MODELING AND NONLINEAR NOISE REDUCTION FOR SPATIO–TEMPORAL SYSTEMS

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Abstract— To analyse spatio–temporal time series (STTS) many tools have been developed. Recent work ([7, 6, 1]) has shown that the concept of *local states* is a powerful method to analyse STTS. It turns out that this concept enables the application of time series analysis methods originally developed for ordinary (not spacially extended) dynamical systems to time series analysis concerning spatio–temporal systems. In this paper we will discuss the special task of noise reduction in STTS.

I. Introduction

A spatio–temporal evolution of a real quantity s can be described by considering s as a function of space and time. Such spatio–temporal evoluting quantities are subject to data analysis in almost every branch of science. So many powerful techniques have been developed concerning data analysis of spatio–temporal evolutions, i.e. to describe the properties of the unknown functional relationship between s and space and time. Usually values of s are given as *snapshots*, i.e. a temporal sequence of spacial patterns $\{\mathbf{s}^n\}_{n=1...N}$. Each pattern $\mathbf{s}$ consists of M elements (pixels) and is therefore represented as a M-dimensional vector with elements s^n_m $(m = 1, ..., M)$. These pixels are considered as a spatial sample of s taken instantaneously at a time instant t_n.

Most of the techniques applied so far to STTS are based on decompositions into spatial modes that constitute orthogonal bases in a high dimensional vector space. Examples for such basic modes are the well known Fourier modes, wavelets or bases that are computed from the data for example by means of a Karhunen–Loève transformation (KLT) (also called Proper Orthogonal Decomposition, Singular Value Decomposition, Empirical Orthogonal Eigenfunctions) [9]. The STTS $\{\mathbf{s}^n\}_{n=1...N}$ is projected onto these spatial modes and the resulting projection coefficients contain all essential information about the dynamics. In particular for the KLT very often only a few coefficients are significantly different from zero and therefore the underlying process can be described in a low dimensional subspace (see Ref. [8] for a succesful application of this approach).

However, not always can a linear decomposition into basic modes yield a low-dimensional description of the data, even in cases, where the STTS is governed by a low dimensional attractor. If the temporal evolution consists for example of structures that move periodically along the spatial axes it is not possible to describe this process using a few *linear* modes only. In these cases, *any* linear decomposition can therefore *not* provide an efficient description of the dynamics [5]. modeling a STTS can also be viewed as a system identification task and only recently a promising approach for extracting PDEs from data has been presented by Voss et al. [11] and Bär et al. [3].

An alternative to decomposition into global linear modes or identifying global nonlinear models is the construction of local models. The main idea behind this approach is to consider s as a spatial function (i.e. a map from the space into the reals), which is subject to dynamical laws in its entity. This is, the change of s is a functional of s itself:

$$\frac{\partial s}{\partial t} = \mathcal{L}s,$$

where $\mathcal{L}$ is an operator acting on spatial functions. Furthermore we require $\mathcal{L}$ to be a *local* law, i.e. the change of s on an arbitrary small part of the space is determined by the values of s on this small part. F.e. partial derivatives w.r.t. spatial variables are local. Finally we require $\mathcal{L}$ to be homogenous, i.e. $\mathcal{L}$ commutes with the spatial shift. This means that the dynamical law according to which s evolves is the same on the whole space. This assumption of homogeneity is of course not fulfilled by typical real world experimental systems (for example due to boundary effects) but it may be viewed as a good approximation for many cases. A generalization for nonhomogeneous systems is possible and will be discussed briefly in the conclusion.

As a first step the (local) state of the system in a small region of space is represented by a vector $\mathbf{x}$. As soon as the local states are (re)constructed they can be used for subsequent analysis or, as in the example given below, for noise reduction of the STTS.

In the following we will discuss only the case of (spatially) one dimensional spatial pattern. Generalizations for higher dimensional cases are straight forward.

