Really useful mathematics

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> And what is that to me whose mind is full of indicies and surds, x squared plus 7 x plus 53 equals eleven thirds
> Lewis Carol

> ”You boil it in sawdust: you salt it in glue:
> You condense it with locusts and tape
> Still keeping one principal object in view–
> To preserve its symmetrical shape.” Lewis Carol

I. Matrix algebra

Matrices are not commutative in multiplication, i.e.

$$AB \neq BA$$

This can be determined by considering the dimensions of each matrix. For example if we write $A_{2\times 3}$ to indicate that it has 2 rows and 3 columns, then

$$A_{2\times 3} B_{3\times 5} = C_{2\times 5}$$

is realised by cancelling the inner 3's resulting in a 2 by 5 matrix.

If a matrix is square and not singular it has an inverse such that

$$AA^{-1} = A^{-1}A = I$$

Where I is the identity matrix. However some matrices are ill-conditioned so that the inverse is difficult to compute. Matrix packages such as matlab often measure a condition number to indicate whether the matrix inverse is sensitive with respect to the accuracies of the computer.

Matrices are associative so that

$$A + AB = A(I + B) \neq A + BA = (I + B)A$$

The transpose of a matrix interchanges rows and columns.
A matrix is symmetric if $B = B^T$
A matrix is orthogonal if $B^{-1} = B^T$
A matrix is positive definite if $x^T Bx$ is positive for all values of the vector x
The scalar $(Ax)^T(Ax)$ will always be zero or positive. Thus if the matrix B can be partitioned such that $B = A^T A$ it will be postive definite.
Note : stuff on Symmetric Positive definite
Note : stuff on eigen values/vectors

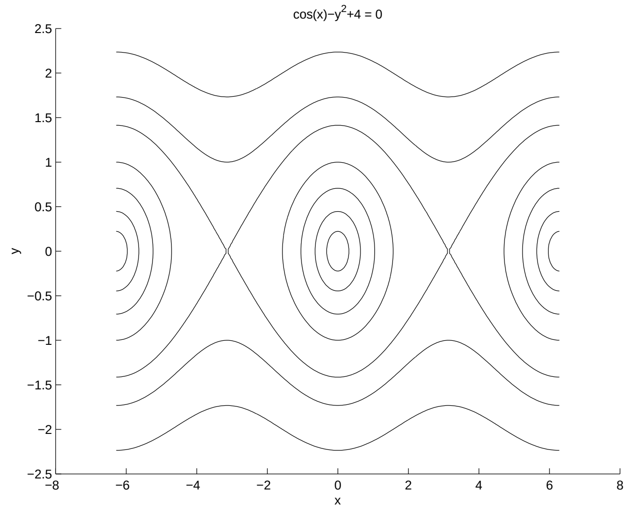

Fig. 1. Pendulum phase plane

II. 2x2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

inverse

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

eigenvalues

$$\lambda = \begin{bmatrix} 1/2\,a + 1/2\,d + 1/2\,\sqrt{a^2 - 2\,ad + d^2 + 4\,bc} \\ 1/2\,a + 1/2\,d - 1/2\,\sqrt{a^2 - 2\,ad + d^2 + 4\,bc} \end{bmatrix}$$

A. pendulum equations

Given a mass on a rod the equations of motion are

$$\ddot{\theta} l = g \sin\theta \qquad (1)$$

Using the identity $\ddot{\theta} = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}$
Integrate eqn 1 to get

$$\omega^2 = 2\frac{g}{l}\sin\theta + \omega_0^2$$

See figure 1 for phase plane

III. DISTRIBUTIONS

$\mu_X = \mathrm{E}\,(X)$

<u>Normal (gaussian) μ, σ^2</u>

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right] \\ &\qquad -\infty < x < \infty \\ \mathrm{E}X &= \mu,\ \mathrm{Var}X = \sigma^2 \\ \mathrm{mgf} &= \exp\left[t\mu + \frac{1}{2}\sigma^2 t^2\right] \end{aligned}$$

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$$

$$\mathrm{erf}(\infty) = 1$$

<u>Bernouilli</u> Let X=1 if event A occurs 0 otherwise

$$\mathrm{P}(X=1) = p,\ \mathrm{P}(X=0) = 1-p$$

ie

$$\mathrm{P}(X=k) = p^k(1-p)^{1-k} \qquad k = 0,1$$

$$\mathrm{E}X^k = \mathrm{E}X = p$$

$$\mathrm{mgf} = \mathrm{E}e^{tx} = 1 + pt + \frac{pt^2}{2!} + ... = (1-p) + pe^t$$

