State space equations, the important stuff

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1. INTRODUCTION

Both continuous and sampled time systems are considered in state space. The continuous time state space equations are usually given in the general form as

$$\begin{aligned}\dot{\underline{x}} &= A(\underline{x}) + B(\underline{u})\\ \underline{y} &= C(\underline{x}) + D(\underline{x})\end{aligned}$$

(The sampled time variant should become obvious!). In the following we will consider only linear systems where A,B and C are matrices, and only strictly proper systems where $D = 0$. (That is a system where all the gain of the system tends to zero as the frequency approaches infinity). We will only consider SISO systems so that y and u are scalars and the length of the vector $\underline{x}$ is the order of the system. Thus the equation will be of the form

$$\begin{aligned}\dot{\underline{x}} &= A\underline{x} + Bu\\ y &= C\underline{x}\end{aligned}$$

The general solution to the first of the above equations is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{1}$$

Likewise in the sampled time domain we can describe systems as functions of the sample number n so they are in the form

$$\begin{aligned}\underline{x}_{n+1} &= \mathsf{G}\underline{x}_n + \mathsf{H}u_n\\ y_n &= C\underline{x}_n\end{aligned}$$

Put into the s domain to get

$$\begin{aligned}s\underline{\mathsf{x}} - \underline{x}(0) &= A\underline{\mathsf{x}} + B\mathsf{u}\\ \mathsf{y} &= C\underline{\mathsf{x}}\end{aligned}$$

and z domain we get

$$\begin{aligned}z\underline{\mathsf{x}} - \underline{x}(0) &= \mathsf{G}\underline{\mathsf{x}} + \mathsf{H}\mathsf{u}\\ \mathsf{y} &= C\underline{\mathsf{x}}\end{aligned}$$

Note: in most cases x, y, z, etc are implied from context as either $x(t)$ (continuous time), x_n (sampled time), $x(s)$ (Laplace domain), $x(z)$ (z domain). Check the TEX source for confirmation.

We will consider some of the following continuous SISO (single input single output) transfer functions.

First order

$$\frac{y}{u} = \frac{K}{s+a} \tag{2}$$

Second order

$$\frac{y}{u} = \frac{K}{s^2+as+b} \tag{3}$$

$$\frac{y}{u} = \frac{K}{(s+p_1)(s+p_2)} \tag{4}$$

Second order with zero

$$\frac{y}{u} = \frac{K(s+r)}{s^2+as+b} \tag{5}$$

Third order

$$\frac{y}{u} = \frac{K(s+r)}{s^3+as^2+bs+c} \tag{6}$$

$$\frac{y}{u} = \frac{K(s^2+rs+q)}{s^3+as^2+bs+c} \tag{7}$$

or defined with poles as

$$\frac{y}{u} = \frac{K}{(s+p_1)(s+p_2)(s+p_2)} \tag{8}$$

etc

2. STATE-SPACE

We can put the above transfer functions into state-space form. Trivially for first order (eq 2) that is

$$\begin{aligned} \dot{x} &= [-a]x + Ku \\ y &= x \end{aligned} \tag{9}$$

2.1. Controlable canonical form

For second and higher orders we can arrange the states in several ways, the first is the controlable canonical form.

The all pole second order system of eq 3 is thus expressed in controlable canonical form as

$$\begin{aligned} \underline{\dot{x}} &= \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ K \end{bmatrix} u \\ y &= [1 \quad 0] \underline{x} \end{aligned} \tag{10}$$

The single zero given in system 5 changes the output equation to

$$y = [1 \quad r] \underline{x}$$

Third order Controlable canonical form

$$\begin{aligned} \underline{\dot{x}} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} u \\ y &= [1 \quad 0 \quad 0] \underline{x} \end{aligned} \tag{11}$$

The single zero shown in system 6 changes the output equation to

$$y = [1 \quad r \quad 0] \underline{x}$$

2.2. Observable canonical form

Two examples of observable canonical form are

<u>Second order</u> (system 3)

$$\begin{aligned} \underline{\dot{x}} &= \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix} \underline{x} + \begin{bmatrix} K \\ 0 \end{bmatrix} u \\ y &= [0 \quad 1] \underline{x} \end{aligned}$$