II. Reconstruction of local states

The entire STTS may be represented by a $N \times M$-matrix S as shown in Fig. 1a. The state of the system at position m and time n is reconstructed in analogy to the delay embedding of scalar time series [10]. Here we use the center element s_m^n, some of its neighbors and the corresponding values in the past to construct the *state vector*

$$\begin{aligned} \mathbf{x}^{n,m} \quad = \quad & (s^n_{m-IK}, ..., s^n_m, ..., s^n_{m+IK}, ..., \\ & s^{n-JL}_{m-IK}, ..., s^{n-JL}_m, ..., s^{n-JL}_{m+IK}) \end{aligned} \tag{1}$$

where I is the number of spatial neighbors, J is the number of temporal neighbors (in the past), K is the spatial shift which has a similar meaning as the time delay L (time lag) known from the delay embedding of scalar time series. This construction is visualized in Fig. 1a for $I = 1$, $J = 3$, $K = 2$ and $L = 2$. The dimension d of the state vector $\mathbf{x}^{n,m} \in \mathbb{R}^d$ is given by $d = (J + 1)(1 + 2I)$. Spatial boundaries of the dynamical process generating the STTS can be taken into account by constructing the corresponding states separately in a way analogous to that described above. In principle all states $\mathbf{x}^{n,m}$ containing boundary values of the STTS have to be devided into different classes where each class is characterized by the distance of the center of the region from the boundary. For predicting or modeling only states of the relevant class have then to be used. This decomposition into classes can be implemented in a discrete and in a continuous way. For details we refer to [6, 7].

III. Noise reduction by local linear projections

As an application of the local state reconstruction we consider noise reduction of the STTS. We use the method of local linear projections well known from ordinary time series analysis (see f.e. [2]). The main assumption behind common nonlinear noise reduction methods is the following: The Takens delay embedding yields an embedding of the state space M of the dynamical system under consideration, i.e. M is reconstructed in the reconstruction space $\mathbb{R}^d$ as a differential submanifold. So in general the reconstructed state vectors do not fill the entire reconstruction space but lie in a manifold of lower dimension than the state space. But if the reconstruction vectors are corrupted by noise the situation changes. Due to the random perturbations the reconstructed state vectors will jump off of the submanifold and form a cloud around the manifold. The idea now is to project back the noisy state vectors orthogonally on the manifold M (see figure 2). Note that projecting orthogonally is a consequence of the least–mean–square principle.

