Normal Hyperbolicity and Robust Synchronization

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Abstract— We demonstrate, using examples of chaos synchronization, that normal hyperbolicity is a necessary and sufficient condition for synchronization manifold to be smooth and persistent under small perturbations. The same examples, however, show that in real applications, when parameter mismatch is not arbitrary small, normal hyperbolicity is *not* sufficient to give quantitative bounds for deformations of the synchronization manifold. In other words, we show that even in the case of normal hyperbolicity two almost identical systems may cause large synchronization errors.

I. INTRODUCTION

Synchronization between dynamical systems has been an active research field in all scientific and technical disciplines since the time of Huygens. It is a phenomenon of interest to fields ranging from celestial mechanics to laser physics, from electronics to communications, and from biophysics to neuroscience [1]. In every-day life, synchronization means correlated in-time behavior between different processes. Indeed, according to the Oxford Dictionary, the most general definition of synchronization is "to agree in time" and "to happen at the same time" [2].

Synchronization in chaotic dynamics [3] has attracted much attention during the last years because of its role in understanding the basic features of man-made and natural systems. Natural language for description of chaos synchronization is in terms of invariant, stable, and robust manifolds [4, 5]. In other words, only synchronization phenomena that are described with stable and robust manifolds can be observed in physical experiments. The physical notion of a robust phenomenon contains two separate issues: persistence under small perturbations of the system(s) and persistence under small noise. In this paper we address only the question of synchronization that is robust with respect to small perturbations of the dynamical systems involved.

II. ROBUSTNESS AND NORMAL HYPERBOLICITY

In order to be physically meaningful and experimentally observable any synchronization phenomenon has to be robust, i.e. the synchronization manifold $M = \{(\boldsymbol{x},\boldsymbol{y}) : \boldsymbol{x} = \boldsymbol{y}\}$ and its stability properties have to be persistent with respect to (small) perturbations of the dynamical systems involved. Therefore, we not only have to ask "Under what condition is M a stable manifold?" but also "Under what condition is M persistent under perturbations?"

In this paper we address the second question that may be answered in terms of normal hyperbolicity [6, 7, 8]. The conditions for stability and normal hyperbolicity can be expressed in terms of Lyapunov exponents. Let m be the dimension of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ and $\mathcal{A}$ be a chaotic attractor ($\mathcal{A} \subset M$). Furthermore, let ρ be an ergodic invariant measure supported in $\mathcal{A}$. Then there exist m tangential LEs (equal to the LEs of $\mathcal{A}$ considered as an attractor of a dynamical system restricted to M) and m normal LEs. We write $\lambda_{max}(\rho)$ for the largest normal LE and $\mu_{min}(\rho)$ for the smallest tangential LE. We define $\lambda_{max} = \sup \cup_{\rho\in E}\lambda_{max}(\rho)$, $\lambda_{min} = \inf \cup_{\rho\in E}\lambda_{max}(\rho)$, $\mu_{max} = \sup \cup_{\rho\in E}\mu_{min}(\rho)$, and $\mu_{min} = \inf \cup_{\rho\in E}\mu_{min}(\rho)$, where E is the set of all ergodic invariant probability measures supported in $\mathcal{A}$. The invariant manifold is stable iff

$$\lambda_{max} < 0 \tag{1}$$

and the stable manifold is normally hyperbolic if

$$\lambda_{max} < \mu_{min}. \tag{2}$$

If the contraction towards the synchronization manifold is sufficiently strong and if this is true for all trajectories embedded in the chaotic attractor $\mathcal{A}$, the manifold is persistent under perturbations.

We now illustrate the importance of normal hyperbolicity for chaos synchronization using two examples.

