Trace Identities and Universal Estimates for Eigenvalues of Linear Pencils*

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Abstract

We describe the method of constructing the spectral trace identities and the estimates of eigenvalue gaps for the linear self-adjoint operator pencils $A - \lambda B$.

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1 Introduction

Since 1950's, mathematicians have devoted a lot of efforts to the construction of the so-called *universal eigenvalue estimates* for elliptic boundary value problems. As an illustration, we recall the original result of Payne, Pólya

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and Weinberger [PayPoWe], who have shown in 1956 that if $\{\lambda_j\}$ is the set of (positive) eigenvalues of the Dirichlet boundary value problem for the Laplacian in a domain $\Omega \subset \mathbb{R}^n$, then

$$\lambda_{m+1} - \lambda_m \leq \frac{4}{mn} \sum_{j=1}^{m} \lambda_j \tag{PPW}$$

for each $m = 1, 2, \ldots$. The term *universal estimate* is used in this context since (PPW) does not involve any information on the domain Ω apart from the dimension n.

The Payne–Pólya–Weinberger inequality has been significantly improved over the course of several decades; similar universal estimates have been also obtained in the spectral problems for operators other then the Euclidean Dirichlet Laplacian (or Schrödinger operator), e.g. higher order differential operators in $\mathbb{R}^n$, operators on manifolds, systems like Lamé system of elasticity etc. We refer the reader to the important paper [HaSt] and the recent survey [Ash] for more details.

In the recent paper [LePar], we have shown that the majority of the universal eigenvalue estimates known so far can be easily obtained from a certain operator trace identity which is valid not only for boundary value problems for PDEs but, under minimal conditions, for a general self-adjoint operator acting in a Hilbert space. Namely, we have proved the following

Theorem 1.1. *Let H and G be self-adjoint operators such that $G(D_H) \subseteq D_H$. Let λ_j and ϕ_j be eigenvalues and eigenvectors of H. Then for each j*

$$\begin{aligned} \sum_k \frac{|\langle [H,G]\phi_j, \phi_k\rangle|^2}{\lambda_k - \lambda_j} &= -\frac{1}{2}\langle [[H,G],G]\phi_j, \phi_j\rangle \\ &= \sum_k (\lambda_k - \lambda_j)|\langle G\phi_j, \phi_k\rangle|^2. \end{aligned} \tag{1.1}$$

This result implies

Theorem 1.2. *Under conditions of Theorem 1.1,*

$$-(\lambda_{m+1} - \lambda_m) \sum_{j=1}^{m} ([[H,G],G]\phi_j, \phi_j) \leq 2 \sum_{j=1}^{m} \|[H,G]\phi_j\|^2 . \tag{1.2}$$

The inequalities (1.2) may be viewed as a family of estimates of the spectral gap $\lambda_{m+1} - \lambda_m$ for the operator H, with different estimates being obtained for each choice of the operator G. The particular choice of G depends, of course, on the problem, and cannot be prescribed. In [LePar], we give numerous examples of concrete estimates for boundary value problems for elliptic PDEs (and systems of PDEs) obtained using Theorem 1.2. In particular, the classical estimate (PPW) follows from Theorem 1.2 by taking G to be an operator of multiplication by the coordinate x_l, summing the resulting equalities over l, and using some elementary bounds.

The aim of this paper is to extend Theorems 1.1 and 1.2 to the spectra of *linear self-adjoint operator pencils* of the type $A - \lambda B$. The statement of the problem and the main results are collected in Section 2; the proofs are in Section 3.

2 Statement of the Problem and Main Results

We consider a linear operator pencil

$$\mathcal{P}(\lambda) = A - \lambda B\,. \tag{2.1}$$

acting in a Hilbert space $\mathcal{H}$ equipped with the scalar product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. Here, A and B are self-adjoint operators such that $D_A \subseteq D_B$. We identify the domain of the pencil $D_\mathcal{P}$ with D_A.

By the spectrum of $\mathcal{P}$ we understand the set of complex values λ for which $\mathcal{P}(\lambda)$ is not invertible. Throughout this paper we assume, for simplicity, that the spectrum consists of isolated real eigenvalues $\lambda_1 \le \lambda_2 \le \dots$ (counted with multiplicity) which may accumulate only to $+\infty$, and that the system of corresponding eigenfunctions u_j such that $\mathcal{P}(\lambda_j)u_j = 0$, is complete in $\mathcal{H}$ (the case of a pencil with a continuous spectrum can be treated as well, cf. Remark 2.5 in [LePar]). We refer to [GoKr] for a general discussion of the spectral theory of operator pencils.

An important property, which we shall use later on, is the fact that, under a technical condition $\langle Bu_j, u_j\rangle \ne 0$ (which is satisfied automatically if, e.g., $B > 0$), the eigenfunctions can be assumed to be normalised by the relations

$$\langle Bu_j, u_k\rangle = \delta_{jk}\,. \tag{2.2}$$

Indeed, multiplying the equality $\mathcal{P}(\lambda_j)u_j = 0$ by u_k in $\mathcal{H}$, and using the fact that A and B are self-adjoint, we get

$$\langle Au_j, u_k\rangle = \langle u_j, Au_k\rangle = \lambda_j\langle Bu_j, u_k\rangle = \lambda_k\langle u_j, Bu_k\rangle\,,$$

which implies (2.2).