<u>Third order</u> (system 7)

$$\begin{aligned} \underline{\dot{x}} &= \begin{bmatrix} 0 & 0 & -c \\ 1 & 0 & -b \\ 0 & 1 & -a \end{bmatrix} \underline{x} + \begin{bmatrix} K \\ Kr \\ Kq \end{bmatrix} u \\ y &= [0 \quad 1] \underline{x} \end{aligned}$$

2.3. Diagonal form

This is based on pole zero form equations.

system 4

$$\begin{aligned} \underline{\dot{x}} &= \begin{bmatrix} -p_1 & 0 \\ 0 & -p_2 \end{bmatrix} \underline{x} + \begin{bmatrix} K \\ K \end{bmatrix} u \\ y &= [p_1 - r \quad -p_2 + r] \underline{x} \end{aligned}$$

We can write system 6,7 or 8 in the form

$$\frac{y}{u} = \frac{c_1}{(s+p_1)} + \frac{c_1}{(s+p_1)} + \frac{c_1}{(s+p_1)}$$

We can then put it into diagonal form as

$$\begin{aligned} \underline{\dot{x}} &= \begin{bmatrix} -p_1 & 0 & 0 \\ 0 & -p_2 & 0 \\ 0 & 0 & -p_3 \end{bmatrix} \underline{x} + \begin{bmatrix} K \\ K \end{bmatrix} u \\ y &= [c_1 \quad c_2 \quad c_3] \underline{x} \end{aligned}$$

Where there are multiple repeated poles then a more general variant known as Jordan Canonical form is used.

3. CONVERSION TO SAMPLED TIME SYSTEM

We can use a zero order hold to find the equivalent sampled data system to a particular state space continuous time system.

Broadly the method uses the solution to $\dot{x} = Ax$ which is $x(t) = e^{At}x(0)$ (ignoring input for the moment) by setting $x(0)$ to x_n and using a sampling time of Δ we can calculate $x_{n+1} = e^{A\Delta}x_n$ and hence see that $G = e^{At}$. The more general conversion is via the equation in the form $P = e^{F\Delta}$

Where $F = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ and $P = \begin{bmatrix} \mathsf{G} & \mathsf{H} \\ 0 & 0 \end{bmatrix}$

We will use the obvious approximation for matrix exponent where $e^x = I + x + x^2/2! + x^3/3!....$ Alternatively a Padé approximation might be considered. But any matrix exponent approximations need to keep the matrix norm small to avoid calculation errors (See Moler and Van Loan 2003).

3.0.1. Sampled state version of the control canonical form

We can use the second order state space system 10 as

$$A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \tag{12}$$

$$B = \begin{bmatrix} 0 \\ K \end{bmatrix} \tag{13}$$

Expanding two terms of the exponential is trivial so the digital equivalent is

$$\mathsf{G} = \begin{bmatrix} 1 & \Delta \\ -\Delta\, b & 1 - \Delta\, a \end{bmatrix} \tag{14}$$

$$\mathsf{H} = \begin{bmatrix} 0 \\ \Delta\, K \end{bmatrix} \tag{15}$$

Three term exponential expansion is

$$\mathsf{G} = \begin{bmatrix} 1 - 1/2\,\Delta^2 b & \Delta - 1/2\,\Delta^2 a \\ -\Delta\, b + 1/2\,\Delta^2 ab & 1 - \Delta\, a - 1/2\,\Delta^2 b + 1/2\,\Delta^2 a^2 \end{bmatrix} \tag{16}$$

$$\mathsf{H} = \begin{bmatrix} 1/2\,\Delta^2 K \\ \Delta\, K - 1/2\,\Delta^2 aK \end{bmatrix} \tag{17}$$

Now for a third order system The two term expansion is

$$\mathsf{G} = \begin{bmatrix} 1 & \Delta & 0 \\ 0 & 1 & \Delta \\ -\Delta\, c & -\Delta\, b & 1 - \Delta\, a \end{bmatrix} \tag{18}$$

and the three term expansion is

$$\mathsf{G} = \begin{bmatrix} 1 & \Delta & 1/2\,\Delta^2 \\ -1/2\,\Delta^2 c & 1 - 1/2\,\Delta^2 b & \Delta - 1/2\,\Delta^2 a \\ -\Delta\, c + 1/2\,\Delta^2 ac & -\Delta\, b - 1/2\,\Delta^2 c + 1/2\,\Delta^2 ab & 1 - \Delta\, a - 1/2\,\Delta^2 b + 1/2\,\Delta^2 a^2 \end{bmatrix} \tag{19}$$

$$\mathsf{H} = \begin{bmatrix} 0 \\ 1/2\,\Delta^2 K \\ \Delta\, K - 1/2\,\Delta^2 aK \end{bmatrix} \tag{20}$$

If Δ is sufficiently small then the matrix norm will be small and a two or three term expansion should surfice.