<u>Binomial (n,p)</u> (n independent binomial trials) (out of n things choose r)

$$\mathrm{P}(X=r) = \binom{n}{r}p^r(1-p)^{n-r} \qquad r = 0,1...n$$

$$\mathrm{E}X = np,\ \mathrm{Var}X = np(1-p)$$

$$\mathrm{mgf} = (pt + (1-p))^n$$

<u>Multinomial $(n,$p)</u> k outcomes with probability $p_1...p_k$ $\sum_i p_i = 1$

$$\mathrm{P}(X_1 = f_1 ...\mathrm{and} X_k = f+k) = \frac{n!}{f_1!...f_k!}p_1^{f_1}...p_k^{f_k}$$

where $\sum_i f_i = n$

$$\mathrm{mgf} = \left(p_1 e^{t_1} + ...p_k e^{t_k}\right]^n$$

<u>Geometric p</u>(number of failures before first success)

$$\mathrm{P}(X=r) = (1-p)^r p \qquad r = 0,1,...$$

$$\mathrm{E}X = \frac{1-p}{p},\ \mathrm{Var}X = \frac{1-p}{p^2}$$

$$\mathrm{mgf} = \frac{p}{1-t(1-p)}$$

<u>Poisson λ</u>

$$\mathrm{P}(X=r) = e^{-\lambda}\lambda^r/r! \qquad r = 0,1...$$

$$\mathrm{E}X = \lambda = \mathrm{Var}X$$

$$\mathrm{mgf} = e^{\lambda(t-1)}$$

IV. FIELDS

V. STOKES THEROM.

$$\oint \overline{F}.\vec{dl} = \int_s (\nabla\wedge\overline{F}).\vec{ds}$$

VI. DIVERGENCE THEROM (GAUSS'S LAW).

$$\oint_{\text{surface}} \overline{F}.d\vec{S} = \int_{\text{vol}} \nabla.\overline{F}dV$$

$\nabla.\nabla s = \nabla^2 s$
$\nabla.\nabla\wedge\overline{v} = 0$
$\nabla\wedge\nabla\overline{v} = 0$
$\nabla\wedge\nabla\wedge\overline{v} = \nabla(\nabla.\overline{v}) - \nabla^2\overline{v}$
what is del squared in this case?

Potentials

Scalar potential ϕ can be defined where $\nabla\wedge\overline{v} = 0$ as $v = \nabla\phi$.

Vector potential A can be defined where $\nabla.\overline{v} = 0$ as $v = \nabla\wedge\overline{A}$.

VII. MAXWELL'S EQUATIONS

Differential form of Maxwell's equations (Ramo, Whinnery and van Duzer pg 235)

$$\nabla.\overline{D} = \rho$$

$$\nabla.\overline{B} = 0$$

$$\nabla\wedge\overline{E} = -\frac{\partial\overline{B}}{\partial t}$$

$$\nabla\wedge\overline{H} = \overline{i} - \frac{\partial\overline{D}}{\partial t}$$

NTS: Need to put in the integral form as well!

Auxilary Equations,

Force on charge q moving with velocity $\vec{v}$

$$f = q(\overline{E} + \vec{v}\wedge\overline{B})$$

current density (ohms law)

$$\overline{i} = \sigma\overline{E}$$

where $\overline{i}$ is current density

$$D = \epsilon\overline{E} = \epsilon_r\epsilon_0\overline{E}$$

$$B = \mu\overline{H} = \mu_r\mu_0\overline{H}$$

also wave propogation velocity v is given by

$$v = 1/\sqrt{\epsilon\mu}$$

Retarded potentials

$$\nabla^2\phi - \mu\epsilon\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon}$$

$$\nabla^2\overline{A} - \mu\epsilon\frac{\partial^2\overline{A}}{\partial t^2} = -\mu\overline{i}$$

$$\overline{B} = \nabla\wedge\overline{A}$$

$$\overline{E} = -\nabla\phi - \frac{\partial\overline{A}}{\partial t}$$

VIII. Formulae

A. Markov process

A Markov process is one that is for any finite set of time points $l < m < n$

$$P(X_n = k|X_l = h, X_m = j) = P(X_n = k|X_m = j)$$

B. Bayes

$$P(A|B)P(B) = P(A \cap B) = P(B|A)P(A)$$

or

$$\begin{aligned} P(A, B|M) &= P(A \cap B|M) \\ &= P(A|M).P(B|A, M) \end{aligned}$$

