VIBRATIONS OF SHELLS CONTACTING FLUID: ASYMPTOTIC ANALYSIS

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ABSTRACT

We consider free and forced harmonic vibrations of a thin elastic shell filled with or immersed into fluid. We construct the asymptotics of the eigenfrequencies and scattering frequencies in the problems of free vibrations, and of the solutions of non-homogeneous problems, using the relative shell thickness as the main asymptotic parameter.

0. Introduction

The main object studied in this paper is the problem of harmonic vibrations (free or forced) of a thin elastic shell which is either filled with an inviscid compressible fluid (the *interior* problem) or is immersed into an unbounded inviscid compressible fluid (the *exterior* problem).

We give an extended review of the results obtained in these problems by the authors and their collaborators during the last fifteen years. These results have been mostly published in Russia and are not widely available in the West.

The vibrating shell-fluid system is characterized by many parameters: the characteristic linear size of the shell R^*, Young's modulus E^*, the density ρ_s^* and Poisson's ratio ν of the material of the shell, shell thickness h^*, fluid density ρ_{fl}^*, sound velocity in the fluid c_{fl}^*, and frequency of vibrations ω^*. (We denote all dimensional parameters by an asterisk; we shall switch to dimensionless parameters in Section 2.)

Our approach is asymptotic, with the relative shell thickness $h = h^*/R^*$ playing the rôle of the main asymptotic parameter. We extensively use modern mathematical techniques such as the theory pseudodifferential operators and advanced results in spectral asymptotics. Most of our results (but not all) are rigorously proved (for example, we show that the formal asymptotic expansions indeed converge asymptotically to the exact solutions), however we do not present the details of the proofs

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here in order to simplify matters; a rigorous mathematical analysis of these problems can be found in the works by A. G. Aslanyan, A. L. Goldenveizer, M. Levitin, V. B. Lidskii, D. Vassiliev *et. al.* [2,4–6,8,9,13–16,19–23].

1. Basic equations

1.1. Coordinate system and dimensionless parameters

Let us denote by Γ the closed bounded infinitely smooth middle surface of our shell. The surface Γ divides $\mathbb{R}^3$ into two parts: a bounded domain G_i (interior of the shell) and an unbounded domain G_e (exterior of the shell). In the interior problem the domain G_i is occupied by fluid and the domain G_e is occupied by vacuum, whereas in the exterior problem the domain G_i is occupied by vacuum and the domain G_e is occupied by fluid.

We denote by $\mathbf{x} = \mathbf{x}^*/R^* = (x_1, x_2, x_3)$ the dimensionless Cartesian coordinates in $\mathbb{R}^3$, and by $\alpha = \alpha^*/R^* = (\alpha_1, \alpha_2)$ the dimensionless local coordinates on Γ associated with the lines of curvature. By $\mathbf{n} = \mathbf{n}^*/R^*$ we denote the unit normal vector to Γ pointing inside G_e; we assume that the main trihedron $\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}$ is chosen in such a way that the basis vectors form the right triple. Let $A_1(\alpha)$, $A_2(\alpha)$ denote the (dimensionless) coefficients of the first quadratic form of the surface Γ, $dS^2 = {A_1}^2 d{\alpha_1}^2 + {A_2}^2 d{\alpha_2}^2$, and let $R_1^{-1}(\alpha)$, $R_2^{-1}(\alpha)$ denote the (dimensionless) curvatures of the surface Γ; the signs of the curvatures are chosen in such a way that they are positive for a convex shell. The surface element of Γ is computed as $dS = A_1 A_2 d\alpha_1 d\alpha_2$, and the volume element is denoted $dV = dx_1 dx_2 dx_3$.

Further on we shall use the following dimensionless parameters characterizing the shell-fluid system:

$$
\begin{aligned}
h &:= h^*/R^* && \text{— relative shell thickness,}\\
\rho &:= \rho^*_{fl}/\rho^*_s && \text{— relative density,}\\
c &:= c^*_{fl}/\sqrt{E^*/\rho^*_s} && \text{— relative sound velocity,}\\
\omega &:= \omega^* R^*/c^*_s && \text{— dimensionless frequency.}
\end{aligned}
$$

1.2. Shell equations in vacuum

Let us first consider the vibrations of a shell in vacuum. We shall describe the shell deformations by the (dimensionless) vector of shell displacements

$$\mathbf{u}(\alpha) \equiv \mathbf{u}^*/R^* \equiv (u_1, u_2, u_3) \equiv u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{n}\,;$$

it is important to note that u_3 is the displacement in the *normal* direction.

Suppose that the vibrations of the shell are excited by a given exterior force with amplitude

$$\mathbf{g}(\alpha) \equiv \mathbf{g}^*/E^* \equiv (g_1, g_2, g_3) \equiv g_1\mathbf{e}_1 + g_2\mathbf{e}_2 + g_3\mathbf{n}\,.$$

The vector-function $\mathbf{g}$ is an arbitrary function from $\mathbf{L}^2(\Gamma)$, i.e., it is not necessarily smooth. Note that there are methods which allow to consider extremely non-smooth loads, for example distributions from the class $\mathcal{D}'(\Gamma)$.

We always assume the time dependence of the shell displacements and the exterior force in the form $e^{-i\omega t}$; we omit this time factor further on.

After the separation of the time factor, the vibrations of a shell in vacuum are described by the system of three partial differential equations on Γ:

$$h\sum_{i=1}^{3}\mathcal{L}_{ij}u_i = h\omega^2\,u_j + g_j\,, \qquad j=1,2,3\,. \tag{1.1}$$

The $\mathcal{L}_{ij}$ are the linear differential operators of shell theory which have the form

$$\mathcal{L}_{ij} = \frac{h^2}{12}n_{ij} + \ell_{ij}\,, \qquad i,j=1,2,3\,. \tag{1.2}$$

Here n_{ij} and ℓ_{ij} are the moment and membrane operators respectively. We recall explicit expressions for ℓ_{ij} from Goldenveizer–Lidskii–Tovstik[3]:

$$\begin{aligned}
\ell_{ii} =& -\frac{1}{1-\nu^2}\frac{1}{A_i}\frac{\partial}{\partial\alpha_i}\frac{1}{A_iA_j}\frac{\partial}{\partial\alpha_i}A_j - \frac{1}{2(1+\nu)}\frac{1}{A_j}\frac{\partial}{\partial\alpha_j}\frac{1}{A_iA_j}\frac{\partial}{\partial\alpha_j}A_i \\
&- \frac{1}{1+\nu}R_i{}^{-1}R_j{}^{-1}\,, \qquad i=1,2,3\,,\\
\ell_{ij} =& -\frac{1}{1-\nu^2}\frac{1}{A_i}\frac{\partial}{\partial\alpha_i}\frac{1}{A_iA_j}\frac{\partial}{\partial\alpha_j}A_i + \frac{1}{2(1+\nu)}\frac{1}{A_j}\frac{\partial}{\partial\alpha_j}\frac{1}{A_jA_i}\frac{\partial}{\partial\alpha_i}A_j\,, \qquad i,j=1,2\,,\\
\ell_{i3} =& -\frac{1}{1-\nu^2}\frac{1}{A_i}\frac{\partial}{\partial\alpha_i}\left(R_i{}^{-1}+R_j{}^{-1}\right) + \frac{1}{1+\nu}\frac{1}{A_iR_j}\frac{\partial}{\partial\alpha_i}\,, \qquad i=1,2\,,\\
\ell_{3i} =& \frac{1}{1-\nu^2}\frac{1}{A_iA_j}\left(R_i{}^{-1}+R_j{}^{-1}\right)\frac{\partial}{\partial\alpha_i}A_j - \frac{1}{1+\nu}\frac{1}{A_iA_j}\frac{\partial}{\partial\alpha_i}\frac{A_j}{R_j}\,, \qquad i=1,2\,,\\
\ell_{33} =& \frac{1}{1-\nu^2}\left(R_1{}^{-2}+2\nu R_1{}^{-1}R_2{}^{-1}+R_2{}^{-2}\right)\,.
\end{aligned}$$

The operators n_{ij} depend on the choice of the particular variant of shell theory. We assume that our n_{ij} satisfy the following natural conditions:

$$\operatorname{ord} n_{ij} \leqslant 2 \qquad \text{for} \qquad i,j\leqslant 2\,,$$

$$\operatorname{ord} n_{ij} \leqslant 3 \qquad \text{for} \qquad i+j<6\,,$$

$$n_{ij} = \frac{1}{1-\nu^2}\Delta_\Gamma^2 + \widetilde{n}_{33}\,, \qquad \operatorname{ord}\widetilde{n}_{33}\leqslant 2\,,$$

where ord stands for the order of the operator and Δ_Γ denotes the surface Laplacian on Γ. We also assume that the matrix differential operator $\left(n_{pq}\right)_{i,j=1}^{3}$ is formally self-adjoint and non-negative with respect to the standard $\mathbf{L}^2(\Gamma)$-inner product on vector-functions.