If the manifold M is given (f.e. by a set of nonlinear

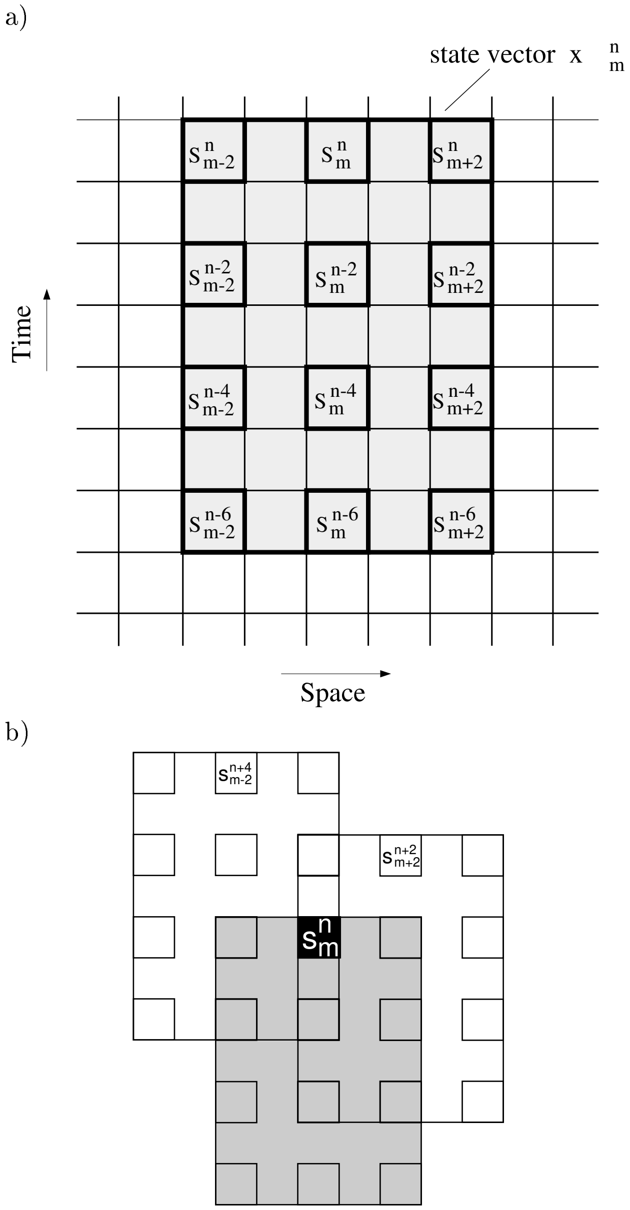

Figure 1: (a) Local reconstruction of states from regions of the spatio-temporal time series $S = \{\mathbf{s}^n\}^{n=1,...,N} = \{s^n_m\}^{n=1,...,N}_{m=1,...,M}$. (b) each element of an STTS (black square) enters several local states.

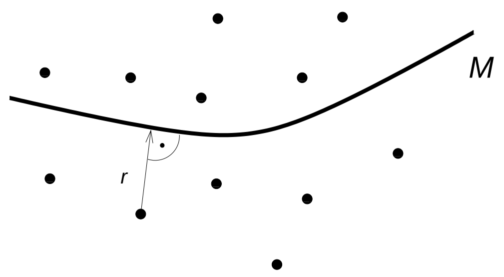

Figure 2: The noisy state vectors (•) lie in the vincinity of a manifold M. By projecting them orthogonally on M along r noise reduction is achieved.

equations), then at least in principle the projections r can be calculated for every noisy state vector. But in the following we will assume that no such equations are given. Therefore we must reconstruct M from the noisy data itself. Although several methods are possible, we will restrict our attention to local linear reconstruction. That is, we try to approximate the manifold M by a set of affine subspaces. These subspaces are constructed as follows:

1.: Estimate the dimension k of M.

2.: For every noisy state vector $\mathbf{x}^n$ find a specified number of nearest neighbors. Denote this set by $\mathcal{U}_n$.

3.: Now find the affine subspace S_n of dimension k having the following property:

$$\sum_{x \in \mathcal{U}_n} \|x - \Pi_n x\|^2 = \min!,$$

where Π_n is the orthogonal projection on S_n. The new noise reduced state vector eventually is given by

$$\hat{\mathbf{x}}^n = \delta \cdot \Pi_n \mathbf{x}^n + (1-\delta) \cdot \mathbf{x}^n,$$

where δ is a damping factor.

Step 2 and 3 are repeated for each n giving a new set of state vectors $\hat{\mathbf{x}}^n$. For step 2 we used an efficient nearest neighbor search algorithm desribed in detail in [4]. Furthermore it turns out that S_n can be found as follows: After calculating the empirical mean

$$\mu^n := \frac{1}{|\mathcal{U}_n|} \sum_{x \in \mathcal{U}_n} x$$

and the covariance matrix

$$\Gamma^n_{ij} := \frac{1}{|\mathcal{U}_n|} \sum_{x \in \mathcal{U}_n} x_i x_j - \mu^n_i \mu^n_j$$

one calculates the eigenvalues $\lambda^n_i, i = 1 \ldots d$ of Γ^n and corresponding eigenvectors $a^n_i, i = 1 \ldots k$ to the k largest eigenvalues. Then Π_n can be expressed as

$$\begin{aligned} \Pi_n : \mathbb{R}^d &\mapsto S_n; \\ x &\mapsto \mu^n + \sum_{i=1}^{k} \langle x - \mu^n | a^n_i \rangle \, a^n_i . \end{aligned}$$

To find k one may apply a common fractal dimension estimator. However, we found that an adaptive rule yields much better results. Therefore we calculated (for each n) the minimal k such that:

$$\frac{\sum_{i=1}^{k} \lambda^n_i}{\sum_{i=1}^{d} \lambda^n_i} > 0.7.$$

Note that d is the dimension of the reconstruction space and λ^n_i are the eigenvalues.