C. Convolution

$$f_1 \otimes f_2 = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

D. Fourier Transform.

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-jwt} dt$$

$$f(\omega) = \int_{-\infty}^{\infty} F(\omega) e^{jwt} d\omega$$

DFT

$$S(k) = \sum_{n=0}^{N-1} W_N^{nk} s(n), k = 0 \rightarrow N-1, \text{where } W_X^x = e^{-j2\pi x/X}$$

E. Laplace

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds$$

Residue Theorem: If $f(z)$ is analytic within and on C except for a finite number of poles

$$\int_C f(z) dz = 2\pi i S$$

Where S is the sum of the residues at the poles within C

Residue

$$a_{-1} = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \big((z-a)^n f(z)\big)$$

Final value Theorem:

$$f(\infty) = \lim_{s \to 0} sF(s)$$

Initial value Theorem:

$$f(0^+) = \lim_{s \to \infty} sF(s)$$

$f(t)$	$F(s)$
unit impulse $\delta(t)$	1
unit impulse $\delta(t-t_0)$	e^{-st_0}
Unit step u(t)	$\frac{1}{s}$
$u(t_0 - t)$	$\frac{e^{st_0}}{s}$
ramp of kt	$\frac{k}{s^2}$
$\frac{df(t)}{dt}$	$sF(s) - f(0)$
e^{-at}	$\frac{1}{s+a}$
$e^{-t/T}$	$\frac{T}{sT+1}$
$f(at)$	$\frac{1}{a}F(\frac{s}{a})$
$f * g$	FG
$A \sin \omega t$	$\frac{A\omega}{s^2+\omega^2}$
$A \cos \omega t$	$\frac{As}{s^2+\omega^2}$

Laplace of cosine arch $1 - \cos(2\pi t/T)$

$$\frac{1}{s} - \frac{e^{-sT}}{s} - \frac{s}{s^2+\omega^2} + \frac{se^{-sT}}{s^2+\omega^2}$$

Note : Could generate this in maple with the latex command, need terms for second order systems, plus can we space out the table contents

F. z-transform

$$F(z) = \sum_{n=-\infty}^{\infty} f[n] z^{-n}$$

where $z = e^{j\omega t}$

Shifting Theorem. if $f[n] \leftrightarrow F(z)$
then $f[n-m] \leftrightarrow z^{-m} F(z)$

Convolution Theorem. if $f_1[n] \leftrightarrow F_1(z)$ and $f_2[n] \leftrightarrow F_2(z)$ then $f_1 \otimes f_2 \leftrightarrow F_1 F_2$

Final value Theorem:

$$f[\infty] = \lim_{z \to 1} (z-1) F(z)$$

Initial value Theorem:

$$f[0] = \lim_{z \to \infty} F(z)$$

Not yet available

G. State space equations

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} + \mathbf{Du} \end{aligned}$$

The continuous solution is then

$$Y = C(sI - A)^{-1} Bu$$

which in the time domain becomes

$$x(t) = \Phi(t) x(0) + \int_0^t \Phi(t-\tau) Bu(\tau) d\tau$$

for time invariance

$$\Phi(t) = e^{At}$$

state space of a 2nd order system

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\varsigma\omega_n \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u \\ y &= [1\ 0]\mathbf{x} + [0]u\end{aligned}$$

$$\omega = \sqrt{\frac{K}{M}}$$

$$\varsigma = \sqrt{\frac{B^2}{4KM}}$$

Note : Specify this as a sampled system

$$\Phi = e^{A\triangle}$$

H. Padé

Delays in a continuous system. If

$$L(t) = C(t-\tau)$$

then

$$L(s) = C(s)e^{-s\tau}$$

An approximation can be made for time delays using the padé formula.

$$e^{-s\tau} \approx \frac{1-s\tau/2}{1+s\tau/2}$$

(eg Richards pg 270)

Check matlab or wikipedia (probably) for more terms.

Leads to approximation for sampled data systems

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

Zero order hold for a signal $y(s) \mapsto y(z)$ is

$$G_0 = \frac{1-z^{-1}}{s}$$

(Brogan)

I. Lyapunov stability

A Lyapunov Function is of the form $V = V(x)$ for a non-linear dynamic system that can be expressed as

$$\dot{x} = f(x)$$

If

•V is a scalar function with continuous 1st derivative, and

•$\dot{V} \leq 0$ for all trajectories, then B is a subspace of the state space where $\dot{V} = 0$. All system trajectories tend to largest invariant set $\in B$

J. Light source scattering model

$$\frac{P_r}{P_o} = \frac{k\cos^n\theta}{r^3}$$

Brightness $= R\cos(i)I + W(i)\cos^n(s)$ First term corresponds to Lamberts Law. In second term n=1 is rough, n=10 is smooth.

