EC933-G-AU INTERNATIONAL FINANCE – LECTURE 5

ASSET MARKETS AND RISK SHARING: ANALYTICAL INTRODUCTION OF UNCERTAINTY

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ABSTRACT. This lecture continues to develop the analytical basics of microfounded open-economy models. Having incorporated trade across *time* into a standard simplified framework of international economic interdependence, we now turn to trade across *states of nature*. The two problems seem at first different, yet they have a common underlying structure. This similarity is employed further in the lecture, where we focus on some parallels (i.e., analogy, if not equivalence) between results derived in dynamic models with no uncertainty and stochastic models with no dynamics. Section 1 begins by a *small* open-economy *real* model with *two* states of nature in the second period and introduces a number of key concepts. Section 2 extends the setting to a *two-country multiple-state* global economy. Both these sections abstract from *capital market imperfections*, and section 3 then sketches what are the main problems with risk sharing under imperfections related to sovereign risk, hidden information and moral hazard.

Date: 1 March 2006 (First draft: 7 November 2004).

This set of lecture notes is preliminary and incomplete. It is based on parts of the four textbooks suggested as essential and supplementary reading for my graduate course in international finance at Essex as well as on the related literature (see the course outline and reading list at http://courses/essex.ac.uk/ec/ec933/). The notes are intended to be of some help to the students attending the course and, in this sense, many aspects of them will be clarified during lectures. The present second draft may be developed and completed in future revisions. The responsibility for any errors and misinterpretations is, of course, only mine. Comments are welcome, preferably by e-mail at mihailov@essex.ac.uk and/or a_mihailov@hotmail.com.

Contents

1. A Stochastic Two-Period Real Model of a Small Open Economy

The models considered in our previous lectures abstracted from issues related to *uncertainty* or *risk*. It is now time to see how the realistic possibility of *multiple* "states of nature" (also called "states of the world") at each date could tractably be handled in the microfounded framework we started building in lecture 4. As in the preceding lecture, the notes that follow draw heavily on Obstfeld and Rogoff's (1996) excellent graduate textbook.

1.1. Assumptions.

Our starting point is the *same* set of assumptions as those in section 1.1.1 of lecture 4. To them, we add the following *additional* assumptions, which make the similar 2-period SOE model with no uncertainty of the previous lecture *different* from the *stochastic* (that is, with uncertainty) 2-period SOE model here in lecture 5.

(1) A first difference is that now we allow for a richer set-up in which not just one, certain, but *two* states of nature are possible at date 2, and the actual realisation of any of them is uncertain (from the perspective of date 1); these two states
   (a) occur randomly, according to a specified (i.e., known to agents) probability distribution;[1] we assume that state s occurs with probability $\pi(s)$, for $s = 1, 2$;
   (b) and differ only in their associated endowment (or, more generally, output or income) levels in the terminal period 2, $y_2(1)$ and $y_2(2)$.

(2) A second key simplification in the present section is that economic agents have *sufficient* foresight to prearrange, by explicit or implicit contracts, for trades in assets that protect them at least partially against future contingencies affecting their well-being.

As we have done previously,

- let us again assume a *constant* population size, convenient to be normalised at 1, so that the endowment (output or income) and consumption of the representative individual can be identified with (i.e., are the same as) national aggregate endowment (output or income) and consumption;
- it will be furthermore assumed that the *representative* individual:
  - has *known* (thus, certain) income y_1 in the first period;
  - and starts out with *zero* net foreign assets.

1.2. State-Contingent Consumption Plans.

It is important to understand and model in an appropriate way one key difference between lifetime consumption under certainty (as in lecture 4) and under uncertainty (as in this lecture). It is that an individual with uncertain future endowment (output, income) cannot predict his optimal consumption level exactly. He could instead only plan (*ex ante*) a range of consumption levels, each *contingent* on the state of nature that can actually materialise. Such desired *contingency plans* for consumption are at the centre of decision making under uncertainty.

In general, which of the many planned consumption levels will be actually chosen (*ex post*) in a given *period of time*, t, will depend on the observed *state of nature* in that period of time, often denoted s_t, and on the *history* of relevant (economic) outcomes up to, and including, the last period, often denoted $s^t \equiv (..., s_0, s_1, s_2, ..., s_{t-1}, s_t)$. To this entirely *backward*-looking determination of consumption economists usually add also a *forward*-looking component of perhaps equal importance, namely the (rational) *expectations* of agents about their future income, hence consumption.

In terms of the simple 2-period stochastic model we build up here, $c_2(s)$, for $s = 1, 2$, shall denote the two state-contingent consumption plans for date 2 (made ex ante, i.e., from the perspective of date 1).

[1] Note that the literature sometimes distinguishes "risk" from "uncertainty" or rather *Knightean* – due to Knight (1921) – *uncertainty*. The latter has been more recently also termed *ambiguity* and has been distinguished from "uncertainty" or, equivalently, "risk". Knightean uncertainty or ambiguity relaxes the assumption of known (knowable) probability distribution for the stochastic process(es) inherent in economic decision making. In this lecture we would not consider such more realistic but more complicated cases, the focus remaining on modelling fixed probabilities.