3.1. Sampled space pole zero (diagonal) form

$$A = \begin{bmatrix} -p_1 & 0 \\ 0 & -p_2 \end{bmatrix} \tag{21}$$

$$B = \begin{bmatrix} K \\ K \end{bmatrix} \tag{22}$$

The generalised digital system for diagonal form is easier to recreate since all the terms are on the diagonal and relate to the exponent expansion Hence we go straight to the four term expansion

$$\mathsf{G} = \begin{bmatrix} 1 - \Delta\, p_1 + 1/2\, \Delta^2 {p_1}^2 - 1/6\, \Delta^3 {p_1}^3 & 0 \\ 0 & 1 - \Delta\, p_2 + 1/2\, \Delta^2 {p_2}^2 - 1/6\, \Delta^3 {p_2}^3 \end{bmatrix} \tag{23}$$

$$\mathsf{H} = \begin{bmatrix} \Delta\, K - 1/2\, \Delta^2 p_1\, K + 1/6\, \Delta^3 {p_1}^2 K \\ \Delta\, K - 1/2\, \Delta^2 p_2\, K + 1/6\, \Delta^3 {p_2}^2 K \end{bmatrix} \tag{24}$$

Since $\mathsf{G} = \begin{bmatrix} e^{-\Delta P_1} & 0 \\ 0 & e^{-\Delta P_2} \end{bmatrix}$ and $\mathsf{H} = \Delta K \begin{bmatrix} e^{-P_1 K} \\ e^{-P_2 K} \end{bmatrix}$

4. EXAMPLES

4.1. A mass spring damper system

Two forms are discussed, the first is typified by a piano string in that a force (eg from a hammer, drumstick, bow, finger) is used to impart a force (often impulse like) to the system. This is shown in figure 1, but note that the force shown is the force the system imparts on the world! In this category are highly oscillatory systems such as any musical instrument that can be hit, plucked, tooted or bowed; as well as damped systems such as thermal heat dissipation.

The second is the shock absorber where a displacement at one end of the system (ie the road) results in displacement at the other (ie the driver and passengers), and is shown in figure 2.

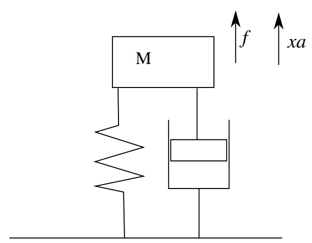

Figure 1. mass-spring-damper (Piano string)

4.1.1. State space of a 2nd order system

For the piano string model, the upward forces on mass m are $-kx_a - b\dot{x}_a + f$ so from Newton we get

$$-kx_a - b\dot{x}_a + f = m\ddot{x}$$

This we can put into CCF by writing $\dot{x}_2 = \ddot{x}_a$ ie $x_2 = \dot{x}_a$ thus emphasising the vector nature of $\underline{x}$

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} f \\ y &= [1\ 0] \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \end{aligned}$$

or by substituting

$$\omega = \sqrt{\frac{k}{m}}$$

$$\varsigma = \frac{b}{2m\omega}$$

we can see it is equivalent to 12

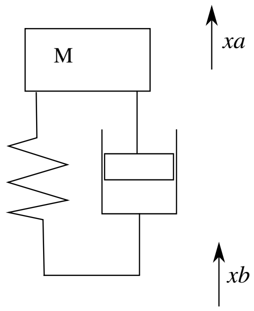

Figure 2. mass-spring-damper (Shock absorber)

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\varsigma\omega_n \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u$$

For the shock absorber model, the upward forces on mass m are $k(x_b - x_a) + b(\dot{x}_b - \dot{x_a})$ so from Newton we get

$$k(x_b - x_a) + b(\dot{x}_b - \dot{x}_a) = m\ddot{x}$$

Hence the state space equations are

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{b}{m} \end{bmatrix} f \\ y &= [1 \quad k/b] \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \end{aligned}$$