The simplest possible choice of the moment operators n_{ij} is

$$n_{ij} = 0 \qquad \text{for} \qquad i + j < 6\,, \qquad \widetilde{n}_{33} = 0\,. \tag{1.3}$$

This choice corresponds to the so-called *technical shell theory*; without loss of generality we shall assume this variant throughout the paper.

1.3. Fluid equations

We shall describe the motion of an ideal fluid (occupying either G_i or G_e) by the potential of displacements

$$\psi(\mathbf{x}) = \psi^*(\mathbf{x}^*/R^*)/R^*\,.$$

The fluid velocity is easily expressed in terms of ψ:

$$\mathbf{v}(\mathbf{x}) = -i\omega\,\mathbf{grad}\,\psi(\mathbf{x})$$

($\mathbf{grad}\,\psi$ is the vector of fluid displacements, and the additional factor $-i\omega$ results from the differentiation with respect to time). It is well-known that the vibrations of fluid are governed by the Helmholtz equation

$$\Delta\psi + \frac{\omega^2}{c^2}\psi = 0\,. \tag{1.4}$$

In the case of the exterior problem the potential ψ should satisfy the additional radiation condition at infinity. This condition has the simplest form for real values of ω (Sommerfeld radiation condition):

$$\frac{\partial\psi}{\partial r} - \frac{i\omega}{c}\psi = o\left(r^{-1}\right) \qquad \text{as} \qquad r \to \infty\,, \tag{1.5}$$

$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. We discuss the radiation condition for non-real ω in Section 4.

In Section 7 we shall briefly discuss the equations of a viscous compressible fluid and the effects of viscosity on shell–fluid vibrations.

1.3. Equations for a shell contacting fluid

For a shell interacting with fluid we shall modify Eqs. (1.1) in order to take into account the presence of fluid. The equations now can be written in a matrix form as

$$h\mathcal{L}\mathbf{u} = h\omega^2\mathbf{u} + \mathbf{g} \pm \rho\omega^2\psi|_\Gamma\mathbf{n}\,, \tag{1.6$^\pm$}$$

where the plus sign corresponds to the interior problem and the minus sign corresponds to the exterior problem. The additional term $\pm\rho\omega^2\psi|_\Gamma\mathbf{n}$ which appears in

Eqs. ($1.6^{\pm}$) by comparison with Eqs. (1.1) gives the pressure of the fluid upon the shell.

In addition, the fluid potential should satisfy the non-penetration condition on Γ

$$\left.\frac{\partial\psi}{\partial n}\right|_{\Gamma} = u_3 \tag{1.7}$$

($\left.\frac{\partial\psi}{\partial n}\right|_{\Gamma}$ is the derivative of ψ with respect to the exterior normal to Γ).

Now, we are able to state our four main problems.

- **Free vibrations, interior problem** – Eqs. (1.4) in G_i, (1.6^+) ($\mathbf{g} = 0$) and (1.7) on Γ.
  Find the eigenfrequencies ω *such that the problem has a non-trivial solution* $\mathbf{u}$, ψ.
- **Forced vibrations, interior problem** – Eqs. (1.4) in G_i, (1.6^+) and (1.7) on Γ.
  Given $\omega > 0$ *and* $\mathbf{g}(\alpha)$, *find the solution* $\mathbf{u}$, ψ.
- **Free vibrations, exterior problem** – Eqs. (1.4) and the radiation condition in G_e, (1.6^-) ($\mathbf{g} = 0$) and (1.7) on Γ.
  Find the resonance frequencies ω *such that the problem has a non-trivial solution* $\mathbf{u}$, ψ.
- **Forced vibrations, exterior problem** – Eqs. (1.4) and (1.5) in G_e, (1.6^-) and (1.7) on Γ.
  Given $\omega > 0$ *and* $\mathbf{g}(\alpha)$, *find the solution* $\mathbf{u}$, ψ.

In the next Sections we shall give rigorous mathematical statements of the above problems, specifying the function classes and other conditions.

2. Mathematical statement and general properties of the interior problem

2.1. Mathematical statement of the interior problem

Let us consider the Hilbert space $\mathcal{H}$ of the quadruples of functions

$$\mathbf{f} = (u_1(\alpha), u_2(\alpha), u_3(\alpha), \psi(\mathbf{x}))\,;$$

we identify the elements of $\mathcal{H}$ which differ by the term $(0, 0, 0, \text{const}\,)$. (Sometimes we shall write $\mathbf{f} = (\mathbf{u}, \psi)$ for brevity.) We introduce the inner product on $\mathcal{H}$ by the formula

$$\left(\mathbf{f}^{(1)}, \mathbf{f}^{(2)}\right)_{\mathcal{H}} = h \iint\limits_{\Gamma} \mathbf{u}^{(1)} \cdot \overline{\mathbf{u}^{(2)}}\, dS + \rho \iiint\limits_{G_i} \mathbf{grad}\,\psi^{(1)} \cdot \overline{\mathbf{grad}\,\psi^{(2)}}\, dV \tag{2.1}$$

(the bar denotes complex conjugation). This inner product and the corresponding norm have a natural physical meaning: $\|\mathbf{f}\|_{\mathcal{H}} = (\mathbf{f}, \mathbf{f})_{\mathcal{H}}$ is a kinetic energy of the vibrating shell–fluid system (up to the factor $\omega^2/2$). The Hilbert space $\mathcal{H}$ is easily identified with the space $\mathbf{L}^2(\Gamma) \times H^1(G_i)$.

Now, using (1.4), we can rewrite the interior problems as

$$\mathcal{A}_i \mathbf{f} = \lambda \mathbf{f} + \frac{1}{h}\mathbf{F}\,, \tag{2.2}$$

where $\lambda = \omega^2$ is a new spectral parameter, the matrix 4×4 differential operator $\mathcal{A}_i$ is defined as

$$\mathcal{A}_i = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} & 0 \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} & 0 \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} & -\frac{\rho c^2}{h}\Theta_0 \Delta \\ 0 & 0 & 0 & -c^2\Delta \end{pmatrix}, \tag{2.3}$$

on quadruples satisfying the condition

$$\Theta_1 \psi = u_3\,; \tag{2.4}$$

Θ_0 and Θ_1 are the operators mapping $\psi(\mathbf{x})$ into $\psi|_\Gamma(\alpha)$ and $\left.\frac{\partial\psi}{\partial n}\right|_\Gamma(\alpha)$, respectively, and $\mathbf{F} = (g_1, g_2, g_3, 0)$.

Obviously, in case of the free vibrations one should set in (2.2) $\mathbf{F} = \mathbf{0}$, and consider the corresponding spectral problem.

2.2. Spectral properties of the operator $\mathcal{A}_i$

We state the main properties of the operator $\mathcal{A}_i$; detailed proofs can be found in Aslanyan–Lidskii–Vassiliev[2].

