when y = 27 and 0

So max at x = 2, min at x = 5.

At x = 5 y = 0, so repeated root.

At x = 0, y = -25;

at x = 0 gradient = 60. So



Using Newton Raphson

$f(x) = y = 2x^3-21x^2+60x-25$; $f'(x) = \frac{dy}{dx} = 6x^2 - 42x + 60$

Repeated roots at x = 5, third root between 0 and 1

Let $x_0 = 0$, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-25}{60} = 0.4167$

$x_2 = x_1 - \frac{-3.5012}{43.4417} = 0.4971$

$x_3 = x_2 - \frac{-0.1186}{40.6053} = 0.500$

$x_4 = x_3 - \frac{-0.0002}{40.5001} = 0.500$

Could have found last root by dividing y by $x^2 - 10x + 25$

Summary

We have looked at maxima and minima and optimisation

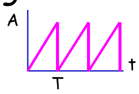
We have looked at Newton Raphson, an application of differentiation, for finding roots of equations.

Next week – we will consider exponentials and sinusoids.

Exercise – so don't forget integration

Sawtooth can be described as

$saw(t) = \frac{A}{T}t$ in the range t = 0 to T



$Mean(saw) = \frac{1}{T}\int_0^T \frac{A}{T}t\,dt =$

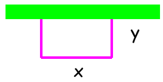
Tutorial – Week 4 – Q1 and 2

4.1. For $y = 2x^3 + 3x^2 - 72x + 5$;
Find maxima and minima.
Confirm whether each is max/min.
Sketch curve from x = -5..5.

4.2. For $y = 20 + 12x - 3x^2 - 2x^3$, find x such that $\frac{dy}{dx} = 0$.
Evaluate y at these and relevant values to sketch graph for values of x between -3 and +3. Find all roots of y - use Newton Raphson where appropriate.

Tutorial – Week 4 – Q3 and 4

4.3 100m of fence are to be built against a hedge, as shown in the figure.



Derive an expression for the area enclosed as a function of x only, and hence find what x and y should be if the enclosed area is to be maximised.

4.4. Newton-Raphson is to be used on $f(t) = t^3 - 4t^2 - 11t + 30$:
a) What is f '(t)?
b) Why would you not estimate a root starting at t = -1 ?
c) Find a root starting at t = 4 accurate to 4dp
d) Find a root starting at t = -2 accurate to 4dp

Tutorial – Week 4 – Q5 and 6

4.5 Searching sorted data is quicker than unsorted data, but it takes time to sort the data. Suppose a computer sorts n out of 100 items and searches these and the unsorted data separately. The time taken to find data is given by

$$t = 0.0250n^2 + 4.55(100-n) + 9*n^{1/2}$$

Show t is max/min at n = 1 and n = 81 respectively.

4.6 A cantilever of length L bends under its own weight.
At distance x from its base the droop y is given by

$y = k\left(x^4 - 4Lx^3 + 4L^2x^2\right)$ k is a constant

Find the value of x where the droop is maximised.

Tutorial – Week 4 – Hints

4.1 Find max/min using first and second differentials.
Calculate appropriate values and so sketch curve.
4.2 Again find max/min, evaluate and sketch.
One of these gives a repeated root. Your sketch should show a good starting value for the final root.
4.3 Find expression for y, thence one for A with x only.
Then differentiate …
4.4 Find f'(t); find f'(-1); then apply method.
4.5 Find dt/dn and d^2t/dn^2 at these values.
4.6 Find dy/dx and values of x (as function of L) where dy/dx is zero; evaluate d^2y/dx^2 at these values.

