Matrix quick reference guide/databook etc

Matrix and vector definitions

A matrix is a two dimensional array that contains expressions or numbers. In general matrices will be notated as bold uppercase letters, sometimes with a trailing subscript to indicate its size.

For example:

$$\mathbf{A}_{4\times 3} = \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

is a matrix with 4 rows and 3 columns containing 12 articles (the location of which is given by the article subscript).

A vector is a collection of symbols or numbers, and can be represented as a matrix with a single column. Vectors are sometimes notated as a bold lowercase symbol, or with a short line or arrow that may be below or above the symbol.

For example

$$\mathbf{b} = \vec{b} = \bar{b} = \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = [b_1, b_2, b_3, b_4]^T$$

is a vector with 4 elements. Note that ^{T} denotes transpose.

Matrix operations

1. Addition:

$\mathbf{C} = \mathbf{A} + \mathbf{B}$ such that $c_{ij} = a_{ij} + b_{ij}$ ($\mathbf{A}$ and $\mathbf{B}$ must be the same size)

2. Transpose:

The transpose of a matrix interchanges rows and columns.

$\mathbf{C} = \mathbf{A}^T$ such that $c_{ij} = a_{ji}$

3. Identity matrix:

The identity matrix $\mathbf{I}$ is square and has 1s on the major diagonal, elsewhere 0s.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

4. Multiplication:

The definition of matrix multiplication is $\mathbf{C} = \mathbf{AB}$ such that $c_{ij} = \sum_k a_{ik} b_{kj}$

(The number of columns of $\mathbf{A}$ must be the same as number of rows of $\mathbf{B}$)

Matrices are associative so that

$$A + AB = A(I + B) \neq A + BA = (I + B)A$$

Matrices are not commutative in multiplication, i.e.

$$AB \neq BA$$

Example: let

$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

Then $\mathbf{C} = \mathbf{AB} = \mathbf{A}_{4\times3}\mathbf{B}_{3\times2}$

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

Subscript notation

To work out if two matrices will multiply it is useful to subscript the matrix variable with row and column information, thus

This can be determined by considering the dimensions of each matrix. For example if we write $A_{2\times3}$ to indicate that it has 2 rows and 3 columns, then

$$A_{2\times3}B_{3\times5} = C_{2\times5}$$

is realised by cancelling the inner 3's resulting in a 2 by 5 matrix.

5. Multiplication of partitioned matrices

Any matrix multiplication can also be carried out on sub-matrices and vectors as long as each sub-matrix/vector is a valid matrix calculation. (You can colour in the individual articles in the equation above to demonstrate.)

For example if $A = \left[\ \mathbf{D} \mid \underline{f}\ \right]$ and $B = \left[\begin{array}{c} \mathbf{G} \\ \hline \underline{h}^T \end{array}\right]$ then

$$AB = \begin{bmatrix} \mathbf{DG} \\ \underline{f}\underline{h}^T \end{bmatrix}$$

6. Matrix inverse:

If a matrix is square and not singular it has an inverse such that

$$AA^{-1} = A^{-1}A = I$$

Where I is the identity matrix.

$\mathbf{AA}^{-1} = \mathbf{I}$

$\mathbf{A}^{-1}$ exists only if $\mathbf{A}$ is square and not singular. $\mathbf{A}$ is singular if the determinant $|\mathbf{A}| = 0$.

A simple example is

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = ad - bc$$

$$\mathbf{A}^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{|\mathbf{A}|} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$$

Computation of the inverse can be done by computing the matrix determinant of $|A|$, the matrix of minors, the co-factor, and the ajoint/adjugate. Thus if X is the ajoint/ajugate of A

$$XA = AX = I|A|$$

Computing the determinant and the ajoint requires computing a matrix of minors and hence the co-factor matrix. The co-factor matrix is then simply the transpose of the ajoint/ajugate.

Note on numerical computation

Some matrices are do not have independent rows so do not have an inverse. This can be tested by computing a matrix rank.

Some matrices are ill-conditioned so that the inverse is difficult to compute. A measure called the condition number, defined as the ratio of the maximum singular to the minimum singular value, indicate whether the matrix inverse will be sensitive with respect to the accuracy of the computer.

In Matlab the condition number can be calculated as

```
>> max(svd(A))/min(svd(A))
```

or via the cond command

Note on Cayley-Hamilton Theorem

The Cayley-Hamilton Theorem says that 'Every square matrix satisfies its own characteristic equation.'