III. BAKER MAP

In the literature on identical chaos synchronization stability of the invariant manifold is the only requirement for synchronization. This is due to the fact that for identical synchronization, the synchronization manifold, $\boldsymbol{x} = \boldsymbol{y}$, is smooth even when it is not stable. We show now, through an example, that slight mismatch of the parameters in the case $\lambda_{max} > \mu_{min}$ may cause the invariant manifold to loose its smoothness and become a fractal set. For our example,

$$\begin{aligned} \boldsymbol{x}(n+1) &= \boldsymbol{f}[\boldsymbol{x}(n)], \\ \boldsymbol{y}(n+1) &= \tilde{\boldsymbol{f}}[\boldsymbol{y}(n)] + \mathbf{h}[x(n), \boldsymbol{y}(n)], \end{aligned} \tag{3}$$

we use $\boldsymbol{f}$ and $\tilde{\boldsymbol{f}}$ to be the baker map

$$\begin{aligned} f_1[x_1, x_2] &= \begin{cases} \alpha x_1 & \text{if} \quad x_2 < a_1 \\ \alpha + \beta x_1 & \text{if} \quad x_2 \geq a_1 \end{cases} \\ f_2[x_1, x_2] &= \begin{cases} x_2/a_1 & \text{if} \quad x_2 < a_1 \\ (x_2 - a_1)/a_2 & \text{if} \quad x_2 \geq a_1. \end{cases} \end{aligned}$$

with different parameters and the feedback linearization coupling

$$\mathbf{h}[\boldsymbol{x}, \boldsymbol{y}] = \tilde{\boldsymbol{f}}[\boldsymbol{x}] - \tilde{\boldsymbol{f}}[\boldsymbol{y}] + A(\boldsymbol{y} - \boldsymbol{x}).$$

As we shall see below the dynamics restricted to the invariant manifold is governed by $\boldsymbol{f} = (f_1, f_2)$, and therefore we denote the LEs of $\boldsymbol{f}$ with μ (tangential LEs). For the baker map, a_1, a_2, α and β are positive real numbers (parameters) such that $\alpha \leq \beta$, $\alpha + \beta = 1$, $a_1 + a_2 = 1$ and $\alpha \neq a_1$. The chaotic attractor of the baker map has a natural invariant measure that is uniform in x_2 and varies wildly in the x_1 direction. It is easy to see that the LEs of this attractor with respect to the natural measure are $\mu_1(\rho_{nat}) = a_1 \ln(1/a_1) + a_2 \ln(1/a_2) > 0$ and $\mu_2(\rho_{nat}) = a_1 \ln \alpha + a_2 \ln \beta < 0$ [9]. In addition one can show that for all ergodic measures ρ, $\ln \alpha \leq \mu_2(\rho) \leq \ln \beta$. Hence, the smallest negative LE of the baker map is given by $\mu_{min} = \ln \alpha$, and the largest negative LE by $\mu_{max} = \ln \beta$.

We choose A to be a stable 2×2 matrix. Using this active passive decomposition [10] of the drive system we obtain a response system that synchronizes globally with the drive. We further assume that $a_{12} = 0$ so that the eigenvalues of A are its diagonal elements. We also assume (without loosing generality) that $0 < a_{11} < a_{22} < 1$. If we define $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{x}$, then the dynamics of (3) can be rewritten as

$$\begin{aligned} \boldsymbol{x}(n+1) &= \boldsymbol{f}[\boldsymbol{x}(n)], \\ \boldsymbol{z}(n+1) &= A\boldsymbol{z} + \tilde{\boldsymbol{g}}[\boldsymbol{x}(n)], \end{aligned} \tag{4}$$

where $\tilde{\boldsymbol{g}} = \tilde{\boldsymbol{f}} - \boldsymbol{f}$. Since $\tilde{\boldsymbol{g}}$ is a bounded function, we write $\tilde{\boldsymbol{g}} = \epsilon \boldsymbol{g}$, where $\epsilon = \max \tilde{\boldsymbol{g}}$. Normal LEs are $\ln a_{11}$ and $\ln a_{22}$, while tangential LEs are those of the baker map. Condition (2) can be rewritten as $a_{22} < \alpha$.