1.3. **The Consumer's Problem under Uncertainty.**

1.3.1. *Lifetime Expected Utility.* The usual assumption in models with uncertainty is that the consumer's satisfaction is measured on the initial date by lifetime *expected* utility, i.e., by *average* lifetime utility *given* the chosen contingency plans for future consumption. Let c_1 denote consumption on date 1. c_1 must be chosen *before* uncertainly is resolved, and thus cannot depend on the state of nature that occurs. The latter is observed by agents on date 2.

The representative individual lifetime expected utility on date 1 is

$$U_l \equiv \underbrace{\pi(1)\{u(c_1)+\beta u[c_2(1)]\}}_{\text{if } s=1} + \underbrace{\pi(2)\{u(c_1)+\beta u[c_2(2)]\}}_{\text{if } s=2}.$$

By the definition of probability, $\pi(1)+\pi(2)=1$, so – replacing $\pi(2)$ above by $1-\pi(1)$ and expanding the expression – one could see that it can also be written as

$$U_l \equiv u(c_1) + \underbrace{\beta\overbrace{\{\pi(1)u[c_2(1)]+\pi(2)u[c_2(2)]\}}^{\text{would be simply } u(c_2) \text{ under certainty (cf. lecture 4)}}}_{\equiv\beta E_1[u(c_2)],\text{ i.e., } expected \text{ (ex ante) utility of consumption on date 2}}. \tag{1.1}$$

An implicit assumption in (1.1) is that the utility function $u(c)$ does not depend on the realised state of nature, i.e., utility is not state-dependent. By analogy with the invariance of period utility across *time* we assumed in lecture 4, we may now say that (period) utility is here also invariant (or stable) across *states* of nature. This assumption allows us to leave $u(c)$ unindexed (either by date t or by state s), which in general need not be the case. An obvious example is that when a person unexpectedly falls ill, his relative preference for various commodities may well change.

1.3.2. *Arrow-Debreu Securities and Complete Asset Markets.* It is the type of asset market structure, namely *complete* asset markets, posited by the Arrow (1953, 1964) – Debreu (1959) paradigm, that makes the choice of consumption in different *states* exactly analogous to the choice of consumption on different *dates* (or periods) or, still, to the choice of *different* consumption goods on a *single* date (or period). The main assumptions of the Arrow-Debreu paradigm (of complete markets), as they apply to the simple two-period analytical framework considered here, are as follows.

(1) There is a worldwide market in which people can buy *contingent claims.*
(2) These contingent claims have period 2 payoffs that vary according to the exogenous shocks which materialise in period 2, i.e., their payoffs depend on the state of nature that has actually occurred, s (itself defined in terms of a *particular* combination of shock realisations in period 2).
(3) More specifically, let us define **the Arrow-Debreu (A-D) security for state of nature** s to have the following *payoff structure*: the owner (who has bought) the A-D security receives – and, symmetrically, the issuer (who has sold) the A-D security pays – 1 unit of output on date 2 if state s (*prespecified* in the explicit or implicit contract implied by the purchase/sale of the A-D security) occurs on date 2 but nothing in all other states. We next assume, as in the Arrow-Debreu paradigm, that there exists a *competitive* market in A-D securities for *every* possible state.
(4) We may also allow for borrowing and lending, i.e., people to sell and buy *noncontingent* (also called riskless or risk-free) assets, such as bonds that pay for sure $1+r$ per unit on date 2 regardless of the state of nature. As in lecture 4, r denotes the (net) real rate of interest when the payoff is certain, that is, the riskless (or safe) RIR. Yet if A-D securities for every possible state are available for purchase/sale, the bond market turns out to be *redundant*, in the sense that its elimination would not affect the economy's equilibrium. To take an evident example, consider our two-date two-state model: in it, the simultaneous purchase of $1+r$ units (that is, number or quantity, implying perfect divisibility) of state 1 A-D securities and $1+r$ units of state 2 A-D securities on date 1 assures a payoff of $1+r$ output units on date 2 no matter which of the two potential states has actually materialised, just as buying on date 1 a one-period (risk-free) bond

would have done. Bonds therefore add nothing to the trading opportunities people have once a *full* set of Arrow-Debreu contingent claims can be traded. The example provides a simple but clear illustration of how prices of more complicated assets (e.g., multi-period bonds or options) could be constructed once one knows the *primal* Arrow-Debreu prices. What is meant by **complete asset markets** is exactly that people can trade *an A-D security* corresponding to *every* future state of nature.

It may appear unrealistic to assume that markets for Arrow-Debreu securities exist: after all, no one has seen such asset prices quoted in standard sources, say, *Financial Times* or *Wall Street Journal*!... However, virtually all real-world assets have state-contingent payoffs: some of these assets, e.g., stocks and stock options, are traded in organised markets (stock exchanges) whereas others, e.g., insurance contracts of various types, are traded on a case-by-case basis. And in most circumstances it can be shown that repeated trading in familiar, real-life securities such as stocks is capable of replicating the allocations that arise if a complete set of A-D contingent claims were traded instead. That is why, despite the fact that Arrow-Debreu securities are stylised theoretical constructs, inexistent in the real world outside, thinking in terms of as if trading them helps clarify the economic roles of the more complex securities tracked daily in the financial press (as we show below).