The Characteristic equation is the solution to

$$|sI - A| = 0$$

This gives a way to calculating the inverse of A by substituting into the characteristic equation and multiplying by A^{-1}.

Algebra rules :

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (Addition is commutative)

$\mathbf{AB} \neq \mathbf{BA}$ (Multiplication not commutative)

$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ (Associative)

$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ (Associative)

$\mathbf{AI} = \mathbf{AI} = \mathbf{A}$

$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ (for non-singular and square matrix)

$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Matrix decomposition

Matrix types

$\mathbf{C} = \mathbf{C}^T$ such that $c_{ij} = c_{ji}$ (If $\mathbf{C}$ is symmetric it must be square)
A matrix is symmetric if $B = B^T$
A matrix is skew symmetric if $B = -B^T$
A matrix is orthogonal if $B^{-1} = B^T$
A matrix is positive definite if $x^T Bx$ is positive for all values of the vector x
The scalar $(Ax)^T(Ax)$ will always be zero or positive. Thus if the matrix B can be partitioned such that $B = A^T A$ it will be positive definite.

Orthogonal matrices

If a vector is transformed by an orthogonal matrix then Euclidean metrics are conserved. That is if $L(x) = x^T x$ and $y = Ax$ then evidently $L(y) = y^T y = (Ax)^T Ax = x^T A^T Ax$

If the measures are conserved then $L(x) = L(y)$ so $A^T A = I$ and hence orthogonality requires $A^T = A^{-1}$

Eigenvalues and Eigenvectors

$$|\lambda I - A| = 0$$

- If A is real then complex Eigenvalues, if they occur, will be as complex conjugate pairs.
- A symmetric matrix has only real Eigenvalues and the Eigenvectors are orthogonal.

Given the Eigenvalues, the Eigenvectors are such that

$$AV = V\text{diag}\lambda_i$$

Where the columns of V are the Eigenvectors and diag lambda is a diagonal matrix of the eigenvalues.

If A is symmetric and $[V, D]$ are the Eigenvectors and diagonalised Eigenvalues such that $AV = VD$. Then since $V^{-1} = V^T$, $A^{-1} = V^T D^{-1} V$ and since D is diagonal the inverse is now trivial...

Rank vs Eigenvalues and Eigenvectors

Singular value decomposition

For a matrix A we can partition A such that $A = UDV^T$ where $U^T U = I$, $V^T V = I$ and D is diagonal. The diagonal elements of D are known as singular values σ

$$\sigma_i = \sqrt{\lambda_i(A^T A)}$$

where $\lambda_i(A^T A)$ is are the Eigenvalues of $A^T A$

See [2] for a good overview.

Transfer functions

Interesting identity

$$A(I + A)^{-1} = (I + A)^{-1} A$$

Things to do with 2x2 matrices

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

determinate $|A| = ad - bc$

inverse

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \left[\begin{array}{cc} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array} \right]$$

Eigenvalues

$$\lambda = \left[\begin{array}{c} 1/2\,a + 1/2\,d + 1/2\,\sqrt{a^2 - 2\,ad + d^2 + 4\,bc} \\ 1/2\,a + 1/2\,d - 1/2\,\sqrt{a^2 - 2\,ad + d^2 + 4\,bc} \end{array} \right]$$

Useful Matlab functions

A*B A+B A\B A/B	matrix multiplication, addition and division (left and right)
eye(n)	create an identity matrix size n
svg(A)	compute the singular value decomposition matrices of A
eig(A)	compute the Eigenvalues and Eigenvectors of A
rank(A)	compute rank (number of independent rows or columns in A)
cond(A)	compute condition number (an indication of numerical error in the inverse)
det(A)	compute determinant
inv(A)	compute inverse
pinv(A)	compute pseudo inverse = $(AA^T)^{-1}A^T$
adjoint(A)	compute adjoint/adjugate
eigshow	demonstration of Eigenvalues, (can drag the x vector)

References

[1] Cleve B. Moler. *Numerical Computing with MATLAB*. SIAM, 2004. DOI: 10.1137/1.9780898717952. URL: https://uk.mathworks.com/moler/index_ncm.html.

[2] Cleve Moler. *Professor SVD*. http://www.mathworks.com/tagteam/35906_91425v00_clevescorner.pdf. 2006.

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