Theorem 2.1. *The operator $\mathcal{A}_i$ is symmetric; moreover, $(\mathcal{A}_i\mathbf{f}, \mathbf{f})_{\mathcal{H}} \geqslant 0$.*

Indeed, simple integration by parts with account of Eq. (2.4) gives

$$\begin{aligned}\left(\mathcal{A}_i \mathbf{f}^{(1)}, \mathbf{f}^{(2)}\right)_{\mathcal{H}} &= h \iint\limits_{\Gamma} \mathcal{L}\mathbf{u}^{(1)} \cdot \overline{\mathbf{u}^{(2)}}\, dS \\ &\quad + \rho c^2 \iiint\limits_{G_i} \Delta\psi^{(1)}\overline{\Delta\psi^{(2)}}\, dV\,.\end{aligned}$$

Since the operator $\mathcal{L}$ is symmetric and non-negative, the same is true for $\mathcal{A}_i$.

Theorem 2.2. *The operator $\mathcal{A}_i$ is an essentially selfadjoint operator in $\mathcal{H}$ with a compact resolvent $(\mathcal{A}_i + I)^{-1}$.*

One of the most important corollaries of Theorems 2.1 and 2.2 is

Theorem 2.3. *The operator $\mathcal{A}_i$ has a discrete spectrum consisting of positive eigenvalues λ_j, $j = 1, 2, \dots$, which have the only limit point $+\infty$.*

In physical terms, Theorem 2.3 implies that the spectrum of eigenfrequencies $\omega_j = \sqrt{\lambda_j}$ of the interior problem is discrete. (Note that due to the symmetry of the problem $-\omega_j$ are also eigenfrequencies; for definiteness we consider only the half-plane $\operatorname{Re}\omega \geqslant 0$ throughout the paper.)

2.3. Existence and uniqueness of the solution of the non-homogeneous problem

Another important corollary of Theorems 2.1 and 2.2 is

Theorem 2.4. *Suppose that $\lambda = \omega^2$ does not coincide with any of the eigenvalues λ_j. Then for any $\mathbf{g}(\alpha) \in \mathbf{L}^2(\Gamma)$ the interior problem of forced vibrations (2.2)–(2.4) has the unique solution $(\mathbf{u}(\alpha), \psi(\mathbf{x}))$ such that $\mathbf{u} \in \mathbf{H}^{2,2,4}(\Gamma)$ and $\psi \in H^3(G_i)$.*

3. Asymptotics of eigenfrequencies for the interior problem

In this Section we construct the asymptotics of eigenfrequencies for the interior problem with respect to the main asymptotic parameter - the relative shell thickness $h \ll 1$. We assume that the relative density ρ and the relative sound velocity c are of order $\sim h^0$. We work in the *low frequency range* $\omega \sim 1$. Our analysis starts with the case of a shell of revolution which can be studied in a very detailed manner.

3.1. Decomposition of the spectrum into three series for a shell of revolution

Let us consider a shell of revolution Γ formed by the rotation of the curve $x_1 = X(s)$, $x_2 = Y(s)$, $0 \leqslant s \leqslant L$, around the x_2-axis, where $X \geqslant 0$, $Y(L) > Y(0)$, $(X'_s)^2 + (Y'_s)^2 = 1$, and s denotes a variable arc length of a meridian. We choose s and the angular coordinate φ as the coordinates $\alpha = (\alpha_1, \alpha_2)$ on the shell, and α_3 (a signed distance from a point to the shell; note that α_3 is negative for points in G_i) as the third coordinate in the vicinity of the shell. Then, the coefficients of the first quadratic form of Γ are $A_1 = 1$, $A_2 = X(s)$, and the principle curvatures of the shell are $R_1^{-1} = X'_s Y''_{ss} - Y'_s X''_{ss}$ and $R_2^{-1} = Y'_s / X$.

All the results of this Section are obtained for a (sufficiently small) fixed *number of waves along the parallel*. More precisely, this means that we study the eigenforms with the angular behaviour $\exp(im\varphi)$, $m \sim 1$, and separate this factor in the initial equations.

One of the most important results in the asymptotic theory of shells containing fluid is the following:

> *As $h \to +0$, the eigenfrequencies of the interior problem for a shell of revolution can be decomposed into three series*:

- **tangential**, *with the prevalence of the tangential components of the vector of shell displacements* $\mathbf{u}$ *in the corresponding eigenform* $(\mathbf{u}, \psi)$;

- **fluid**, *with the prevalence of the fluid potential* ψ *in the corresponding eigenform* $(\mathbf{u}, \psi)$;
- **flexural**, *with the prevalence of the normal component of the vector of shell displacements* $\mathbf{u}$ *in the corresponding eigenform* $(\mathbf{u}, \psi)$.

In the above classification, the prevailing component of the eigenform $\mathbf{f} = (\mathbf{u}, \psi)$ is the one which gives the main asymptotic contribution in the kinetic energy $\frac{\omega^2}{2}(\mathbf{f}, \mathbf{f})_{\mathcal{H}}$.

3.2. Asymptotic formulae for tangential and fluid eigenfrequencies for a shell of revolution

The eigenfrequencies of the tangential series can be asymptotically found as the eigenfrequencies ω_j^{tn} of the *auxiliary tangential problem*

$$\begin{pmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{pmatrix} \begin{pmatrix} u_1^{tn} \\ u_2^{tn} \end{pmatrix} = \rho(\omega^{tn})^2 \begin{pmatrix} u_1^{tn} \\ u_2^{tn} \end{pmatrix} . \tag{3.1}$$

The operator $\mathcal{L}^{tn} = (\ell_{ij})_{i,j=1}^2$ appearing in the left-hand side of (3.1) is simply the operator of the two-dimensional elasticity but written in the curvilinear metric of the shell. This operator is elliptic, symmetric and non-negative, so the ω-spectrum of the problem (3.1) is discrete and symmetric with respect to the origin. (Note that $\omega = 0$ is an eigenfrequency of the problem (3.1) only for a shell of revolution; the multiplicity of this eigenfrequency is three for a sphere and one in all other cases, and the corresponding eigenforms describe the rotation of a shell as a solid body.) We enumerate the positive eigenfrequencies of (3.1) in increasing order with account of multiplicity:

$$0 < \omega_1^{tn} \leqslant \omega_2^{tn} \leqslant \dots .$$

Then, for sufficiently small j, the ω_j^{tn} give the eigenfrequencies of the tangential series of the full shell–fluid system with accuracy $O(h)$, $h \to +0$.

The eigenfrequencies of the fluid series can be asymptotically found as the eigenfrequencies ω_j^{fl} of the *auxiliary fluid problem*

$$c^2 \Delta \psi^{fl} + \omega^{fl} \psi = 0 \qquad \text{in } G_i\,, \qquad \psi|_\Gamma = 0\,. \tag{3.2}$$

The problem (3.2) is simply the Dirichlet problem for the Helmholtz equation, and it is well-known that its ω-spectrum consists of real non-zero eigenfrequencies and is symmetric with respect to zero. We enumerate the positive eigenfrequencies of (3.2) in increasing order with account of multiplicity:

$$0 < \omega_1^{fl} \leqslant \omega_2^{fl} \leqslant \dots .$$

Then, for sufficiently small j, the ω_j^{fl} give the eigenfrequencies of the fluid series of the full shell–fluid system with accuracy $O(h)$, $h \to +0$.

VASSILIEV[20] constructed simple iteration procedures which allow to obtain the eigenfrequencies and corresponding eigenforms of the tangential and fluid series for the full problem with arbitrary accuracy, starting from the eigenfrequencies and eigenfunctions of (3.1) and (3.2). However, in most practical situations even the initial approximations described above provide adequate results.

Note that the eigenfrequencies of both problems (3.1) and (3.2) are distributed on the real axis quite sparsely, with relative density $O(1)$. The vast majority of the eigenfrequencies of the full problem belongs to a very dense flexural series which we consider in the next subsection.