For STTS exactly the same formulae can be applied to the local states $\mathbf{x}^{n,m}$ instead of $\mathbf{x}^n$ yielding new noise reduced local states $\hat{\mathbf{x}}^{n,m}$. In this case we obtain $1+2I$ estimates of s^n_m since each s^n_m is contained in $1+2I$ different local states. These estimates are averaged using an overlap–add approach as illustrated in Fig. 1b.

IV. Numerical examples

We tested a lattice of diffusively coupled logistic maps. The logistic map is given by:

$$f(x) = \alpha x \cdot (x-1)$$

and the lattice by

$$s^{n+1}_m = f(0.5 \cdot s^n_m + 0.25 \cdot (s^n_{m-1} + s^n_{m+1}))$$

for $m = 2 \ldots M-1$ and $n = 1 \ldots N$. The boundary conditions were $s^n_1 = s^n_m = 0.5$ for all times $n = 1 \ldots N$.

We used the following parameters for the construction of the local states (cf. equation (1)):

$$\begin{aligned} I &= 1 \quad K = 0 \quad M = 100 \\ J &= 1 \quad L = 0 \quad N = 200 \end{aligned}$$

This yields a dimension $d = 6$ for the reconstruction space. We made two simulations with an additive noise of SNR = 26.02 dB. For the first we used a fixed $k = 5$, for the second we used the adaptive rule. The results are shown in Fig. 3. The adaptive rule for k shows better performance than the fixed k. The improvement of the SNR increases for the first steps but then decreases and eventually the SNR is worse than for the noisy input. This phenomenon is well known from ordinary time series analysis. It is due to the linearisation errors which are amplified from iteration to iteration. We admit that the performance is worse than someone who is familiar with noise reduction methods for ordinary time series may expect. However, one has to keep in mind that 20000 points are not very much in a 6–dimensional space. First simulations suggest that the performance considerably with more data points.

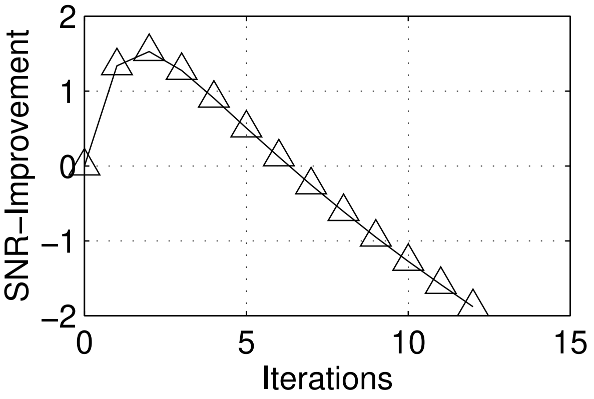

Figure 3: SNR–Improvement for fixed $k = 5$ (triangles) and adaptive rule (squares)

V. Conclusion

The numerical examples show that the local projection technique well known from ordinary nonlinear time series analysis applied to locally reconstructed states of a spatially extended dynamical system is a powerful method for noise reduction in such systems. Local reconstruction of states may also serve as a starting point for deriving a mathematical model of the underlying dynamics and subsequent bifurcation analysis. The schemes discussed in this paper may be generalized in different directions. The dimension of the reconstructed states can be reduced if the STTS stems from a dynamical system that possesses additional spatial symmetries that can be exploited when constructing the state vectors. For the one dimensional case we may for example assume that the system is invariant with respect to spatial reflection. Furthermore, concerning the noise reduction techniques presented, the effects of such symmetries on the empirical mean and covariance seem to merit consideration. However, if the dynamical equations are unknown, usually very few is known about the spatial symmetries of the system as well. If the process generating the STTS is not spatially homogeneous one may just add to the dimension d of the reconstruction space the number d_S of spatial dimensions of the problem (i.e. $d_S = 1, 2$ or 3) and work then in the extended $d + d_S$ dimensional space. Another modification of the scheme proposed concerns the selection of spatio-temporal neighbors of the center element s_m^n for reconstructing the local state $\mathbf{x}_m^n$. At this point one may take into account that any physical information spreads with some maximum speed. Instead of using a rectangular region of the matrix S a triangle (i.e. a "light cone") may be more efficient for reconstructing local states. This feature of locality may also be viewed as a motivation for the concept of local states and seems to be important for a rigorous mathematical justification of the presented method.

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