1.3.3. *Budget Constraints with Arrow-Debreu Securities.* We are now prepared to turn back to our objective of analysing a SOE under uncertainty. Assuming *complete* asset markets, as we do throughout sections 1 and 2, is *equivalent* to allowing the representative individual to hedge risks by trading in a *full* set of Arrow-Debreu contingent claims.

Let $b_2(s)$ denote the representative individual's *net* purchase of – units (number, quantity) of – state s A-D securities on date 1. We keep on to the convention from our previous lecture, following Obstfeld and Rogoff's (1996) book, that *stocks of assets* (as well as the *rate of interest*) are (time-)indexed according to the *start* of the period they are carried over to (or announced to be valid for, in the case of interest rates): thus, $b_2(s)$ is the stock of A-D claims the representative individual in the small open economy holds (immediately) at the *beginning* of period 2, which should be the same as that held (immediately) at the *end* of period 1.

Let also $\frac{p(s)}{1+r}$ denote the world (real) price, quoted in terms of date 1 consumption, of a claim to one output unit to be delivered on date 2 if, and only if, state s occurs. Thus $p(s)$ is the price of date 2 consumption, *conditional* on state s in terms of *certain* date 2 consumption.[2] Since this price is determined in the world market, it is exogenously given to the SOE.

As usual in an exchange economy, the value (price multiplied by quantity) of a country's *net* accumulation of assets on (or, rather, at the end of) date 1 must equal the difference between its income and consumption on (at the end of) that same date 1:

$$\underbrace{\overbrace{\underbrace{\frac{p(1)}{1+r}b_2(1)}_{\text{PV of insurance } \textit{if state } 1 \text{ on date } 2} + \underbrace{\frac{p(2)}{1+r}b_2(2)}_{\text{PV of insurance } \textit{if state } 2 \text{ on date } 2}}^{\text{would be simply } b_2 \text{ (with } b_1\equiv 0) \text{ under certainty (cf. lecture 4)}}}_{\text{PV of } \textit{total} \text{ insurance for the uncertainty of date } 2} \equiv \underbrace{y_1 - c_1}_{\text{date 1 net saving}}. \tag{1.2}$$

Note that the (real) prices $p(1)$ and $p(2)$ are "deflated" by the risk-free (real) interest rate by which the market (objectively) discounts the future one period ahead. Recall also, in view of what was already said, that we need not explicitly consider a bond market, *in addition to* the Arrow-Debreu setting implied by (1.2); the reason is that bonds are redundant given the *two* A-D securities available for the *only* two possible states on date 2 in our simple stochastic model.

How does, next, the future unfold, resolving (revealing) uncertainty, in the context of the analytical framework we are building here? Well, when date 2 arrives, the (actual or realised

[2] Obstfeld and Rogoff (1996), p. 273, footnote 6, stress that this notation is adopted for *two* reasons: (i) to remind that transactions in A-D securities transfer purchasing power across time *as well as* across states and (ii) to render the resulting budget constraints and Euler equations (see further down) in a form that is *easily compared* with their certainty analogues (see the respective expressions in lecture 4).

or materialised) state of nature is observed, and the country will be able to consume its endowment (or output or income) net of any payments on its *state* s- (i.e., the *unique* realised state) *contingent* assets:[3]

$$c_2(s) = y_2(s) + b_2(s), \quad s = 1, 2. \tag{1.3}$$

Using equations (1.3) – for $s = 1$ and $s = 2$ – to eliminate $b_2(1)$ and $b_2(2)$ in the asset accumulation identity (1.2), one can derive the intertemporal budget constraint for this *Arrow-Debreu* economy (i.e., economy enjoying *complete* asset markets):

$$\underbrace{c_1 + \frac{\overbrace{p(1)\,c_2(1) + p(2)\,c_2(2)}^{\text{would be } c_2 \text{ under certainty (cf. lecture 4)}}}{1+r}}_{\text{PV of lifetime (state-)}contingent\ consumption} = \underbrace{y_1 + \frac{\overbrace{p(1)\,y_2(1) + p(2)\,y_2(2)}^{\text{would be } y_2 \text{ under certainty (cf. lecture 4)}}}{1+r}}_{\text{PV of lifetime (state-)}contingent\ income}. \tag{1.4}$$

Equation (1.4) is the present value intertemporal budget constraint under uncertainty. It states that the date 1 present value of the SOE's uncertain consumption stream must equal the date 1 present value of its uncertain income, where (state-)contingent quantities are evaluated at world Arrow-Debreu prices. In the present model context, international asset markets allow the country to smooth consumption not only across time (as in lecture 4) but also across states of nature.

As indicated by the brackets over the numerators in the above expression – and by the similar remarks inviting comparisons with the analogous formulas in lecture 4 in most of the other numbered equations thus far – the expressions in the same model but under uncertainty in the present lecture are (slight) variations of the respective expressions under certainty in the preceding one. Such comparisons are insightful, and reveal a common logical structure which underlies the two problems, dynamic optimisation under certainty and stochastic optimisation under complete asset markets (both static or dynamic). That is why we shall purposefully, partly also for didactic reasons, insist on observing further down in these notes the similarities and differences in the dynamic vs stochastic versions of the baseline model considered in lectures 4 and 5.