3.3. Asymptotic formulae for flexural eigenfrequencies for a shell of revolution

For a shell of revolution the eigenfrequencies ω_j^{fx} can be asymptotically found from the following equation:

$$\frac{1}{\pi}\int_0^L k(s,\omega_j^{fx},h)\,ds = j + |m| - \frac{1}{2} + \delta_{0m}\,, \tag{3.3}$$

where

$$\delta_{lm} = \begin{cases} 1 & \text{for} \quad m = l\,, \\ 0 & \text{for} \quad m \neq l\,, \end{cases}$$

and k is the maximal real root of the six order algebraic equation

$$\frac{h^3}{12(1-\nu^2)}k^4 + hR_2{}^{-2}(s) = h(\omega_j^{fx})^2 + \frac{\rho(\omega_j^{fx})^2}{k}\left(1 + \frac{R_2{}^{-1}(s)}{2k}\right)\,. \tag{3.4}$$

Here $m \sim 1$ is the number of waves along the parallel, and $j \sim h^{-3/5} \gg 1$ is the sequential number of the flexural eigenfrequency ω_j^{fx}.

The asymptotic equation (3.3) determines ω_j^{fx} with accuracy $O\left(j^{-2}\right)$; note that $\omega_{j+1}^{fx} - \omega_j^{fx} \sim j^{-1}$, so the asymptotics allows us to separate eigenfrequencies.

3.4. Asymptotic formulae for eigenfrequencies for a shell of an arbitrary shape

When Γ is not a shell of revolution our results are considerably less precise. Instead of producing asymptotic formulae for the jth eigenfrequencies in all the three series as we did for a shell of revolution, here we obtain a formula for the counting function $N_h(\omega)$, which is defined as the number of eigenfrequencies less than a given number ω.

As $h \to +0$, we have

$$N_h(\omega) = \frac{1+o(1)}{8\pi^2}\int_\Gamma \left(\int_0^{2\pi} k^2(s,\omega,h,\theta)\,d\theta\right)\,dS\,, \tag{3.5}$$

where k is the maximal real root of the six order algebraic equation

$$\frac{h^3}{12(1-\nu^2)}k^4 + hK^2(\alpha_1,\alpha_2,\theta) = h\omega^2 + \frac{\rho\omega^2}{k}\left(1 + \frac{K(\alpha_1,\alpha_2,\theta)}{2k}\right), \tag{3.6}$$

$$K(\alpha_1,\alpha_2,\theta) = R_1^{-1}(\alpha_1,\alpha_2)\sin^2\theta + R_2^{-1}(\alpha_1,\alpha_2)\cos^2\theta\,. \tag{3.7}$$

The natural analogy between Eqs. (3.3)–(3.4) and (3.5)–(3.7) suggests that we are in fact looking for the eigenfrequencies of the flexural series. Indeed, the tangential and fluid eigenfrequencies occur so seldom in our frequency range that their contributions into $N_h(\omega)$ are "lost" in the asymptotically negligible remainder of (3.5). However, we can still find these eigenfrequencies in the generic situation using the procedures of subsection 3.2.

The jth eigenfrequency ω_j^{fx} of the flexural series can be asymptotically found by inverting formula (3.5):

$$j = N_h(\omega_j^{fx}) + 0 \sim \frac{1}{8\pi^2}\int\limits_{\Gamma}\left(\int\limits_0^{2\pi} k^2(s,\omega_j^{fx},h,\theta)\,d\theta\right)dS\,. \tag{3.8}$$

In the generic situation the asymptotic equation given above determines ω_j^{fx} with accuracy $o\left(j^{-1}\right)$ (see Vassiliev[20] for details); note that $\omega_{j+1}^{fx} - \omega_j^{fx} \sim j^{-2}$, so the asymptotics does not allow us to separate eigenfrequencies, unlike the case of a shell of revolution.

Note that rigorous mathematical proofs of the asymptotic formulae of this Section required deep excurses into the modern theory of pseudodifferential operators and spectral asymptotics; the interested reader can find the proofs in Vassiliev[19–22].

4. Mathematical statement and general properties of the exterior problem

4.1. $\mathbf{L}^2$*-spectrum of the exterior problem*

As in the case of the interior problem we can write the exterior problem in the operator form

$$\mathcal{A}_e\mathbf{f} = \lambda\mathbf{f} + \frac{1}{h}\mathbf{F}\,, \tag{4.1}$$

where the matrix 4×4 differential operator $\mathcal{A}_e$ is defined as

$$\mathcal{A}_e = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} & 0 \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} & 0 \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} & \frac{\rho c^2}{h}\Theta_0\Delta \\ 0 & 0 & 0 & -c^2\Delta \end{pmatrix}, \tag{4.2}$$

and differs from the operator $\mathcal{A}_i$ by the sign of the term $\frac{\rho c^2}{h}\Theta_0\Delta$ (*cf*. Eqs. (3.2)–(3.3)), and the new spectral parameter λ and the right-hand side $\mathbf{F}$ are as in Section 2. As before, the operator $\mathcal{A}_e$ acts on quadruples $(\mathbf{u},\psi)$; of course the potential $\psi(\mathbf{x})$ is defined now in the exterior G_e of the shell.

Similarly to the interior problem we can introduce the Hilbert space

$$\mathcal{H} = \left\{\mathbf{f} = (\mathbf{u}(\alpha),\psi(\mathbf{x})) : \mathbf{u} \in \mathbf{L}^2(\Gamma), \psi \in H^1_{\mathrm{loc}}(G_e),\ \mathbf{grad}\,\psi \in \mathbf{L}^2(G_e)\right\}$$

with the scalar product

$$\left(\mathbf{f}^{(1)},\mathbf{f}^{(2)}\right)_{\mathcal{H}} = h\iint\limits_{\Gamma} \mathbf{u}^{(1)}\cdot\overline{\mathbf{u}^{(2)}}\,dS + \rho\iiint\limits_{G_e} \mathbf{grad}\,\psi^{(1)}\cdot\overline{\mathbf{grad}\,\psi^{(2)}}\,dV\,. \tag{4.3}$$

Note that $H^1_{\mathrm{loc}}(G_e)$ denotes a *local* Sobolev space; the introduction of such a space is necessary because the integral $\iiint\limits_{G_e} |\psi|^2\,dV$ might be infinite. This does not cause any difficulties since the potential ψ does not have a physical meaning itself, only its gradient does. As before we identify the quadruples which differ by $(0,0,0,\mathrm{const}\,)$.

We can first define the formally self-adjoint operator $\mathcal{A}_e$ using (4.2) on sufficiently smooth elements of $\mathcal{H}$ satisfying the non-penetration condition (2.4) and such that the support of ψ is compact, and then extend it to a non-negative self-adjoint operator with some domain $D(\mathcal{A}_e) \subset \mathcal{H}$. A standard argument using Weyl's sequences shows that the λ-spectrum of the operator $\mathcal{A}_e$ constructed in this way fills the half-line $[0,+\infty)$. Of course, the non-discreteness of the spectrum is due to the unboundedness of the domain G_e.

4.2. Meromorphic continuation of the resolvent and scattering frequencies

The fact that the spectrum of the exterior problem considered as a spectral problem for the operator $\mathcal{A}_e$ in the Hilbert space $\mathcal{H}$ is continuous and fills the half-line $[0,+\infty)$ (or, in terms of frequency ω, the whole real axis $(-\infty,+\infty)$) is not very informative from the mechanical point of view. This is due to two important factors.

First, the choice of functions in $\mathcal{H}$ (or, more exactly, in the set $D(\mathcal{A}_e)$ which is a dense subset of $\mathcal{H}$) is too narrow: for example the functions $\psi(\mathbf{x}) \not\equiv 0$ which satisfy the Helmholtz equation with real ω and the radiation condition (1.5) do not belong to $\mathcal{H}$ because $\mathbf{grad}\,\psi \notin \mathbf{L}^2(G_e)$.