4.2. The Harris and Wolpert Human model 1998

The transfer function used by Harris and Wolpert is probably of the form $1/s(Js+b)(sT_1+1)(sT_2+1)$ In state space this would be

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{b}{T_1\,T_2\,J} & \left(-1 - \frac{b(T_1+T_2)}{J}\right) {T_1}^{-1}{T_2}^{-1} & -\frac{T_1+T_2}{T_1\,T_2} - \frac{b}{J} \end{bmatrix} \tag{25}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{T_1\,T_2\,J} \end{bmatrix} \tag{26}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \tag{27}$$

Or in diagonal form

$$NewA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -{T_1}^{-1} & 0 & 0 \\ 0 & 0 & -{T_2}^{-1} & 0 \\ 0 & 0 & 0 & -\frac{b}{J} \end{bmatrix} \tag{28}$$

$$NewB = \begin{bmatrix} b^{-1} \\ -\frac{1}{(T_2-T_1)T_1\,(bT_1-J)} \\ \frac{1}{T_2\,(JT_1-JT_2-bT_1\,T_2+bT_2{}^2)} \\ \frac{b^2}{J(bT_2-J)(bT_1-J)} \end{bmatrix} \tag{29}$$

$$NewC = \begin{bmatrix} 1 & -T_1{}^3 & -T_2{}^3 & -\frac{J^3}{b^3} \end{bmatrix} \tag{30}$$

Putting the control canonical form into state space with a two term expansion gives.

$$A = \begin{bmatrix} 1 & \Delta & 0 & 0 \\ 0 & 1 & \Delta & 0 \\ 0 & 0 & 1 & \Delta \\ 0 & -\frac{\Delta b}{T_1T_2J} & \frac{-\Delta}{T_1T_2}\left(\frac{1+b(T_1+T_2)}{J}\right) & -\Delta(1+\frac{T_1+T_2}{T_1T_2}+\frac{b}{J}) \end{bmatrix} \tag{31}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta + (1+\frac{T_1+T_2}{T_1T_2}+\frac{b}{J})\Delta^2 \end{bmatrix} \tag{32}$$

H and W give the following values for the arm $J = .25Kgm^2$ $b = .2Nmsrad^{-1}$ $T1 = 0.04s$ $T_2 = 0.03s$

4.3. Fly fishing rod

As can be seen from the figure on page 198 of ref 3 :-) and also in Figure 3 we can string mass spring damper systems together. We can form the state vector as $\underline{x} = [x_1 \quad \ldots \quad x_n \quad \dot{x}_1 \quad \ldots \quad \dot{x}_n]^T$. The general equation then becomes $\ddot{x}_i = (k_ix_{i-1} + b_i\dot{x}_{i-1} - (k_i + k_{i+1})x_i - (b_i + b_{i+1})\dot{x}_i + k_{i+1}x_{i+1} + b_{i+1}\dot{x}_{i+1})/m_i$ and so an example of a 4 node system would be

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -(k_1+k_2)/m_1 & k_2/m_1 & 0 & 0 & -(b_1+b_2)/m_1 & b_2/m_1 & 0 & 0 \\ k_2/m_2 & -(k_2+k_3)/m_2 & k_3/m_2 & 0 & b_2/m_2 & -(b_2+b_3)/m_2 & b_3/m_2 & 0 \\ 0 & k_3/m_3 & -(k_3+k_4)/m_3 & k_4/m_3 & 0 & b_3/m_3 & -(b_3+b_4)/m_3 & b_4/m_3 \\ 0 & 0 & k_4/m_4 & -k_4/m_4 & 0 & 0 & b_4/m_4 & -b_4/m_4 \end{bmatrix}$$