1.3.4. *Full Insurance: Optimal or Not?* Let us, for a moment, return to equation (1.4) and define, in its context, what is *complete* (or *full*) *insurance*. Suppose that the SOE's output is extremely low in state 1 and extremely high in state 2. Then, by *going "short"*, i.e., *becoming a net seller*, in state 2 securities ($b_2(2) < 0$) **and** *going "long"*, i.e., *becoming a net buyer*, in state 1 securities ($b_2(1) > 0$), the country can smooth consumption across *states*. In such a way, a consumer could assure himself of a completely *nonrandom* period 2 consumption level by, for example, selling his future state 1 output at its world market price, i.e., for $\frac{p(1)y_2(1)}{1+r}$, in *bonds* **and** his future state 2 output at its world market price too, i.e., for $\frac{p(2)y_2(2)}{1+r}$, in *bonds*: that strategy guarantees the safe date 2 consumption level:

$$c_2 = p(1)\,y_2(1) + p(2)\,y_2(2). \tag{1.5}$$

This is, in essence, the economic role of financial hedging. But, as we shall see below, a strategy of *full insurance* is *not necessarily optimal*!

1.3.5. *Optimal Behaviour and Model Equilibrium.* In lecture 4, we showed how one can use the *Lagrangian* function method (of undetermined coefficients, or multipliers) to solve for optimal behaviour and the subsequent equilibrium allocations whenever the constraints of the optimisation problem are linear. The similar model under uncertainty here can be handled by forming the Lagrangian too. But instead of repeating the method, we shall now take an alternative route

[3] Obstfeld and Rogoff (1996), p. 275, footnote 7, duly remark that in the present setting a person's period 2 income can fall short of required payments on the state-contingent securities issued in period 1 only if this shortfall is *planned.* Such a plan would be fraudulent, but – to restrict our analysis to consistent behaviour – we assume that people *do not plan to violate* their intertemporl budget constraints.

that leads to the same results (this alternative route could have also been applied to solve the optimisation problem under certainty in lecture 4). It consists in transforming the constrained optimisation, as initially set up in the preceding pages of this lecture, into an unconstrained optimisation. To accomplish this, we usually express one or more variables from the constraint(s) and substitute them back in the objective function. Having thus transformed the optimisation problem into an unconstrained one, what remains is to equate the FONCs w.r.t. the choice (or decision or, still, control) variables to zero and find an appropriate way of expressing (analytically) and interpreting (intuitively) the efficiency conditions coming out of the described algebraic manipulation. This is what we do next.

The small country's optimal saving and portfolio allocations maximise lifetime expected utility (1.1) subject to constraint (1.4). As we said, one approach to the solution of this constrained optimisation problem will be to transform it into an unconstrained one. To do so, we use equations (1.2) and (1.3) to express the consumption levels in equation (1.1) as function of asset choices:

$$U_l = u\underbrace{\left[y_1 - \frac{p(1)}{1+r}b_2(1) + \frac{p(2)}{1+r}b_2(2)\right]}_{=c_1,\text{ from (1.2)}} +$$

$$+\pi(1)\beta u \underbrace{[y_2(1)+b_2(1)]}_{=c_2(1),\text{ from (1.3) with } s=1} + \pi(2)\beta u \underbrace{[y_2(2)+b_2(2)]}_{=c_2(2),\text{ from (1.3) with } s=2}.$$

Written in a more compact way, the maximisation problem is:

$$\max_{b_2(s)} U_l = u\left[y_1 - \sum_{s=1}^{2}\frac{p(s)}{1+r}b_2(s)\right] + \sum_{s=1}^{2}\pi(s)\beta u\left[y_2(s)+b_2(s)\right]. \tag{1.6}$$

The first-order conditions are:

$$\frac{\partial U_l}{\partial b_2(s)} = 0, \qquad s = 1,2 \Leftrightarrow$$

$$\Leftrightarrow \frac{p(s)}{1+r}u'(c_1) = \pi(s)\beta u'[c_2(s)], \qquad s=1,2. \tag{1.7}$$

As you could verify by comparison, (1.7) is closely related to the intertemporal Euler equation in lecture 4, although it now pertains to an A-D security rather than a risk-free bond. The LHS of (1.7) is the *cost*, in terms of date 1 marginal utility of consumption, $u'(c_1)$, of acquiring the Arrow-Debreu security for state s; the RHS of (1.7) is the *expected discounted benefit* from having an additional unit of consumption, $u'[c_2(s)]$, in state s on date 2. Similarly to what we did with the intertemporal Euler equation in lecture 4, we can also rearrange (1.7) so as to express the *marginal rate of substitution* between c_1 and $c_2(s)$ as being *equal to* the *relative price* between the two goods:

$$\frac{\pi(s)\beta u'[c_2(s)]}{u'(c_1)} = \frac{p(s)}{1+r}, \qquad s=1,2. \tag{1.8}$$