On the other hand, the choice of functions in $\mathcal{H}$ used in the spectral problem (4.1) for the description of possible right-hand sides $\mathbf{F}$ in (4.1) is too wide: in the original mechanical problem we were interested in loads acting only upon the shell and not upon the fluid, especially infinitely far from the shell.

To correct the situation we shall change the statement of the exterior spectral problem following Vainberg[18]. Namely, let us first consider the resolvent of the exterior problem

$$\mathcal{R}_\omega = (\mathcal{A} - \omega^2 I)^{-1}\,, \tag{4.4}$$

defined in the sense of subsection 4.1 as a bounded operator acting in $\mathcal{H}$. Note that for the convenience of further analysis we have switched in (4.4) from the spectral parameter λ to the frequency ω. The change of the spectral parameter $\lambda = \omega^2$ maps the complex λ-plane with a cut along the non-negative real semi-axis onto the upper complex ω-half-plane. Therefore, we are considering (initially) the operator $\mathcal{R}_\omega$ for $\operatorname{Im}\omega > 0$. In this half-plane the operator $\mathcal{R}_\omega$ is a *holomorphic* operator-valued function of ω with values in $\mathcal{H}$.

The general construction of VAINBERG[18] (see also the very detailed description in Chapter 9 of SANCHEZ-HUBERT–SANCHEZ-PALENCIA[15]) allows us to continue the resolvent through the continuous spectrum on the whole ω-complex plane. For each ω, the resulting operator which we still denote $\mathcal{R}_\omega$ acts from the Hilbert space

$$\mathcal{H}_{\min} = \left\{\mathbf{f} = (\mathbf{u}(\alpha), \psi(\mathbf{x})) : \mathbf{u} \in \mathbf{L}^2(\Gamma), \psi \in H^1_{\text{loc}}(G_e), \text{supp } \mathbf{grad}\,\psi \subset G_\tau\right\}$$

with the scalar product

$$\left(\mathbf{f}^{(1)}, \mathbf{f}^{(2)}\right)_{\mathcal{H}_{\min}} = \left(\mathbf{f}^{(1)}, \mathbf{f}^{(2)}\right)_{\mathcal{H}}$$

into the Hilbert space

$$\mathcal{H}_{\max} = \left\{\mathbf{f} = (\mathbf{u}(\alpha), \psi(\mathbf{x})) : \mathbf{u} \in \mathbf{L}^2(\Gamma), \psi \in H^1_{\text{loc}}(G_e), \exp(-|\mathbf{x}|^2/2)\,\mathbf{grad}\,\psi \in \mathbf{L}^2(G_e)\right\}$$

with the scalar product

$$\left(\mathbf{f}^{(1)}, \mathbf{f}^{(2)}\right)_{\mathcal{H}_{\max}} = h \iint\limits_{\Gamma} \mathbf{u}^{(1)} \cdot \overline{\mathbf{u}^{(2)}} dS + \rho \iiint\limits_{G_e} \exp(-|\mathbf{x}|^2/2)\,\mathbf{grad}\,\psi^{(1)} \cdot \overline{\mathbf{grad}\,\psi^{(2)}}\, dV.$$

Here $G_\tau = G_e \cap \{|\mathbf{x}| < \tau\}$, and τ is some fixed sufficiently large number such that the shell Γ lies within the ball $\{|\mathbf{x}| < \tau\}$.

It may be shown that $\mathcal{R}_\omega$ is a *meromorphic* function of ω with values in $\mathcal{L}(\mathcal{H}_{\min}, \mathcal{H}_{\max})$ (the space of bounded linear operators acting from $\mathcal{H}_{\min}$ into $\mathcal{H}_{\max}$). Its poles are situated in the lower half-plane $\{\operatorname{Im}\omega < 0\}$ and are of finite multiplicities; there is a finite number of poles in any compact subset of $\mathbb{C}$. These poles are called *scattering frequencies* of the exterior problem, and are the main subject of our study in this and the next Sections.

4.3. Radiation conditions

The construction using the analytic continuation described in the previous subsection is a bit awkward from the practical point of view. In practice, this construction leads to the appearance of the *radiation conditions* which "single out" the required solutions and allow to analyze the problem effectively. We gave the radiation condition for the real frequencies ω in Section 1 (see Eq. (1.5)). Now we shall describe a construction of the radiation conditions for all $\omega \in \mathbb{C}$.

Consider a sphere $\Gamma' = \{|\mathbf{x}| = \tau\}$, where τ is large enough so that the shell Γ lies within this sphere. We denote the domain bounded by G and Γ' by ${G_e}'$, and the exterior of the sphere Γ' by ${G_e}''$. Obviously, $G_e = {G_e}' \cup \Gamma' \cup {G_e}''$. Let $\psi(\mathbf{x})$ be a solution in G_e of the Helmholtz equation (1.4) with some complex frequency $\omega \neq 0$. Expanding $\psi(\mathbf{x})$ for $\mathbf{x} \in \overline{{G_e}''}$ to a Fourier series in spherical harmonics we get

$$\psi(\mathbf{x}) = \sum_{j=0}^{+\infty} \sum_{m=-j}^{j} a_{jm}(r) Y_j^m(\alpha')\,, \tag{4.5}$$

where the functions $a_{jm}(r)$ satisfy the spherical Bessel equation

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} a_{jm} - \frac{j(j+1)}{r^2} a_{jm} + \frac{\omega^2}{c^2} a_{jm} = 0\,,$$

and, therefore,

$$a_{jm}(r) = a_{jm}^{(1)} h_j^{(1)}\left(\frac{\omega r}{c}\right) + a_{jm}^{(2)} h_j^{(2)}\left(\frac{\omega r}{c}\right)\,. \tag{4.6}$$

Here $a_{jm}^{(1)}$, $a_{jm}^{(2)}$ are constants, $h_j^{(1)}(\cdot)$ and $h_j^{(2)}(\cdot)$ are the spherical Hankel functions of the first and the second kind, respectively, $r = |\mathbf{x}|$, $\alpha' = (\theta, \varphi)$ are the spherical coordinates on Γ',

$$Y_j^m(\alpha') = \frac{1}{2\tau}\sqrt{\frac{2j+1}{\pi} \cdot \frac{(j-|m|)!}{(j+|m|)!}} P_j^{|m|}(\cos\theta)\exp(im\varphi)$$

are the spherical functions normalized with respect to the scalar product

$$(v, w)' = \tau^2 \int_0^{2\pi}\int_0^{\pi} v\overline{w} \sin\theta \, d\theta \, d\varphi$$

(recall that τ is the radius of the sphere Γ').

Assuming that ω is real, let us substitute the function (4.5) into the Sommerfeld radiation condition (1.5). Using the asymptotics of the Hankel functions for big values of the argument[1], we obtain the equivalent form of the radiation condition for real ω:

$$a_{jm}^{(2)} = 0\,, \qquad j = 0, 1, 2, \dots\,, \qquad |m| \leqslant j\,. \tag{4.7}$$

The rigorous proof of equivalence between Eqs. (1.5) and (4.7) requires the use of the smoothness of $\psi(\mathbf{x})$.

It is natural to assume that Eqs. (4.7) are the radiation conditions for all ω, including the complex ones.

From the mechanical point of view, the radiation conditions (4.7) allow one to distinguish between two possible types of solutions of the Helmholtz equation: the *outgoing* solutions which correspond to the waves propagating towards infinity and which are singled out by (4.7), and the *incoming* solutions which correspond to the waves propagating from infinity towards the shell and which do not satisfy (4.7).