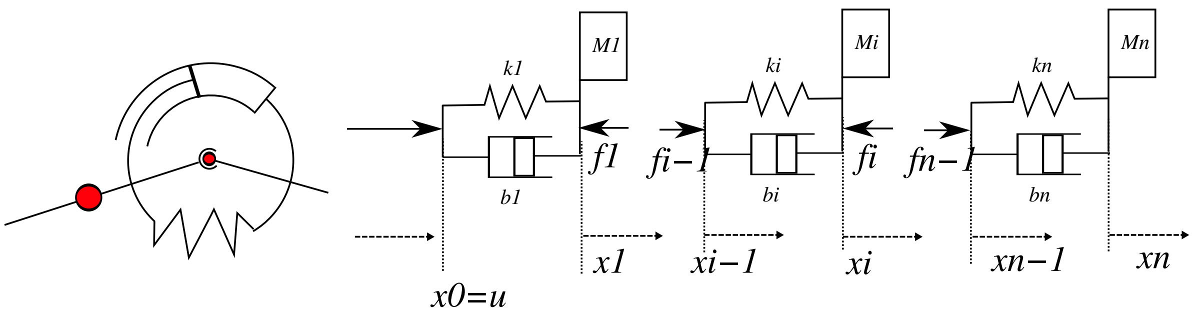

Figure 3. mass-spring-damper: Fly fishing rod

While casting we can ignore gravity!

We can use matlab or octave to generate graphcs of frequency response, step response etc.

5. CHANGING THE STATES

Let J be any square invertable matrix. We can use J to change the states around (scaling, mixing and swapping). let $x = Jx'$ so $J\dot{x}' = AJx' + Bu$, $y = CJx'$. The equivalent state space equation is then $\dot{x} = J^{-1}AJx + J^{-1}Bu$, $y = CJx$.

Now if we know the Eigen vectors and values (D,diag(λ_n) and V) for A we can use J=V (provided V is invertable and A is diagonisable) to transform the system into $\dot{x} = diag(\lambda_i)x + V^{-1}u$, $y = CVx$.

6. EIGEN VALUES AND VECTORS

Recall that eigen values and vectors for any matrix A are those vectors for which the equation $\lambda v = Av$ is true. This can be written in matrix form as $AV = VD$ eg for a 2x2 matrix this would be $A[\,v_1 \quad v_2\,] = [\,v_1 \quad v_2\,][\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}]$

We can again use the system given in eqn 10 and by writing ω^2 for b and $2\zeta\omega$ for a the state space equations become

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\,\omega\,(\zeta) \end{bmatrix} \tag{33}$$

$$B = \begin{bmatrix} 0 \\ K \end{bmatrix} \tag{34}$$

The eigen values are then

$$D = \begin{bmatrix} \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega & 0 \\ 0 & \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega \end{bmatrix} \tag{35}$$

and eigen vectors are

$$V = \begin{bmatrix} -\frac{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega+2\,\omega\,(\zeta)}{\omega^2} & -\frac{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega+2\,\omega\,(\zeta)}{\omega^2} \\ 1 & 1 \end{bmatrix} \tag{36}$$

so the transformed equation has $A = D$ (ie the system poles), $B = \begin{bmatrix} -K \\ K \end{bmatrix}$ and $C = [\,1 \quad 1\,]$ For the sampled data system the Eigen values of G for the two term expansion are thus as follows Two element dig eigen values/vectors

$$D = \begin{bmatrix} 1 - \Delta\,p_1 + 1/2\,\Delta^2 {p_1}^2 - 1/6\,\Delta^3 {p_1}^3 & 0 \\ 0 & 1 - \Delta\,p_2 + 1/2\,\Delta^2 {p_2}^2 - 1/6\,\Delta^3 {p_2}^3 \end{bmatrix} \tag{37}$$

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{38}$$

We can easily check stability against the unit circle

From which it is obvious that the poles are $\begin{bmatrix} e^{-p_1\Delta} & 0 \\ 0 & e^{-p_2\Delta} \end{bmatrix}$

7. CONCATINATION OF SYSTEMS

Given two state space systems in series it is relatively easy to show that the state space of the combined system is

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} A_2 & B_2C_1 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ B_1 \end{bmatrix} u_1 \tag{39}$$

$$y_2 = [\,C_2 \quad 0\,] \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \tag{40}$$

8. THE END

Available on the Internet as URL= `http://www.rdg.ac.uk/~shshawin/dnb`. Comments and corrections to W.S. Harwin, (pick up the address or email from the web page). You are free to make as many copies as needed and use this material in any way you see fit as long as it is not illegal, immoral, or likely to endanger life or property. Notes are currently used in Mechatronics module of Cybernetics degree.

9. REFERENCES

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