1.3.6. *Creating Synthetic Assets from Primal Arrow-Debreu Securities.* We can use equations (1.8) to derive the intertemporal Euler conditions for more complex securities that pay off in more than one state of the world. A usual example of such an asset is a riskless bond, which pays $1+r$ output units on date 2 always (i.e., no matter the particular state that has materialised) for every one output unit worth of bonds purchased on date 1. As noted earlier, we can create a "synthetic" safe bond out of "primal" A-D securities by buying $1+r$ units of the state 1 A-D security on date 1 at the (complete asset markets) price of $\frac{p(1)}{1+r}$ per unit **and**, at the *same* time, buying $1+r$ units of the state 2 A-D security on date 1 at the (complete asset markets) price of $\frac{p(2)}{1+r}$ per unit. Such a "portfolio" yields *sure* delivery of $1+r$ output units on date 2, no matter which of the possible states eventually occurs. For this reason, the "portfolio" or, rather, the *synthetic* sure bond *replicated* (or constructed) by a combination of (here, two) *primal* A-D securities must have the same price as a straight (i.e., not synthetic or replicated) bond paying $1+r$ output units next period (that is, the "portfolio" of A-D securities which is termed a synthetic bond must cost 1 output unit on date 1). Therefore,

$$\underbrace{(1+r)}_{\text{units of state 1 A-D}} \underbrace{\frac{p(1)}{1+r}}_{\text{unit price of state 1 A-D}} + \underbrace{(1+r)}_{\text{units of state 2 A-D}} \underbrace{\frac{p(2)}{1+r}}_{\text{unit price of state 2 A-D}} =$$

$$= \underbrace{1}_{\text{cost in terms of date 1 output units to buy 1 bond}},$$

or, equivalently,

$$p(1) + p(2) = 1. \tag{1.9}$$

Let us note here, in passing (and for future reference), that in the general case when the states of nature s are more than two and belong to some set $\mathcal{S}$ the analogous formula is:

$$\sum_{s=1}^{\mathcal{S}} p(s) = 1. \tag{1.10}$$

We next derive the *stochastic Euler equation for a riskless bond*, (1.11) below, by adding equations (1.7),

$$\underbrace{[p(1) + p(2)]}_{=1,\text{ from (1.9)}} u'(c_1) = (1+r)\beta \underbrace{\{\pi(1) u'[c_2(1)] + \pi(2) u'[c_2(2)]\}}_{\equiv E_1[u'(c_2)],\text{ by definition}},$$

Using the definition of mathematical expectation in the RHS and equation (1.9) in the LHS, we can write the expression above as

$$u'(c_1) = (1+r)\beta E_1[u'(c_2)], \tag{1.11}$$

where $E_1[\cdot]$ is the expectation operator conditional on information known at date 1.

Like we are by now already accustomed to do, let us also present the stochastic Euler equation for the risk-free bond, (1.11), in another way:

$$\frac{\beta E_1[u'(c_2)]}{u'(c_1)} = \frac{1}{1+r}. \tag{1.12}$$

The interpretation of (1.12) is that the (discounted) *expected* marginal rate of substitution of present for future consumption, LHS, *equals* the (relative) price of *certain* future consumption, RHS.

1.3.7. *Actuarially Fair Arrow-Debreu Security Prices: More on Optimal Insurance.* A still further implication of the first-order conditions (1.7) is easily seen when one writes them in a manner explicitly accounting for the two possible states of nature in the second period of the baseline stochastic model we are analysing here. Dividing through the two FONCs, we obtain what was termed a "compact" first-order condition in lecture 4 (check the analogous equation there):

$$\frac{\pi(1) u'[c_2(1)]}{\pi(2) u'[c_2(2)]} = \frac{p(1)}{p(2)}. \tag{1.13}$$

(1.13) states that the *MRS* of state 2 consumption (in the *denominator* of the LHS) for state 1 consumption (in the numerator of the LHS) must equal the *relative price* of state 2 consumption (in the *denominator* of the RHS) for state 1 consumption (in the numerator of the RHS). Recall the static optimality condition familiar from consumer theory in microeconomics, to see that (1.13) is also a variation of it. Moreover, observe that *only when*

$$\frac{\pi(1)}{\pi(2)} = \frac{p(1)}{p(2)} \tag{1.14}$$

does condition (1.13) imply that

$$u'[c_2(1)] = u'[c_2(2)], \text{ hence } c_2(1) = c_2(2) = c_2 = const, \tag{1.15}$$

so that it is *optimal* to *smooth* consumption across *states of nature*. When (1.15) holds, it defines what is said to be *actuarially fair Arrow-Debreu security prices*. *Only* at *actuarially fair* prices a country trading in *complete* asset markets will *fully* insure against *all* future consumption fluctuations. *If* prices are *not* actuarially fair, the country will optimally choose to *tilt* its consumption across *states*. Given two *equiprobable* states, $\pi(1) = \pi(2) = \frac{1}{2}$, for example, the country will *plan* for *relatively lower consumption* in the *state* for which *consumption insurance is relatively expensive*, i.e., $p(s)$ for $s = 1, 2$ is relatively higher. In a more practical context, similarly – if other things are equal – individuals confronted with a higher relative price of, say, auto insurance will buy less of it (lower coverage limits, higher deductible, etc.). It is, again, interesting to compare these results on *consumption tilting across states* (of nature) with the analogous ones on consumption tilting *across dates (or periods)*, i.e., across time, and to understand the parallels as well as the differences.

1.3.8. *The Role of the Coefficient of Relative Risk Aversion (CRRA).* A further analogy – with some variation, of course – concerns the definitions and roles of the *elasticity of intertemporal substitution (EIS) in consumption* in lecture 4 and the *coefficient of relative risk aversion (CRRA)* in the present context with uncertainty.