In order to summarize the mathematical properties of the scattering frequencies, let us introduce the bilinear form

$$\left\langle \mathbf{f}^{(1)}, \mathbf{f}^{(2)} \right\rangle_{\mathcal{H}} = h \iint\limits_{\Gamma} \mathbf{u}^{(1)} \cdot \mathbf{u}^{(2)} \, dS + \rho \iiint\limits_{G_e} \mathbf{grad}\, \psi^{(1)} \cdot \mathbf{grad}\, \psi^{(2)} \, dV \,, \qquad (4.8)$$

which differs from the scalar product (4.3) by the absence of complex conjugation. We shall compute the bilinear form (4.8) not only for elements of $\mathcal{H}$ but also for elements of $\mathcal{H}_{\max}$, that is we shall allow the situation in which $\psi^{(1)}$, $\psi^{(2)}$ grow exponentially as $|\mathbf{x}| \to +\infty$. In this case the volume integral from the right-hand side of Eq. (4.8) requires regularization. We shall evaluate it as

$$\iiint\limits_{G_e} \mathbf{grad}\, \psi^{(1)} \cdot \mathbf{grad}\, \psi^{(2)} \, dV = \lim_{\varepsilon \to +0} \iiint\limits_{G_e} \exp(-\varepsilon |\mathbf{x}| \log |\mathbf{x}|) \, \mathbf{grad}\, \psi^{(1)} \cdot \mathbf{grad}\, \psi^{(2)} \, dV \,.$$

It may be shown that such a regularization is well-defined for any functions $\psi^{(1)}$ and $\psi^{(2)}$ which satisfy the Helmholtz equation with some frequencies $\omega^{(1)}$ and $\omega^{(2)}$ such that

$$\omega^{(1)} + \omega^{(2)} \notin \{ z : \operatorname{Re} z = 0 \,, \ \operatorname{Im} z \leqslant 0 \} \,.$$

Theorem 4.1. *The resolvent of the exterior problem $\mathcal{R}_\omega$ (as an operator acting from $\mathcal{H}_{\min}$ to $\mathcal{H}_{\max}$) admits a meromorphic continuation onto the whole complex ω-plane. The poles of the meromorphic continuation (or the scattering frequencies of the exterior problem) are the values of ω for which the problem (1.4), (1.6$^-$), (1.7), (4.7) has a non-trivial solution (eigenform) $\mathbf{f} = (\mathbf{u}, \psi)$. The pole $\omega = \omega_0$, $\operatorname{Re} \omega_0 \neq 0$, is simple if and only if the eigenforms $\mathbf{f}^{(k)}$, $k = 1, 2, \ldots, l$, corresponding to this scattering frequency can be orthonormalized by the condition $\left\langle \mathbf{f}^{(k)}, \mathbf{f}^{(k')} \right\rangle_{\mathcal{H}} = \delta_{kk'}$. In this case in the vicinity of the resonance*

$$\mathcal{R}_\omega = \frac{1}{\omega_0^2 - \omega^2} \cdot \sum_{k=1}^{l} \mathbf{f}^{(k)} \left\langle \,\cdot\, , \mathbf{f}^{(k)} \right\rangle_{\mathcal{H}} + \widetilde{\mathcal{R}}_\omega \,, \qquad (4.9)$$

where $\widetilde{\mathcal{R}}_\omega$ is the regular part of the resolvent.

5. Asymptotics of scattering frequencies of the exterior problem

Generally, the asymptotic formulae for the scattering frequencies of the exterior problem are quite similar to those for eigenfrequencies of the interior problem, see Section 3. In particular, the same decomposition of the scattering frequencies into three series takes place, and one has to examine these three series separately. The main difference between the interior and exterior problems is that in the latter case the spectrum of scattering frequencies is not real due to radiation, so we have to pay additional attention to the imaginary parts of eigenfrequencies.

5.1. Fluid and tangential scattering frequencies

As in the the case of a Helmholtz equation without a shell[7,17,18], the fluid scattering frequencies are situated in the complex plane quite far from the real line:

$$|\operatorname{Im}\omega^{fl}| = O(1) \qquad \text{as} \qquad h \to +0\,.$$

This happens due to the strong radiation of energy towards infinity by the corresponding eigenmodes. Therefore, as we shall demonstrate in greater detail in the next Section, the fluid scattering frequencies do not generate resonances in the the exterior problem of forced vibrations and are not very interesting from the mechanical point of view. We do not consider them any further.

The tangential series lies at a distance $O(h)$ away from the real line, but in the first approximation remains real and is determined from the same system (3.1) as in the case of the interior problem.

5.2. Flexural scattering frequencies

The flexural scattering frequencies are located very close to the real line: if we consider the low frequency range $\operatorname{Re}\omega^{fl} \sim 1$, then the imaginary part of a flexural scattering frequency is $O(h^\infty)$. This corresponds to the fact that they radiate very little.

One can use the formulae obtained in Section 3 for the flexural eigenfrequencies of the interior problem in order to compute the flexural scattering frequencies of the exterior problem with small modifications. Firstly, one has to invert the signs of all curvatures in Eqs. (3.4) and (3.7). Secondly, one has to understand the counting function $N_h(\omega)$ in Eqs. (3.5), (3.8) as the number of scattering frequencies with the *real part* less than a given ω. Thirdly, one has to remove the term δ_{0m} from the right-hand side of (3.3). Since these changes are elementary, we do not present the resulting formulae, see [15,16,22,23].

In principle the same formulae can be used for higher frequencies[6], however the results for higher frequencies were obtained on the "physical level of rigour" without detailed mathematical proofs.

6. Resonance phenomena

6.1. Statement of the problem

One of the most important problems in the theory of vibrations of shells contacting fluid is to find effective asymptotic formulae for the solutions of the problem of forced vibrations (interior and exterior). In this Section we concentrate on the exterior problem since the interior one is well presented in the accessible literature[15].

In order to make the problem more realistic we introduce an additional dimensionless parameter $\gamma > 0$ which characterizes the damping in the material of the shell. The *internal friction* in the shell prevents the system from developing unrealisticly sharp peaks (resonances) in the amplitude vs. frequency diagram and therefore makes the problem tractable for all ω in the selected frequency range. Recall that in the absence of the internal friction the imaginary parts of flexural eigenfrequencies are very small, of order of $h^{+\infty}$.

With account of the damping in the shell material the shell equation (1.6^-) reads

$$(1 - i\gamma)h\mathcal{L}\mathbf{u} = h\omega^2\mathbf{u} - \rho\omega^2\psi|_\Gamma\mathbf{n} + \mathbf{g}\,. \tag{6.1}$$

Here the real frequency ω varies in the given range $[\Omega_{\min}, \Omega_{\max}]$, $0 < \Omega_{\min} < \Omega_{\max}$, $\Omega_{\min} \sim 1$, $\Omega_{\max} \sim 1$, and a smooth load $\mathbf{g}(\alpha)$ is given. This equation is supplemented by the Helmholtz equation (1.4), the non-penetration condition (1.7) and by the Sommerfeld radiation condition (1.5). We also introduce a technical requirement

$$\frac{\log\gamma}{\log h} \sim 1\,,$$

which is satisfied in most practical situations.

6.2. Iteration process and asymptotic expansion in the absence of tangential eigenfrequencies

Suppose that our frequency range $[\Omega_{\min}, \Omega_{\max}]$ does not contain any of the eigenfrequencies ω_j^{tn} of the auxiliary tangential problem (3.1). Under this additional assumption we shall describe now a formal asymptotic procedure for solving our problem (6.1), (1.4), (1.7), (1.5).

Let us denote by $\mathbf{u}^{(k)}(\alpha)$, $\psi^{(k)}(\mathbf{x})$, $k = 0, 1, 2, \ldots$, the kth asymptotic approximation to the solution $\mathbf{u}(\alpha)$, $\psi(\mathbf{x})$ of the problem (6.1), (1.4), (1.7), (1.5). The asymptotic process for determining these approximations is as follows.