In the ideal case, when the parameters of drive and response are the same the synchronization manifold $\boldsymbol{z} = 0$ is globally stable. If the parameters of the drive and the response system are different, then the synchronization manifold takes the more general form of a generalized synchronization (GS) manifold. One can show that the GS manifold is in this case given by the function

$$\boldsymbol{z} = \epsilon B^{-1} \sum_{j=1}^{\infty} C^{j-1} B \boldsymbol{g}[\boldsymbol{f}^{-j}[\boldsymbol{x}]] \tag{5}$$

where B and $C = diag(a_{11}, a_{22})$ are 2×2 matrices with $b_{11} = b_{22} = 1$, $b_{12} = 0$, and $b_{21} = a_{21}/(a_{22} - a_{11})$. We have shown elsewhere [5] that for $a_{22} < \alpha$ Eq. (5) describes a smooth manifold and we observe robust synchronization (see Fig. 1a). On the other hand, for $a_{11} > \beta$ the manifold (5) has a fractal structure, as can be seen in the Fig. 1b.

We now ask the question: "How small should the perturbation (mismatch of the parameters) be so that the perturbed manifold is close to the original (unperturbed)"? Normal hyperbolicity guarantees only persistence and smoothness, but says nothing about the actual deformation of the manifold due to a perturbation. Using example (4) we shall demonstate that even for fixed, small normal LEs this deformation may become arbitrary large depending on the coupling. The unperturbed manifold is $\boldsymbol{z} = 0$. In order to study the deformation of this manifold we estimate $\max|\boldsymbol{z}|$ for $a_{21} \gg 1$ to be $\max|\boldsymbol{z}| \sim \epsilon a_{21}$. The parameter ϵ is a measure of the parameter mismatch or perturbation and we assume that it is small (but not arbitrary small, because we want to apply this reasoning also to experimental systems). If the parameter becomes a_{21} large the GS manifold (5) deviates significantly from the IS manifold and we observe large synchronization errors (for our system this error grows linearly with a_{21}) although a_{21} *doesn't* influence the normal contraction rates (and LEs) ! Therefore even in the case when the synchronization manifold is normally hyperbolic, the systems may possess a parameter that influences very strongly the synchronization (manifold). To demonstrate this effect we show in Fig. 2 the same graph as in Fig. 1a but now for $a_{21} = 50$. As can be seen the manifold remains smooth (as we expect it, because condition (2) is fulfilled) but deviates significantly from the IS manifold $M = \{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x} = \boldsymbol{y}\}$. Note that in this case $b_{21} = 2500$ and the eigenvectors $[1, -b_{21}]$ and $[0, 1]$ of the matrix A are almost parallel. This type of degeneracy is the deeper reason for the strong sensitivity of the synchronization manifold on parameter mismatch.