Recall from microeconomic theory that an *expected*-utility maximiser will be, formally, risk-*averse*, which will make him *willing* to buy consumption insurance, only if the *period* utility function is strictly *concave*.[4] Under risk aversion, i.e., concavity of utility, such individuals strictly prefer the expected value of a finite gamble than the gamble itself. The key point this subsection will stress is that the *curvature*, that is, the *degree of concavity* of the period utility function, which measures the *extent of risk aversion* (e.g., not whether you are risk averse but *how much risk averse* you are, *relative to* another individual), is at the same time an inverse measure of the individual's portfolio response to changes in Arrow-Debreu prices. The role risk aversion plays is thus, essentially, in determining the demands for state-contingent consumptions.

For more clarity, we largely repeat below the derivation and definitions we did with respect to the EIS but now in the set-up of the baseline stochastic model we deal with in this lecture the purpose is to derive the CRRA. Going back to the across-state compact FONC (1.13), we first *take natural logarithms* from both sides of it:

$$\ln\left[\frac{\pi(1)\,u'[c_2(1)]}{\pi(2)\,u'[c_2(2)]}\right] = \ln\left[\frac{p(1)}{p(2)}\right],$$

$$\underbrace{\ln \pi(1)}_{=const} + \ln u'[c_2(1)] - \underbrace{\ln \pi(2)}_{=const} - \ln u'[c_2(2)] = \ln p(1) - \ln p(2),$$

$$\ln p(1) - \ln p(2) = \ln u'[c_2(1)] - \ln u'[c_2(2)] + \underbrace{\ln \pi(1)}_{=const} - \underbrace{\ln \pi(2)}_{=const}.$$

We next *totally differentiate* the above equation (treating probabilities as constant, since they are fixed, i.e., known to agents by assumption):

$$\frac{d\ln p(1)}{dp(1)}dp(1) - \frac{d\ln p(2)}{dp(2)}dp(2) = \frac{d\ln u'[c_2(1)]}{dc_2(1)}dc_2(1) - \frac{d\ln u'[c_2(2)]}{dc_2(2)}dc_2(2),$$

$$\frac{1}{p(1)}dp(1) - \frac{1}{p(2)}dp(2) = \frac{1}{u'[c_2(1)]}u''[c_2(1)]\,dc_2(1) - \frac{1}{u'[c_2(2)]}u''[c_2(2)]\,dc_2(2),$$

[4] Recall as well that, alternatively, a strictly *convex* period utility characterises the less realistic (or less typical) economic agents who are risk-*lovers*; and that a *linear* period utility function implies risk-*neutral* behaviour.

$$\frac{dp(1)}{p(1)} - \frac{dp(2)}{p(2)} = \frac{1}{u'[c_2(1)]}u''[c_2(1)]\underbrace{\frac{c_2(1)}{c_2(1)}}_{=1}dc_2(1) - \frac{1}{u'[c_2(2)]}u''[c_2(2)]\underbrace{\frac{c_2(2)}{c_2(2)}}_{=1}dc_2(2),$$

$$\underbrace{\frac{dp(1)}{p(1)}}_{\equiv d\ln p(1)} - \underbrace{\frac{dp(2)}{p(2)}}_{\equiv d\ln p(2)} = c_2(1)\frac{u''[c_2(1)]}{u'[c_2(1)]}\underbrace{\frac{dc_2(1)}{c_2(1)}}_{\equiv d\ln c_2(1)} - c_2(2)\frac{u''[c_2(2)]}{u'[c_2(2)]}\underbrace{\frac{dc_2(2)}{c_2(2)}}_{\equiv d\ln c_2(2)},$$

$$\underbrace{d\ln p(1) - d\ln p(2)}_{\equiv d\ln\left[\frac{p(1)}{p(2)}\right]} = \frac{c_2(1)u''[c_2(1)]}{u'[c_2(1)]}d\ln c_2(1) - \frac{c_2(2)u''[c_2(2)]}{u'[c_2(2)]}d\ln c_2(2),$$

$$d\ln\left[\frac{p(1)}{p(2)}\right] = \frac{c_2(1)u''[c_2(1)]}{u'[c_2(1)]}d\ln c_2(1) - \frac{c_2(2)u''[c_2(2)]}{u'[c_2(2)]}d\ln c_2(2). \tag{1.16}$$

The ratios in the RHS above define what is known as the Arrow (1953, 1964) – Pratt (1964) *coefficient of relative risk aversion*:[5]

$$\rho(c) \equiv -\frac{cu''(c)}{u'(c)}. \tag{1.17}$$

If we next assume that this coefficient is constant, $\rho(c) = \rho = const$, equation (1.16) simplifies to:

$$d\ln\left[\frac{p(1)}{p(2)}\right] = \underbrace{\frac{c_2(1)u''[c_2(1)]}{u'[c_2(1)]}}_{=-\rho}d\ln c_2(1) - \underbrace{\frac{c_2(2)u''[c_2(2)]}{u'[c_2(2)]}}_{\equiv\rho}d\ln c_2(2),$$

$$d\ln\left[\frac{p(1)}{p(2)}\right] = -\rho d\ln c_2(1) + \rho d\ln c_2(2),$$

$$d\ln\left[\frac{p(1)}{p(2)}\right] = \rho\left[d\ln c_2(2) - d\ln c_2(1)\right],$$

$$d\ln\left[\frac{p(1)}{p(2)}\right] = \rho d\ln\left[\frac{c_2(2)}{c_2(1)}\right],$$

$$d\ln\left[\frac{c_2(2)}{c_2(1)}\right] = \frac{1}{\rho}d\ln\left[\frac{p(1)}{p(2)}\right].$$