Differentiation and Integration 5
Prof Richard Mitchell

Today we will start to look at sinusoidal signals often encountered and their differential and integral

Some of this can be found in the recommended books

Croft 313-374,1139-1148; James 111-126,509-511,811-817

Stroud 249-276, 620, 825; Singh 168-222,266,363;

Don't forget to attend the tutorials to get practice

Also, extra support is available from

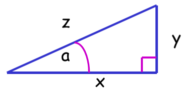
http://www.reading.ac.uk/mathssupport centre

and http://www.mathtutor.ac.uk

Trigonometric Funcs: Sin, Cos, etc

First few slides you should have met – read before lecture – I will clarify any point if needed…

Consider a right angled triangle



Basic Funcs: $\sin(a) = \frac{y}{z}$; $\cos(a) = \frac{x}{z}$; $\tan(a) = \frac{y}{x}$;

Recip. Funcs: $\text{cosec}(a) = \frac{z}{y}$; $\sec(a) = \frac{z}{x}$; $\cot(a) = \frac{x}{y}$;

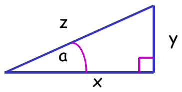
Their inverse functions are

$$a = \sin^{-1}\left(\frac{y}{z}\right) \text{ or } \cos^{-1}\left(\frac{x}{z}\right); \text{ or } \tan^{-1}\left(\frac{y}{x}\right);$$

These are sometimes called arcsin, arccos or arctan

Identities from Pythagoras

Pythagoras: $x^2 + y^2 = z^2$



Div by z^2: $\frac{x^2}{z^2} + \frac{y^2}{z^2} = 1$; so $\cos^2(a) + \sin^2(a) = 1$

Div by x^2: $1 + \frac{y^2}{x^2} = \frac{1}{x^2}$; so $1 + \tan^2(a) = \sec^2(a)$

Sin, Cos Formulae – To Note

sin(a+b) = sin(a) cos(b) + sin(b) cos(a)

so $\sin(a+\frac{\pi}{2}) = \sin(a)\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2})\cos(a) = \cos(a)$

sin(2a) = sin(a+a) = 2 sin(a) cos(a)

sin/cos graphs can show sin(-a) = -sin(a); cos(-a) = cos(a)

sin(a-b) = sin(a) cos(-b) + sin(-b) cos(a)

= sin(a) cos(b) - sin(b) cos(a)

so $\sin(a-\frac{\pi}{2}) = \sin(a)\cos(\frac{\pi}{2}) - \sin(\frac{\pi}{2})\cos(a) = -\cos(a)$

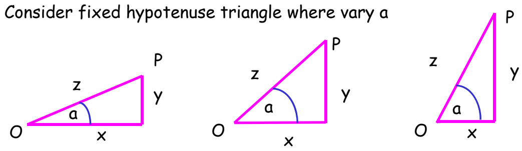
cos(a+b) = cos(a) cos(b) - sin(a) sin(b)

$\cos(2a) = \cos^2(a) - \sin^2(a)$

Other forms exist

How Sin and Cos vary with angle

Consider fixed hypotenuse triangle where vary a



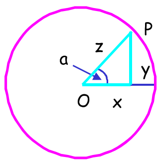
x = z cos(a) and y = z sin(a) : z fixed

cos(a), sin(a) and so x, y change with a

In effect,

P is point moving round circle origin O

So talk of 'circular functions' …



On Radians and sin varying with time

If angle in degrees, one complete cycle occupies 360^o or 2π radians

(perimeter of circle has length 2π * radius)

Engineers measure angles in radians not degrees

$\frac{1}{4}$ cycle is 90^o or $\pi/2$ radians : $\pi/6$ radians = 30^o

1 radian is $180/\pi$ degrees

Engineers have sinusoids which vary with time : sin(ωt)

frequency, f, how many times cycle repeats per second

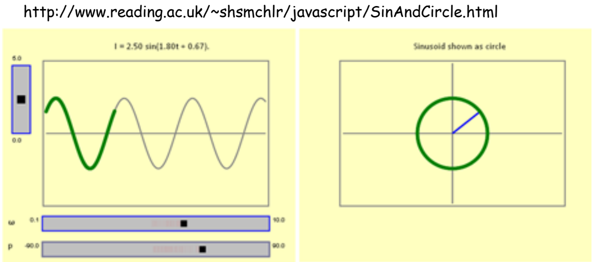
$\omega = 2\pi f$ = angular frequency : ωt is angle in radians