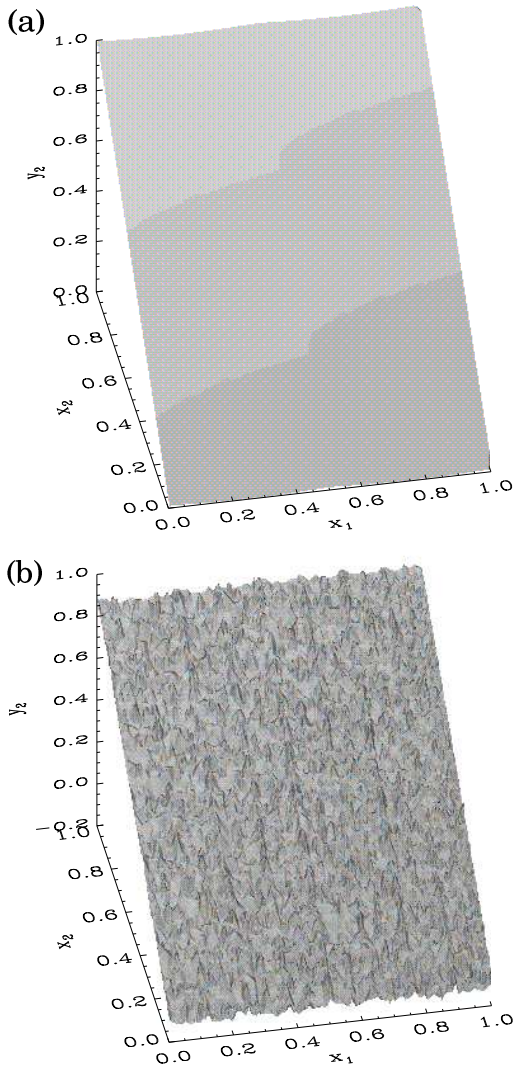

Figure 1: Graph y_2 vs. (x_1, x_2) of the synchronization manifold of the system (3) of two coupled baker maps (3). Parameter values are $\alpha = 0.5$, $\beta = 0.5$, $a_1 = 0.45$ $a_2 = 0.55$, $\tilde{\alpha} = 0.49$, $\tilde{\beta} = 0.51$ and $a_{21} = 1$. (a) Smooth manifold for $a_{11} = 0.1$ and $a_{22} = 0.12$ (b) Fractal manifold for $a_{11} = 0.8$ and $a_{22} = 0.82$.

IV. LORENZ SYSTEM

For our continuous example we use a Lorenz system

$$\begin{aligned} \dot{x}_1 &= \sigma(-x_1 + x_2) \\ \dot{x}_2 &= \rho x_1 - x_2 + x_1 x_3 \\ \dot{x}_3 &= -b x_3 + x_1 x_2 \end{aligned} \tag{6}$$

that drives another Lorenz system with different parameters,

$$\begin{aligned} \dot{y}_1 &= \tilde{\sigma}(-y_1 + y_2) + g_1[\boldsymbol{x}, \boldsymbol{y}] \\ \dot{y}_2 &= \tilde{\rho} y_1 - y_2 + y_1 y_3 + g_2[\boldsymbol{x}, \boldsymbol{y}] \\ \dot{y}_3 &= -\tilde{b} y_3 + y_1 y_2 + g_3[\boldsymbol{x}, \boldsymbol{y}] \end{aligned} \tag{7}$$

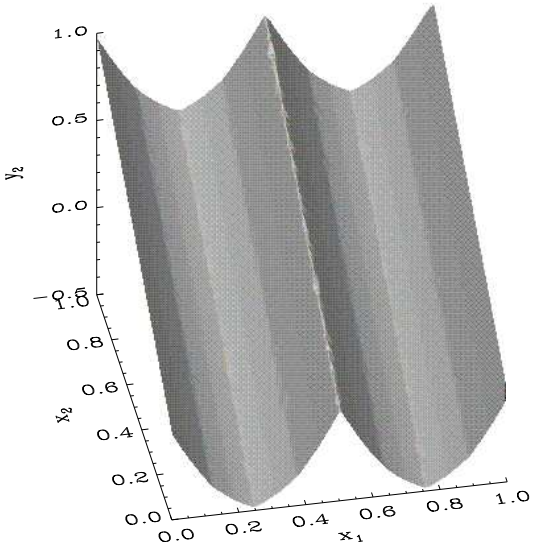

Figure 2: Graph y_2 vs. (x_1, x_2) of the synchronization manifold of the system (3) of two coupled baker maps (3). Parameter values are $\alpha = 0.5$, $\beta = 0.5$, $a_1 = 0.45$ $a_2 = 0.55$, $\tilde{\alpha} = 0.49$, $\tilde{\beta} = 0.51$, $a_{11} = 0.1$ and $a_{22} = 0.12$ and $a_{21} = 50$. The manifold is smooth due to normal hyperbolicity but deviates strongly from the identical synchronization manifold $\mathbf{x} = \mathbf{y}$ (compare Fig. 1a).

where $\mathbf{g}[\boldsymbol{x}, \boldsymbol{y}]$ again denotes a feedback linearization coupling

$$\begin{aligned} g_1[\boldsymbol{x}, \boldsymbol{y}] &= k_1(x_1 - y_1) + \tilde{k}(x_3 - y_3) \\ g_2[\boldsymbol{x}, \boldsymbol{y}] &= x_1 x_3 - y_1 y_3 + k_2(x_2 - y_2) \\ g_3[\boldsymbol{x}, \boldsymbol{y}] &= x_1 x_2 - y_1 y_2 + k_3(x_3 - y_3). \end{aligned} \tag{8}$$