Not yet available

K. Velocity and acceleration

${}^A r_p$ is measured in frame A

The velocity ${}^A\dot{r}_p$ is measured in frame A

ω is the rotational velocity of B measured in frame A

$$^A\dot{r}_p = \dot{r}_{org} + \omega \wedge\ {}^A_BR\ {}^Br_p +\ {}^A_BR\ {}^B\dot{r}_p \qquad \text{(craig 5:13)}$$

if R=I this reduces to the formula in the cued data book

Recursion formula is

$$^{i+1}\omega_{i+1} =^{i+1}_i R\ {}^i\omega_i + \dot{\theta}_{i+1}\ i+1^{\hat{z}}_{i+1}$$

$$^{i+1}v_{i+1} =^{i+1}_i R({}^iv_i +^i \omega_i \wedge^i P_{i+1})$$

(craig 5:45 and 5:47)

$$\begin{aligned}{}^A\ddot{r}_p &= \ddot{r}_{org} +\ {}^A_BR\ {}^B\ddot{r}_p + \dot{\omega}\wedge {}^A_BR\ {}^Br_p \\ &\quad +\omega\wedge(\omega\wedge {}^A_BR\ {}^Br_p) + 2\omega\wedge {}^A_BR\ {}^B\dot{r}_p\end{aligned}$$

(craig 6:10)

for planar motion $\omega \wedge (\omega \wedge r_p)$ becomes $-\omega^2 r_p$

(in craig $\omega =\ {}^A\Omega_B$)

$$^A\dot{\Omega}_C =^A \dot{\Omega}_B + {}^A_BR\ {}^B\dot{\Omega}_C +^A \Omega_B \wedge {}^A_BR\ {}^B\Omega_C$$

(craig 6:15)

L. Cayley's formula

$$A = [I-B]^{-1}[I-B]$$

M. Rodrigues formula:

For spacial displacement $X - x = b\wedge(X + x - 2c) + ds$ using screw axis $L = c + ds$ (McCarthy pg 21)

N. Newton-Euler

$$F = m\ddot{x}$$

$$T = I\dot{\omega} + \omega \wedge I\omega$$

where

$$\omega = \frac{d\theta}{dt}$$

IX. Log arithmetic

$$\log(AB) = \log(A) + \log(B)$$
$$\log(A/B) = \log(A) - \log(B)$$
$$\log(-A) = \log(A) + \log(-1)$$
$$\log(-A) = \log(A) + j(1+2n)\pi \text{for integer n}$$

Adding logs can be done by assume $A > B$ and by then approximating the series expansion (see [thesis ref]) which provides a uselful technique for computations over a large range so as to avoid machine rounding errors.

$$\log(A+B) = \log(A) + \log(1 + B/A)$$

The Mercator series for $\ln 1 + x$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad \text{for} \quad |x| \leq 1 \quad \text{unless} \quad x = -1$$

$$\log(A-B) = \log(A) + \log(1 - B/A)$$

A. Multiple regression:

For N random variables $(y = mx + c)$ of the form $Y = BX$ where

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, X = \begin{bmatrix} 1 & x_1^T \\ \vdots & \vdots \\ 1 & x_N^T \end{bmatrix} B = \begin{bmatrix} c \\ m \\ B_3 \\ \vdots \\ B_p \end{bmatrix},$$

one best estimate is

$$\hat{B} = (X^T X)^{-1} X^T Y$$
$$\hat{\sigma}^2 = \frac{1}{N}(Y - X\hat{B})^T (Y - X\hat{B})$$

(Annette Dobson 1983 pg 47)

X. Conversions and constants

14.7 psi	=	100kPa
1m.p.h.	=	0.45 m/s
μ_0	=	$4\pi \times 10^{-7}$ henrys/metre
ϵ_0	=	8.854×10^{-12} farads/metre
c (speed of light)	$\approx$	3×10^8 m s^{-1} (2.998)
c	=	$1/\sqrt{(\mu_0 \epsilon_0)}$ m/s
1 hopper[1]	=	distance light goes in 1 nS (about 300mm)
G	=	(Gravitational constant)
speed of sound in air	$\approx$	300m/s
γ gamma water	=	$25 - 71$ erg/cm^2 perhaps

A. EM Spectrum

Infra red light is $\approx 720 - 940$ nM.

XI. References

Craig
McCarthy Introduction to Theoretical Kinematics 1990
Dobson 1983
Brogan