If f = 1; as t varies 0..1, ωt varies 0.. 2π one cycle of sin

NB sin(ωt+φ) is a sinusoid shifted in time compared with sin(ωt)

Demonstration of Circular Sin

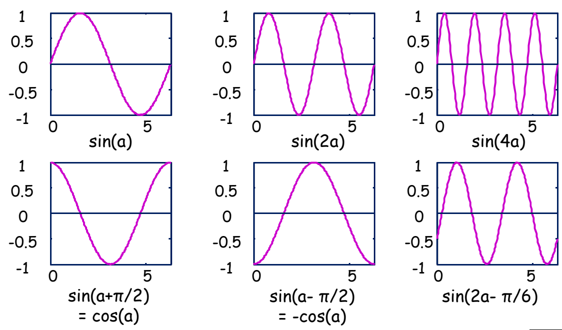
http://www.reading.ac.uk/~shsmchlr/javascript/SinAndCircle.html



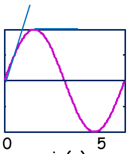
Plots K sin (ωt + p) - use sliders to adjust K, ω, p

Plot Sinusoid and going round circle at same time

Various sin(na+p)



On Differentials – add tangents



sin(a)

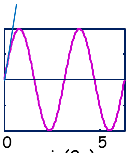
Slope sin(0) = 1

= cos(0)

Slope sin(π/2) = 0

= cos(π/2)

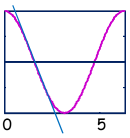
etc



sin(2a)

Slope sin(2*0) = 2

= 2 * cos(0)



sin(a+π/2)
= cos(a)

So $\frac{d(\sin(a))}{da} = \cos(a)$

Sim $\frac{d(\cos(a))}{da} = -\sin(a)$

$\frac{d(\sin(2a))}{da} = 2\cos(2a)$

$\frac{d(\sin(\omega a))}{da} = \omega\cos(\omega a)$

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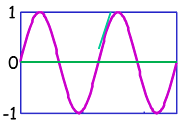
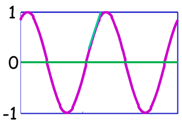
On Shifted Sinusoids





sin(ωa+b) is sin(ωa) shifted

Function and so tangent shifted

When sin shifted, tangents have same slope, but are shifted, so

$$\frac{d(\sin(\omega a + b))}{dt} = \omega * \cos(\omega a + b)$$

As integration reverse of differentiation

$$\int \cos(\omega a + b)\, da = \frac{1}{\omega}\sin(\omega a + b) + c$$

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On Diff and Int of Sin and Cos

$\frac{d(\sin(\omega a+\phi))}{da} = \omega * \cos(\omega a+\phi)$ $\frac{d(\cos(\omega a+\phi))}{da} = -\omega * \sin(\omega a+\phi)$

$\int \sin(\omega a+\phi)\, da = -\frac{1}{\omega} * \cos(\omega a+\phi)+c$ $\int \cos(\omega a+\phi)\, da = \frac{1}{\omega} * \sin(\omega a+\phi)+c$

Can also say $\frac{d(\sin(\omega a+\phi))}{da} = \omega * \cos(\omega a+\phi) = \omega * \sin(\omega a+\phi+\frac{\pi}{2})$

$$\int \sin(\omega a+\phi)\, da = -\frac{1}{\omega} * \cos(\omega a+\phi)+c = \frac{1}{\omega}\sin(\omega a+\phi-\frac{\pi}{2})+c$$

Differentiation: multiply by ω, shift by π/2;

Integration: divide by ω, shift by -π/2

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In Lecture Exercise

Find equation of tangent to 5sin(2t) at t = $\frac{\pi}{6}$

$\frac{d}{dt}(5\sin(2t)) =$

Gradient at t = $\frac{\pi}{6}$ is

Offset is

So tangent equation is

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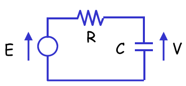
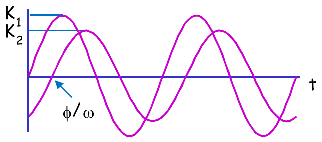
Application : Sinusoid In and Out

sin in to linear system : sin out same freq

True of RC circuit, for instance





Input, E = K_1 sin(ωt).