In the case of identical parameters of drive and response the dynamics of the error $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$ is linear, $\dot{\boldsymbol{z}} = A\boldsymbol{z}$, where A is a 3×3 matrix with $a_{11} = -\sigma - k_1$, $a_{12} = \sigma$, $a_{13} = -\tilde{k}$, $a_{21} = \rho$, $a_{22} = -1 - k_2$, $a_{33} = -b - k_3$ and $a_{23} = a_{31} = a_{32} = 0$. The eigenvalues of this matrix are given by

$$\begin{aligned} \mu_{1,2} &= 0.5\left[a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 - 4a_{12}a_{21}}\right] \\ \mu_3 &= a_{33}. \end{aligned}$$

and do *not* depend on $\tilde{k}$. The normal LEs can be expressed in terms of the eigenvalues μ_i and tangential LEs are those of the Lorenz system. Therefore, the synchronization manifold $\boldsymbol{x} = \boldsymbol{y}$ is globally stable if we choose the constants k_i such that all eigenvalues μ_i of A have negative real parts. The eigenvectors of A are

$$[1, \frac{a_{21}}{\mu_i - a_{22}}, 0],$$

where $i = 1, 2$ and

$$\left[1, \frac{a_{21}}{\mu_3 - a_{22}}, \frac{1}{\tilde{k}}\left(\mu_3 - a_{11} - \frac{a_{12}a_{21}}{\mu_3 - a_{22}}\right)\right]$$

and depend, in contrast to the eigenvalues, on $\tilde{k}$. In particular for very large values of $\tilde{k}$ the third eigenvectors converges to the subspace spanned by the first two eigenvectors. This asymptotic degeneracy is similar to the baker map, and we expect to see large deviation from the synchronization manifold for slightly different parameters even in the case when the synchronization manifold is globally stable and normally hyperbolic. That this indeed is the case, is shown in figure 3 where the root mean square synchronization error of x_1 and y_1 is plotted versus the off-diagonal element $\tilde{k}$. As can be seen the synchronization error due to the parameter mismatch increases almost linearly with $\tilde{k}$ (although the normal contraction rates (Lyapunov exponents) are constant !). For this simulation we use the parameter values $\sigma = 10, \tilde{\sigma} = 10.1, \rho = 28, \tilde{\rho} = 28.1, b = 2.66, \tilde{b} = 2.8, k_1 = 30, k_2 = 40$ and $k_3 = 15$ for which condition (2) for normal hyperbolicity is fulfilled.

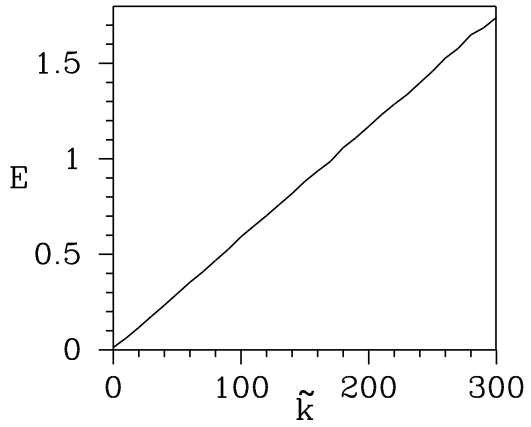

Figure 3: Mean synchronization error E vs. coupling parameter $\tilde{k}$.

V. CONCLUSION

In this paper the problem of robustness of synchronization (manifolds) with respect to (small) perturbations of the underlying dynamical systems has been addressed. It turns out that two different aspects have to be distinguished: (i) persistence of qualititive features like existence or smoothness of the manifold and (ii) boundedness of (quantitative) deformations of the manifold. Both robustness features have been found to be independent from each other. Robustness in the first sense is guaranteed for normally hyperbolic systems and depends on normal contraction rates that can be measured in terms of normal and tangential Lyapunov exponents. The (non-) boundedness of deformations of the manifold does not depend on the contraction rates but is related to contraction (and expansion) directions. Since for many physical systems and applications not only smoothness of the synchronization manifold but also quantitative deviations and differences are of importance normal hyperbolicity seems in this sense not to be a sufficient condition for robust synchronization.

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