Quaternion Rotation for robotics and graphics coordinate transformation

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> Quaternions came from Hamilton and have been an unmixed evil to those who have touched them in any way. Vectors are a useless survival and have never been of the slightest use to any creature. *Lord Kelvin*

Quaternions are beginning to find favour for describing robot kinematics and computer graphics/games. Their advantage over rotation matrices is that they require fewer multiply adds. Their disadvantage over homogenous transforms is that you can't quickly simply group coordinate rotation-translation sequences and multiply them together. In this case one can use the 7 element dual-quaternion where the the dual part gives the vector to the new frame origin [Mcc90], but in most cases the convenience of homogenous transforms wins out. Quaternions still provide a useful analysis and a possible implimentation tool for some robotic and graphic applications.

Quaternions are given by a scalar and a vector ($\mathcal{R}^4$) and their algebra forms a field encompassing both complex arithmetic and vectors. So

$$\begin{array}{lrcl} \text{let} & q & = & q_0 + \bar{\mathbf{q_v}} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \\ \text{and} & r & = & r_0 + \bar{\mathbf{r_v}} = r_0 + \mathbf{i}r_1 + \mathbf{j}r_2 + \mathbf{k}r_3 \end{array}$$

Quaternion multiplication is not commutative thus $qr \neq rq$

$$\begin{aligned} qr &= (q_0 + \bar{\mathbf{q_v}})(r_0 + \bar{\mathbf{r_v}}) \\ &= (q_0 r_0 - \bar{\mathbf{q_v}}.\bar{\mathbf{r_v}}) + (q_0 \bar{\mathbf{r_v}} + r_0 \bar{\mathbf{q_v}} + \bar{\mathbf{q_v}} \wedge \bar{\mathbf{r_v}}) \end{aligned}$$

or by multiplying out in full, subject to

$$\begin{array}{lrcl} & \mathbf{i}^2 & = & \mathbf{j}^2 = \mathbf{k}^2 = -1 \\ \text{and} & \mathbf{ijk} & = & -1 \end{array}$$

The conjugate is defined as

$$q^* = q_0 - \bar{\mathbf{q_v}} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3$$

and the squared norm as

$$|q| = q_0^2 + \bar{\mathbf{q_v}}.\bar{\mathbf{q_v}} = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

A unit quaternion is any quaternion that has a norm of 1.

The quaternion inverse is defined as

$$qq^{-1} = q^{-1}q = 1$$

that is

$$q^{-1} = \frac{q_0 - \bar{\mathbf{q_v}}}{q_0^2 + \bar{\mathbf{q_v}}.\bar{\mathbf{q_v}}}$$

$$= \frac{q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3}{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

which sometimes expressed as the conjugate over the squared norm.

Rotation about a unit vector $\bar{\mathbf{v}}$ given by the direction cosines (l,m,n) through an angle (θ) is given by

$$q' = rqr^{-1}$$

where

$$\begin{aligned} r &= \cos(\theta/2) + \mathbf{i}\mathrm{l}\sin(\theta/2) + \mathbf{j}\mathrm{m}\sin(\theta/2) + \mathbf{k}\mathrm{n}\sin(\theta/2) \\ &= \cos(\theta/2) + \sin(\theta/2)\bar{\mathbf{v}} \end{aligned}$$

since r is a unit quaternion $r^{-1} = r^*$

expanding this generalised rotation in vector notation

$$\begin{aligned} rqr^* &= q_0(r_0^2 + \bar{\mathbf{r_v}}.\bar{\mathbf{r_v}}) + (r_0^2 - \bar{\mathbf{r_v}}.\bar{\mathbf{r_v}})\bar{\mathbf{q_u}} + 2r_0(\bar{\mathbf{r_v}} \wedge \bar{\mathbf{q_u}}) + 2(\bar{\mathbf{q_u}}.\bar{\mathbf{r_v}})\bar{\mathbf{r_v}} \\ \text{if } q_0 = 0 & \\ &= (r_0^2 - \bar{\mathbf{r_v}}.\bar{\mathbf{r_v}})\bar{\mathbf{q_u}} + 2r_0(\bar{\mathbf{r_v}} \wedge \bar{\mathbf{q_u}}) + 2(\bar{\mathbf{q_u}}.\bar{\mathbf{r_v}})\bar{\mathbf{r_v}} \end{aligned}$$

note that if r is a rotation

$$(r_0^2 - \bar{\mathbf{r_v}}.\bar{\mathbf{r_v}}) = \cos(\theta)$$

i.e.

$$rqr^* = \cos(\theta)\bar{\mathbf{q_v}} + \sin(\theta)(\bar{\mathbf{v}} \wedge \bar{\mathbf{q_v}}) + (1 - \cos(\theta))(\bar{\mathbf{q_u}}.\bar{\mathbf{v}})\bar{\mathbf{v}}$$

The above equation is an interesting alternative way to rotate a vector, ie by scaling and summing the three vectors, formed by the rotation axis, the vector to be transformed and the cross product of the pair. Computationally however it is less efficient than homogenous transforms.

In the following $w = r_0, x = r_1, y = r_2, z = r_3$. Convert a quaternion to a rotation matrix as follows

$$R = \begin{Bmatrix} (w^2 + x^2 - y^2 - z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & (w^2 - x^2 + y^2 - z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & (w^2 - x^2 - y^2 + z^2) \end{Bmatrix}$$

or

$$R = \begin{Bmatrix} (1 - 2y^2 - 2z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & (1 - 2x^2 - 2z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & (1 - 2x^2 - 2y^2) \end{Bmatrix}$$

Euler rotation ϕ about Z followed by θ about X followed by ψ about Z is given by the quaternion

$$r = \cos(\theta/2)\cos(\phi/2+\psi/2)+\sin(\theta/2)\cos(\phi/2-\psi/2)i+\sin(\theta/2)\sin(\psi/2-\phi/2)j+\cos(\theta/2)\sin(\phi/2+\psi/2)k$$

Coordinate frame rotation

If the coordinate frame is rotated through θ about (l,m,n),the vector expressed in the new **ijk** coordinate frame is given by

$$q' = r^{-1}qr$$

Because quaternion algebra is only now beginning to regain favour in engineering mathematics there has been little published since Brand in 1947[Bra47].

However Rooney is perhaps the best work on quaternion algebra in recent times[Roo77, Roo78].

Theorem: [Har91] If two quaternions r_1 and r_2 have the same or opposite directions then $r_1 r_2 = r_2 r_1$. This is true because the only non commutative term in the vector expansion $(\bar{\mathbf{v}} \wedge \bar{\mathbf{u}})$ is zero.

References

[Bra47] Louis Brand. *Vector and Tensor Analysis*, chapter 10. Wiley, 1947.

[Roo77] J. Rooney. A survey of representations of spatial rotation about fixed point. *Environment and Planning B*, 4:185–210, 1977.

[Roo78] J. Rooney. A comparison of representations general spatial screw displacement. *Environment and Planning B*, 5:45–88, 1978.

[Har91] W.S. Harwin. Computer Recognition of the Unconstrained and Intentional Head Gestures of Physically Disabled People. *PhD thesis, University of Cambridge*, appendix A, 1991

[Mcc90] J.M. McCarthy An introduction to theoretical kinematics *MIT Press*, 1990