The above equation shows that the *inverse* of the *constant* coefficient of relative risk aversion, $\frac{1}{\rho}$, is also, by definition, the *elasticity of substitution between state-contingent consumption levels with respect to relative Arrow-Debreu prices* (to be distinguished from the EIS in consumption we derived under certainty in lecture 4!). Intuitively, a *high* risk aversion – embodied in a *high* value of ρ ($\rho > 0$ is a usual parameter in *period* utility functions, as we shall see below) – implies an *inelastic* (i.e., $0 < \frac{1}{\rho} < 1$) response of relative demand for consumption insurance to a change in the relative price of insurance.

The class of period utility functions characterised by *constant relative risk aversion (CRRA)* is given by:

$$u(c) = \begin{cases} u(c) = \frac{c^{1-\rho}}{1-\rho}, & \rho > 0, \rho \neq 1 \\ \ln c, & \rho = 1, \end{cases} \tag{1.18}$$

an analytical description that also fits the *isoelastic* utility class if σ, the elasticity of intertemporal substitution (EIS) in consumption, equals $\frac{1}{\rho}$. That is,

[5] Observe that the RHS of formula (1.17) would coincide with the definition of the elasticity of intertemporal substitution (EIS) in consumption of lecture 4 if we had a model with *certain* consumption streams over time.

$$\sigma = \frac{1}{\rho} \Leftrightarrow \rho = \frac{1}{\sigma}. \tag{1.19}$$

The above isoelastic class example illustrates a shortcoming of the *expected* utility framework, in the sense that it does not permit to vary the consumer's aversion to risk and elasticity of intertemporal substitution, two different concepts, independently of each other. However, economic analysis usually accepts this drawback because of the need for *tractability*: the latter is provided by *both* retaining the *expected* utility framework *and* often specialising preferences to the *CRRA* (or *isoelastic*) class. Moreover, as Obstfeld and Rogoff (1996) point out in their textbook, p. 279, footnote 8, it is precisely this class of preferences that has the advantage to be *consistent* with steady long-run rate of consumption growth. Obstfeld and Rogoff also stress it clearly that the reason risk aversion and intertemporal substitutability are *indistinguishable* with CRRA expected-utility preferences is that lifetime utility is assumed *additive* across states (*state separable*) *as well as* across dates/periods (*time separable*), with probabilities weighting the period utility function as applied to different states in the same multiplicative fashion that the temporal discount factor weighs the value of period utility on different dates.

A consumer is said to be risk *neutral* when $u''(c) = 0$, implying $\rho = 0$. As $\rho \to 0$, $\frac{1}{\rho} \to \infty$, so individuals would concentrate all their consumption in states s with $\pi(s) > p(s)$. But this result in the *partial* equilibrium, SOE model described up to here does *not* carry over to *general* equilibrium, because world equilibrium requires date 2 output market to clear state by state: we shall see in section 2 that the only price vector consistent with general equilibrium as $\rho \to 0$ is $\pi(s) = p(s)$, in which case risk-neutral agents are indifferent as to how they allocate their wealth across states of nature.

But before moving to the two-country version of the baseline model we are analysing, let us look at a log-utility example that will clarify the links between our stochastic set-up here and the current account, providing some coherence with earlier lectures as well.

1.3.9. *Consumption Demands and the Current Account in the Log-Utility Special Case.* The special case of *logarithmic utility* has been much exploited in theoretical frameworks. The main purpose when recurring to this otherwise quite restrictive case is to *derive explicitly closed-form solutions*. With log-utility, this turns out straightforward in many economic models. The benefit of analytical clarity is thus often perceived to offset the drawback of reducing the generality of results to this particular class of preferences.

In the context of the present model with uncertainty, specialising utility to the logarithmic class is useful as well in providing interesting insights on the current account. We illustrate them below.

With $u(c) = \ln c$, the lifetime expected utility function (1.1) that the representative individual maximises subject to the lifetime expected budget constraint (1.4) becomes:

$$U_l \equiv \ln c_1 + \pi(1)\beta \ln c_2(1) + \pi(2)\beta \ln c_2(2). \tag{1.20}$$

For a simpler notation, we can also define $\mathcal{W}_l$ to be the present value of lifetime expected resources on date 1, in fact, the RHS of (1.4):

$$\mathcal{W}_l \equiv y_1 + \frac{p(1)\, y_2(1) + p(2)\, y_2(2)}{1+r}. \tag{1.21}$$

Now, with *logarithmic* utility, the Euler equations (1.7) become:

$$\frac{p(1)}{1+r} \underbrace{\frac{1}{c_1}}_{u'(c_1)} = \pi(1)\beta \underbrace{\frac{1}{c_2(1)}}_{u'[c_2(1)]}, \tag{1.22}$$

$$\frac{p(2)}{1+r} \underbrace{\frac{1}{c_1}}_{u'(c_1)} = \pi(2)\beta \underbrace{\frac{1}{c_2(2)}}_{u'[c_2(2)]}. \tag{1.23}$$