By completeness, any element $f \in \mathcal{H}$ can be expanded in a convergent series

$$f = \sum_k \langle Bf, u_k\rangle u_k \,,$$

and for any $f, g \in \mathcal{H}$,

$$\langle Bf, g\rangle = \sum_k \langle Bf, u_k\rangle \cdot \langle u_k, Bg\rangle \,.$$

Thus, by the density argument,

$$\langle f, g\rangle = \sum_k \langle f, u_k\rangle \cdot \langle u_k, Bg\rangle \tag{2.3}$$

In what follows, $[H, G]$ denotes the standard commutator $HG - GH$.

Our main result is the following spectral trace identities for $\mathcal{P}(\lambda)$.

Theorem 2.1. *Let $\mathcal{P}(\lambda) = A - \lambda B$ be a self-adjoint linear operator pencil with discrete spectrum λ_j and eigenfunctions u_j. Let G be an auxiliary self-adjoint operator with domain D_G such that $G(D_{\mathcal{P}}) \subseteq D_{\mathcal{P}} \subseteq D_G$. Then for each j*

$$\sum_k \frac{|\langle[\mathcal{P}(\lambda_j), G]u_j, u_k\rangle|^2}{\lambda_k - \lambda_j} = -\frac{1}{2}\langle[[\mathcal{P}(\lambda_j), G], G]u_j, u_j\rangle. \tag{2.4}$$

and

$$\sum_k (\lambda_k - \lambda_j)|\langle BGu_j, u_k\rangle|^2 = -\frac{1}{2}\langle[[\mathcal{P}(\lambda_j), G], G]u_j, u_j\rangle. \tag{2.5}$$

Remark 2.2. Instead of the condition $G(D(A)) \subseteq D(A)$ we can impose weaker conditions $Gu_j \in D(A)$, $G^2u_j \in D(A)$, $j = 1, \dots$. Moreover, the latter condition can be dropped if the double commutator is understood in the weak sense, i.e., if the right-hand side of (2.4) and (2.5) is replaced by $\langle[\mathcal{P}(\lambda), G]u_j, Gu_j\rangle$ (see (3.5) below).

The trace identities (2.5) imply the following universal estimate, which generalises the Payne–Pólya–Weinberger inequality for linear operator pencils.

Theorem 2.3. *Under conditions of Theorem 2.1,*

$$-(\lambda_{m+1} - \lambda_m)\sum_{j=1}^{m}([[\mathcal{P}(\lambda_j), G], G]u_j, u_j) \le 2\sum_{j=1}^{m} \|B^{-1}[\mathcal{P}(\lambda_j), G]u_j\|^2 \,. \tag{2.6}$$

3 Proof of the Main Results

We start with establishing (2.5).

Obviously, we have

$$[A-\lambda_j B, G]u_j = (A-\lambda_j B)Gu_j\,. \tag{3.1}$$

Therefore,

$$\langle [A-\lambda_j B, G]u_j, Gu_j\rangle = \langle (A-\lambda_j B)Gu_j, Gu_j\rangle\,. \tag{3.2}$$

Since A, B and G are self-adjoint, we have, using (2.3)

$$\begin{aligned}
&\langle (A-\lambda_j B)Gu_j, Gu_j\rangle \\
&\quad = \sum_k \langle (A-\lambda_j B)Gu_j, u_k\rangle\langle u_k, BGu_j\rangle \\
&\quad = \sum_k \langle Gu_j, (A-\lambda_j B)u_k\rangle\langle u_k, BGu_j\rangle \\
&\quad = \sum_k (\lambda_k-\lambda_j)|\langle BGu_j, u_k\rangle|^2.
\end{aligned} \tag{3.3}$$

Using the fact that $[A-\lambda_j B, G]$ is skew-adjoint, the left-hand side of (3.2) can be rewritten as

$$\begin{aligned}
\langle G[A-\lambda_j B, G]u_j, u_j\rangle &= -\langle [[A-\lambda_j B, G], G]u_j, u_j\rangle + \langle [A-\lambda_j B, G]Gu_j, u_j\rangle \\
&= -\langle [[A-\lambda_j B, G], G]u_j, u_j\rangle - \langle u_j, G[A-\lambda_j B, G]u_j\rangle\,,
\end{aligned} \tag{3.4}$$

so

$$\langle G[A-\lambda_j B, G]u_j, u_j\rangle = -\frac{1}{2}\langle [[A-\lambda_j B, G], G]u_j, u_j\rangle \tag{3.5}$$

(notice that $\langle G[A-\lambda_j B, G]u_j, u_j\rangle$ is real, see (3.2) and (3.3)). This proves (2.5).

Since (3.1) implies

$$\langle [A-\lambda_j B, G]u_j, u_k\rangle = \langle Gu_j, (A-\lambda_j B)u_k\rangle = (\lambda_k-\lambda_j)\langle BGu_j, u_k\rangle\,,$$

this also proves (2.4).

We now proceed to the proof of (2.6). Let us sum the equations (2.4) over $j = 1, ..., m$. Then we have

$$\sum_{j=1}^{m} \sum_{k=m+1}^{\infty} \frac{|([\mathcal{P}(\lambda_j), G]u_j, u_k)|^2}{\lambda_k - \lambda_j} = -\frac{1}{2} \sum_{j=1}^{m} ([[\mathcal{P}(\lambda_j), G], G]u_j, u_j)\,. \tag{3.6}$$

Parceval's equality implies that the left-hand side of (3.6) is not greater than $\displaystyle\frac{1}{\lambda_{m+1} - \lambda_m} \sum_{j=1}^{m} \|B^{-1}[\mathcal{P}(\lambda_j), G]u_j\|^2$. This proves (2.6).

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