Output V = K_2 sin(ωt+ϕ).

Same ω, different K and phase shifted by ϕ

Measure Gain K_2/K_1

For most systems, Gain and Phase are functions of ω

For RC circuit: $\frac{K_2}{K_1} = \frac{1}{\sqrt{1+(\omega CR)^2}}$ and $\phi = -\tan^{-1}(\omega CR)$

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Sin+Cos = Phase Shifted Sin

A phase shifted sinusoid can be expressed as the sum of a sin and a cos: as follows

$$K \sin(\omega t+\phi) = K \sin(\omega t)\cos(\phi) + K\cos(\omega t)\sin(\phi)$$
$$= p \sin(\omega t) + q\cos(\omega t)$$

where p = K cos (ϕ), q = Ksin (ϕ)

Clearly $p^2 + q^2 = K^2$

Also $\frac{q}{p} = \frac{K\sin(\phi)}{K\cos(\phi)} = \tan(\phi)$ and so $\phi = \tan^{-1}\left(\frac{q}{p}\right)$

But, $\tan^{-1}$ can be problematic …

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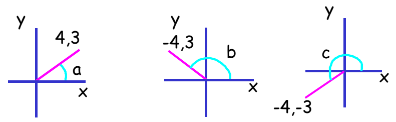
Finding $\tan^{-1}(y/x)$

Can't just calc y/x and press $\tan^{-1}$ key:



If did, a = c!

If x = 0, then $\phi = \pi/2$ else $\phi = \tan^{-1}(\text{abs}(\frac{y}{x}))$;

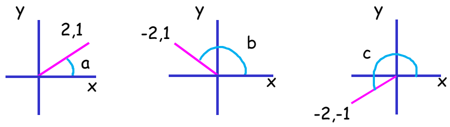
If x < 0, $\phi = \pi - \phi$;

If y < 0, $\phi = 2\pi - \phi$;

Hence arctan(x) & arctan2(y,x) often provided

Exercises

Given that tan(0.464) = 0.5, what are angles a, b and c?

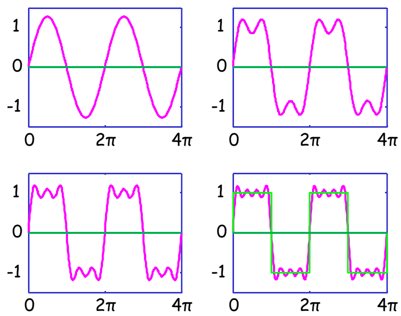


Express $\sqrt{2}\sin(a) - \sqrt{2}\cos(a)$ in the form $K\sin(a + \phi)$

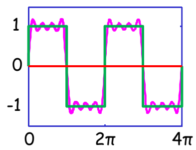
Application : On Repetitive Signals

Any signal that repeats can be represented by a sum of sinusoids of multiple frequencies.

The figure shows many sinusoids approximating a square wave

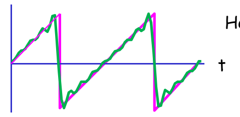


Fourier Series



The square wave was approx by the "Fourier Series":

$$\frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \frac{\sin(7t)}{7}\right)$$



Here a sawtooth is approximated

$$\sin(t) - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} - \frac{\sin(4t)}{4}$$

In general, series of sin and/or cos functions are needed

Integrals for Finding Fourier Series

A repetitive signal f(t) with period T can be expressed by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n2\pi t/T) + \sum_{n=1}^{\infty} b_n \sin(n2\pi t/T)$$

where

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n2\pi t/T)\,dt$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n2\pi t/T)\,dt$$

n is constant integer

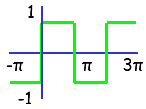
NB a_n and b_n are functions of n

Fourier Series for Square Wave

Square wave : period 2π and is -1 from −π..0, 1 from 0..π

$$\text{So } a_n = \frac{1}{\pi}\int_{-\pi}^{0} -\cos(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi}\cos(nt)\,dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} -\sin(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi}\sin(nt)\,dt$$