Let us set $u_3^{(0)}(\alpha) \equiv 0$ and determine $u_1^{(0)}(\alpha)$, $u_2^{(0)}(\alpha)$ from the system of equations

$$(1 - i\gamma)\begin{pmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{pmatrix}\begin{pmatrix} u_1^{(0)} \\ u_2^{(0)} \end{pmatrix} - \omega^2\begin{pmatrix} u_1^{(0)} \\ u_2^{(0)} \end{pmatrix} = \frac{1}{h}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

Then we determine $\psi^{(0)}(\mathbf{x})$ from the equation

$$\Delta\psi^{(0)} + \frac{\omega^2}{c^2}\psi^{(0)} = 0$$

subject to the conditions

$$\psi^{(0)}\Big|_\Gamma = \frac{1}{\rho\omega^2}\left(g_3 - h(1-i\gamma)\left(\left(\ell_{31} + \frac{h^2}{12}n_{31}\right)u_1^{(0)} + \left(\ell_{32} + \frac{h^2}{12}n_{32}\right)u_2^{(0)}\right)\right),$$

$$\frac{\partial\psi^{(0)}}{\partial r} - \frac{i\omega}{c}\psi^{(0)} = o(r^{-1}) \qquad \text{as} \qquad r\to\infty\,.$$

Now, assume that we have already determined $\mathbf{u}^{(k)}(\alpha)$, $\psi^{(k)}(\mathbf{x})$, for some $k = 0,1,2,\ldots$. Then $\mathbf{u}^{(k+1)}(\alpha)$, $\psi^{(k+1)}(\mathbf{x})$ is determined in the following way. Set

$$u_3^{(k+1)} = \frac{\partial\psi^{(k)}}{\partial n}\Big|_\Gamma\,.$$

The tangential displacements $u_1^{(k+1)}(\alpha)$, $u_2^{(k+1)}(\alpha)$ are determined from the system of equations

$$(1-i\gamma)\begin{pmatrix}\ell_{11} & \ell_{12}\\ \ell_{21} & \ell_{22}\end{pmatrix}\begin{pmatrix}u_1^{(k+1)}\\ u_2^{(k+1)}\end{pmatrix} - \omega^2\begin{pmatrix}u_1^{(k+1)}\\ u_2^{(k+1)}\end{pmatrix} =$$
$$\frac{1}{h}\begin{pmatrix}g_1\\ g_2\end{pmatrix} - \frac{h^2}{12}(1-i\gamma)\begin{pmatrix}n_{11} & n_{12}\\ n_{21} & n_{22}\end{pmatrix}\begin{pmatrix}u_1^{(k)}\\ u_2^{(k)}\end{pmatrix} - (1-i\gamma)\begin{pmatrix}\frac{h^2}{12}n_{13}+\ell_{13}\\ \frac{h^2}{12}n_{23}+\ell_{23}\end{pmatrix}u_3^{(k+1)}\,.$$

Then we determine $\psi^{(k+1)}(x)$ from the equation

$$\Delta\psi^{(k+1)} + \frac{\omega^2}{c^2}\psi^{(k+1)} = 0$$

subject to the conditions

$$\psi^{(k+1)}\Big|_\Gamma = \frac{1}{\rho\omega^2}\Big(g_3 + \rho h\omega^2 u_3^{(k+1)}$$
$$-h(1-i\gamma)\left(\left(\ell_{31} + \frac{h^2}{12}n_{31}\right)u_1^{(k+1)} + \left(\ell_{32} + \frac{h^2}{12}n_{32}\right)u_2^{(k+1)}\right)\Big),$$

$$\frac{\partial\psi^{(k+1)}}{\partial r} - \frac{i\omega}{c}\psi^{(k+1)} = o(r^{-1}) \qquad \text{as} \qquad r\to\infty\,,$$

and so on.

The asymptotic process described above is organized in such a manner that at each stage we have to solve partial differential equations which do not contain the small parameter h in their coefficients; h appears only in the non-homogeneous terms (right-hand sides and boundary conditions). Consequently, our approximations $\mathbf{u}^{(k)}(\alpha)$, $\psi^{(k)}(\mathbf{x})$ are polynomials in h. Moreover, our asymptotic process is such that the correction introduced at each iteration is of an order in h which grows with the number of the iteration. Consequently, the coefficients of our polynomials at any given power of h "stabilize" when the number of the iteration tends to infinity. As a result, our approximations $\mathbf{u}^{(k)}(\alpha)$, $\psi^{(k)}(\mathbf{x})$ turn to

$$u_p(\alpha) \sim \sum_{j=-1}^{+\infty} u_{p,j}(\alpha) h^j \,, \qquad p = 1, 2, \tag{6.2}$$

$$u_3(\alpha) \sim \sum_{j=0}^{+\infty} u_{3,j}(\alpha) h^j \,, \tag{6.3}$$

$$\psi(\mathbf{x}) \sim \sum_{j=0}^{+\infty} \psi_j(\mathbf{x}) h^j \tag{6.4}$$

when $k \to +\infty$; here all the $u_{p,j}(\alpha)$, $p = 1, 2, 3$, and the $\psi_j(\mathbf{x})$ are independent of h.

This is the formal asymptotic expansion for the solution $\mathbf{u}(\alpha)$, $\psi(\mathbf{x})$ of the problem (6.1), (1.4), (1.7), (1.5), in the sense that if we formally substitute Eqs. (6.2)–(6.4) into our system (6.1), (1.4), (1.7), (1.5) and collect the terms with the same power of the small parameter h, the Eqs. (6.1), (1.4), (1.7), (1.5) will be satisfied.

It is worth noting that the functions $u_{p,j}(\alpha)$ and $\psi_j(\mathbf{x})$ depend on the second small parameter, that is on the damping ratio γ. However this dependence is of a regular character in the sense that these functions are holomorphic in γ in a neighbourhood of $\gamma = 0$. This is why we do not write an additional expansion of each $u_{p,j}(\alpha)$ and $\psi_j(\mathbf{x})$ into a power series with respect to γ, it is simply not very interesting. However, we heavily use here the fact that our frequency range does not contain the eigenfrequencies of the tangential problem. The theorem about the asymptotic convergence of the formal expansions (6.2)–(6.4) is given in Vassiliev[23].

6.3. Asymptotic expansion in the vicinity of a tangential eigenfrequency

Let $\omega_j^{tn} \sim 1$ be a simple eigenfrequency of the auxiliary tangential problem (3.1). Then, the exterior problem has a scattering frequency ω_0 such that

$$\operatorname{Re}\omega_0 \sim \omega_j^{tn} \,, \qquad \operatorname{Im}\omega_0 \sim O(h) + O(\gamma) \,.$$

Let us denote by $\mathbf{f}_0 = (\mathbf{u}_0, \psi_0)$ the corresponding eigenform of the exterior problem normalized by the condition $\langle \mathbf{f}_0, \mathbf{f}_0 \rangle_{\mathcal{H}} = 1$. Using Theorem 4.1 and Eq. (4.8) we conclude that for ω close to ω_j^{tn} the solution $(\mathbf{u}, \psi)$ of the exterior non-homogeneous problem has the asymptotics

$$\mathbf{f} = \frac{\iint\limits_{\Gamma} \mathbf{g} \cdot \mathbf{u}_0 \, d\Gamma}{\omega_0^2 - \omega^2} \mathbf{f}_0 + \widetilde{\mathbf{f}} \,, \tag{6.5}$$

where $\widetilde{\mathbf{f}} = (\widetilde{\mathbf{u}}, \widetilde{\psi})$ admits (regular) asymptotic expansions of the type (6.2)–(6.4), see VASSILIEV[23] for proofs. Note, however, that in this case the dependence of the first term in (6.5) on γ is non-trivial.

6.4. Interpretation of the results

The asymptotic expansions (6.2)–(6.4) and (6.5) provide important information about the behaviour of the solution of the exterior non-homogeneous problem in different frequency ranges and under different types of loads.