NB $\sin(n\pi) = 0$; $\cos(n\pi) = 1$ if n even and -1 if n odd

We expect $\frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \frac{\sin(7t)}{7}\right)$

So $a_n = 0$ and $b_n = \frac{4}{n\pi}$ if n odd, else 0

Finding a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -\cos(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi} \cos(nt)\,dt$$

$$= \left[-\frac{\sin(nt)}{n\pi}\right]_{-\pi}^{0} + \left[\frac{\sin(nt)}{n\pi}\right]_{0}^{\pi}$$

$$= \frac{1}{n\pi}[\,-\sin(0) + \sin(-n\pi) + \sin(n\pi) - \sin(0)]$$

$$= \frac{1}{n\pi}[-0 + 0 + 0 - 0] \quad \{ \text{as } \sin(n\pi) = 0 \text{ ... remember n is integer} \}$$

$$= 0$$

And b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} -\sin(nt)\,dt + \frac{1}{\pi}\int_{0}^{\pi} \sin(nt)\,dt$$

$$= \left[\frac{\cos(nt)}{n\pi}\right]_{-\pi}^{0} + \left[-\frac{\cos(nt)}{n\pi}\right]_{0}^{\pi}$$

$$= \frac{1}{n\pi}[\cos(0) - \cos(-n\pi) - \cos(n\pi) + \cos(0)]$$

$$= \frac{2(1-\cos(n\pi))}{n\pi}$$

$\cos(-n\pi) = \cos(n\pi)$

If n is even $\cos(n\pi) = 1$, so $b_n = 0$

If n is odd $\cos(n\pi) = -1$, so $b_n = \frac{2(1--1)}{n\pi} = \frac{4}{n\pi}$

Summary

In this lecture we have looked at the trigonometric functions often encountered in systems

specifically : sin, cos, tan, $\tan^{-1}$, etc.

We have argued graphically their differential

That of sin(3x) is

That of cos(4x) is

By inverse process, $\int \cos 2x\,dx$ is

We have also seen how many sinusoids approx a function

Next week we will look at exponentials and logarithms.

Tutorial – Week 5 - Q1 and 2

5.1 Given $\cos^2(x)+\sin^2(x)=1$, and $\cos(2x)=\cos^2(x)-\sin^2(x)$, derive an expression for cos(2x) not including $\sin^2(x)$.

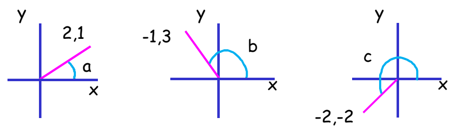
Rearranging your answer, find $\frac{d\left(\cos^2(x)\right)}{dx}$ and $\int \cos^2(x)\,dx$

5.2 For a pendulum oscillating without friction, if θ is the angle of the string from the vertical, $\theta = -0.01\sin(10t)$

a) Find $\frac{d^2\theta}{dt^2}$; b) Show that $\frac{d^2\theta}{dt^2} + 100\,\theta = 0$

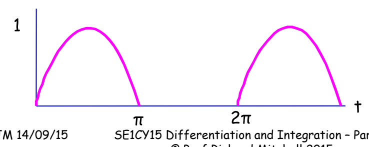
Tutorial – Week 5 – Q3, 4 and 5

5.3 Find angles a, b and c in the following (in degrees):



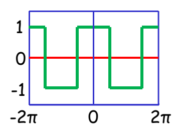
5.4 Express $\sqrt{8}\sin(\omega t) - \sqrt{8}\cos(\omega t)$ in the form $K\sin(\omega t+\phi)$

5.5 Evaluate the mean of the half wave rectified sinusoid:



Tutorial – Week 5 – Q6

5.6 Find the Fourier coefficients a_n and b_n for the square wave.



Tutorial – Week 5 – Hints

5.1 Straightforward; Rearrange to use cos(2a)

5.2 Differentiate twice, evaluate LHS of equation

5.3 Apply the algorithm for each angle

5.4 Find K and then the angle

5.5 and 5.6 Use standard integrals