Before proceeding to this analysis let us recall that we derived our expansions (6.2)–(6.4) and (6.5) under the assumption that the load $\mathbf{g}(\alpha)$ is smooth ($\mathbf{g} \in C^\infty(\Gamma)$). The analysis of the asymptotic formulae (6.2)–(6.5) shows that the flexural scattering frequencies (which constitute the bulk of all scattering frequencies) do not affect our solution. The explanation of this surprising phenomenon is that the eigenforms corresponding to the flexural scattering frequencies are *strongly oscillating* along Γ (namely, their characteristic wavelengths are of order $h^{3/5}$, see VASSILIEV[20,22,23] for details), so the smooth external load $\mathbf{g}$ is asymptotically orthogonal to the eigenforms $\mathbf{u}$ in $\mathbf{L}^2(\Gamma)$.

Therefore, applying a smooth load upon the shell and changing the frequency of vibration, one can observe only the resonances generated by the tangential scattering frequencies. This explain why the bulk of flexural scattering frequencies usually cannot be seen in scattering experiments.

In the non-smooth case ($\mathbf{g} \in \mathcal{D}'(\Gamma)$) the situation is completely different. There is no reason for the load to be asymptotically orthogonal to the eigenforms of the homogeneous problem, so all the $O(h^{-6/5})$ eigenfrequencies of the homogeneous problem are in general excited, which creates an extremely complicated vibration pattern on the shell. The difficulties arising in the case of a non-smooth load can be seen from our asymptotic process: though we gain powers of h with each iteration, the order of singularities increases as well, so one cannot expect asymptotic convergence. Nevertheless, our asymptotic expansions make some sense even in the non-smooth case. Namely, it was shown in VASSILIEV[22,23] that for any $\mathbf{g} \in \mathcal{D}'(\Gamma)$ we have the asymptotic convergence of ψ outside the immediate vicinity of the shell.

The case when the shell is a shell of revolution and the load is concentrated on a circle (i.e. $\mathbf{g}$ is, after the separation of the angular variable, a δ-function or a derivative of a δ-function) was analyzed in GOLDENVEIZER–VASSILIEV[4]; see also VASSILIEV[22] for more details.

Note that the case of non-smooth loads is also very important in applications because it naturally arises when one tries to describe the effect of various stiffening elements supporting the shell, like bulkheads, stringers and stiffening rings. In this case there is a complex interaction between flexural and tangential vibrations modes, see VASSILIEV[22] for details and graphs.

7. Effect of fluid viscosity

7.1. Statement of the problem

In some situations, it is necessary to consider more elaborate models of shell-fluid coupled vibrations than those studied in Sections 1–6. In particular, one may need to take into account the viscosity of the fluid. The authors of this paper have obtained a number of results in this problem; these results are briefly discussed in this Section.

We restrict ourselves to the interior problem which is more motivated by practice.

The scalar Helmholtz equation which describes the vibrations of an ideal compressible fluid should be replaced in the case of a *viscous compressible fluid* by the system of four scalar equations

$$-\rho i\omega \mathbf{v} = \mu \left(\Delta \mathbf{v} + \frac{1}{3} \,\mathbf{grad}\, \mathrm{div}\, \mathbf{v} \right) - \mathbf{grad}\, p \,, \tag{7.1}$$

$$-i\omega \mathbf{v} = -\rho c^2 \mathrm{div}\, \mathbf{v} \,, \tag{7.2}$$

where $\mathbf{v}(\mathbf{x}) = \mathbf{v}^*/c_s^* = (v_1, v_2, v_3)$ is the vector of fluid velocity, $p(\mathbf{x}) = p^*/E^*$ is the fluid pressure, and $\mu = \mu^*/(\rho *_s E^* R^{*2})$ is the dimensionless viscosity coefficients which is considered further on as the additional small parameter. As usually, Eqs. (7.1)–(7.2) are obtained after the separation of the time factor $\exp(-i\omega t)$.

The equations of shell vibrations with account of the presence of the viscous fluid have the form

$$h\mathcal{L}\mathbf{u} = h\omega^2 \mathbf{u} + g - \left. (-p\mathbf{n} + \mu T \mathbf{v}) \right|_\Gamma \,, \tag{7.3}$$

where the operator T is defined in the vicinity of the shell as

$$T\mathbf{v} = \sigma'(\mathbf{v})\mathbf{n} \,,$$

the components of the viscous stress tensor $\sigma'(\mathbf{v})$ being

$$\sigma'_{ij}(\mathbf{v}) = -\frac{2}{3}\delta_{ij} \mathrm{div}\, \mathbf{v} + \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) , \qquad i, j = 1, 2, 3 \,.$$

Finally, Eqs. (7.1)–(7.3) should be supplemented by the boundary conditions

$$\mathbf{v}|_\Gamma = -i\omega \mathbf{u} \,. \tag{7.4}$$

7.2. Effect of viscosity on eigenfrequencies

Taking the viscosity into account changes the structure of the spectrum of free vibrations ($\mathbf{g} = \mathbf{0}$) significantly. In particular, the spectrum is no longer purely discrete: one can show (see LEVITIN[8−11]) that there are two points of the *essential spectrum* which are located on the imaginary ω-axis. The remaining spectrum consists of isolated eigenfrequencies of finite multiplicity, and is located for any $\mu > 0$ within the sector

$$\{\operatorname{Im}\omega \leqslant -C_1\mu, |\operatorname{Re}\omega| \leqslant C_2\mu^{-1}\operatorname{Im}\omega\},$$

where C_1, C_2 are some positive constants independent of μ.

As $\mu \to +0$ the spectrum of eigenfrequencies of the viscous problem splits into two subseries. The eigenfrequencies of the first subseries are localized in the vicinity of the imaginary non-negative ω-semiaxis and do not present any mechanical interest (though there is a vast mathematical literature devoted to their study). The eigenfrequencies of the second subseries tend (as μ goes to zero) to the eigenfrequencies of the inviscid problem (1.4), (1.6^+), (1.7) ($\mathbf{g} = \mathbf{0}$). One of the most important results is that if $\omega_0 > 0$ is a simple eigenfrequency of the inviscid problem, then the corresponding eigenfrequency ω of the viscous problem has an absolutely convergent power series expansion

$$\omega = \omega_0 + \mu^{1/2}\omega_1 + \mu^1\omega_2 + \dots .$$

The leading term of this expansion can be found analytically:

$$\mu^{1/2}\omega_1 = \frac{-i^{3/2}\rho^{1/2}\omega_0^{1/2}\|\mathbf{u}_{0\tau} - \mathbf{grad}_\tau\psi_0\|^2_{L^2(\Gamma)}}{2\left(h\|\mathbf{u}_0\|^2_{L^2(\Gamma)} + \rho\|\mathbf{grad}\,\psi_0\|^2_{L^2(G_i)}\right)}. \tag{7.5}$$

Here $\mathbf{u}_0$, ψ_0 is the eigenform of the inviscid problem corresponding to ω_0, and the subscript τ is used to denote the tangential components of vectors on Γ.

With the account of the asymptotic formulae of Section 3 (as $h \to +0$) Eq. (7.5) allows to estimate the effect of viscosity on various types of eigenfrequencies. In particular,

$$|\operatorname{Im}\omega| = O\left(\mu^{1/2}h^{-1}\right) \qquad \text{for tangential eigenfrequencies,}$$

$$|\operatorname{Im}\omega| = O\left(\mu^{1/2}h^{-3/5}\right) \qquad \text{for flexural eigenfrequencies,}$$

$$|\operatorname{Im}\omega| \leqslant \left(\mu^{1/2}h^{0}\right) \qquad \text{for fluid eigenfrequencies.}$$

Therefore, the effect of dissipation in viscous fluid is most noticeable for the tangential eigenfrequencies and least noticeable for the fluid eigenfrequencies.

We refer the reader to [8–14] for other results in the theory of vibrations of shells contacting viscous fluid, such as the asymptotics of solutions in the problem of forced vibrations, and the exterior problem.

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