Expressing from them the (optimal) date 2 (contingent) consumption demands, we get:

$$c_2(1) = \frac{\pi(1)}{p(1)}\beta(1+r)c_1, \tag{1.24}$$

$$c_2(2) = \frac{\pi(2)}{p(2)}\beta(1+r)c_1. \tag{1.25}$$

Dividing through the above equations provides the (optimal) relation between the contingent consumptions on date 2:

$$\frac{c_2(1)}{c_2(2)} = \frac{\frac{\pi(1)}{p(1)}}{\frac{\pi(2)}{p(2)}} = \frac{\pi(1)}{p(1)}\frac{p(2)}{\pi(2)} = \frac{\pi(1)}{\pi(2)}\frac{p(2)}{p(1)},$$

hence

$$c_2(1) = \frac{\pi(1)}{\pi(2)}\frac{p(2)}{p(1)}c_2(2), \tag{1.26}$$

and

$$c_2(2) = \frac{\pi(2)}{\pi(1)}\frac{p(1)}{p(2)}c_2(1). \tag{1.27}$$

Expressing, in turn, from (1.22) and (1.23) the (optimal) date 1 consumption demand results in:

$$c_1 = \frac{p(1)}{\pi(1)}\frac{c_2(1)}{\beta(1+r)}, \tag{1.28}$$

$$c_1 = \frac{p(2)}{\pi(2)}\frac{c_2(2)}{\beta(1+r)}. \tag{1.29}$$

We can now substitute back for $c_2(1)$ and $c_2(1)$ into the lifetime expected budget constraint (1.4) and – using also the definition of present value lifetime expected wealth (1.21) – solve for c_1:

$$c_1 + \frac{p(1)\overbrace{\frac{\pi(1)}{p(1)}\beta(1+r)c_1}^{=c_2(1)\text{ from (1.24)}} + p(2)\overbrace{\frac{\pi(2)}{p(2)}\beta(1+r)c_1}^{=c_2(2)\text{ from (1.25)}}}{1+r} = \mathcal{W}_l,$$

$$c_1 + \frac{\beta(1+r)\overbrace{[\pi(1)+\pi(2)]}^{\equiv 1}}{1+r}c_1 = \mathcal{W}_l,$$

$$c_1 + \beta c_1 = \mathcal{W}_l,$$

$$c_1 = \frac{1}{1+\beta}\mathcal{W}_l = \frac{1}{1+\beta}\left[y_1 + \frac{p(1)y_2(1) + p(2)y_2(2)}{1+r}\right]. \tag{1.30}$$

We can substitute back for c_1 in (1.22) and (1.23) and solve them for the contingent consumption demands $\frac{p(1)}{1+r}c_2(1)$ and $\frac{p(2)}{1+r}c_2(2)$, respectively:

$$c_2(1) = \frac{\pi(1)}{p(1)}\beta(1+r)\underbrace{\frac{1}{1+\beta}\mathcal{W}_l}_{=c_1},$$

$$\frac{p(1)}{1+r}c_2(1) = \frac{\pi(1)\beta}{1+\beta}\mathcal{W}_l = \frac{\pi(1)\beta}{1+\beta}\left[y_1 + \frac{p(1)y_2(1)+p(2)y_2(2)}{1+r}\right], \tag{1.31}$$

$$c_2(2) = \frac{\pi(2)}{p(2)}\beta(1+r)\underbrace{\frac{1}{1+\beta}\mathcal{W}_l}_{=c_1},$$

$$\frac{p(2)}{1+r}c_2(2) = \frac{\pi(2)\beta}{1+\beta}\mathcal{W}_l = \frac{\pi(2)\beta}{1+\beta}\left[y_1 + \frac{p(1)y_2(1)+p(2)y_2(2)}{1+r}\right]. \tag{1.32}$$

The date 1 consumption demand (1.30) is therefore completely parallel to that for the nonstochastic case under logarithmic preferences, with the only difference being that in place of the known (i.e., certain) date 2 output, y_2, we now have the date 1 *value of random date 2 output at world market prices*, $p(1)y_2(1) + p(2)y_2(2)$. Be careful to also distinguish the latter value, i.e., the *value of random date 2 output at world market prices*, from the *date 2 expected output*, $E_1[y_2] = \pi(1)y_2(1) + \pi(2)y_2(2)$! These values will coincide *only* in the case of *actuarially fair prices*, that is, with $\pi(1) = p(1)$ and $\pi(2) = p(2)$, as evident from comparing the respective expressions.

Finally, using the solution for c_1, (1.30) above, we can express the *date 1 current account* balance under *log* utility (and with *zero* initial net foreign assets, as we assumed in the beginning):

$$CA_1 \equiv y_1 - c_1 = y_1 - \frac{1}{1+\beta}\left[y_1 + \frac{p(1)y_2(1)+p(2)y_2(2)}{1+r}\right] =$$

$$= y_1 - \frac{1}{1+\beta}y_1 - \frac{1}{1+\beta}\left[\frac{p(1)y_2(1)}{1+r} + \frac{p(2)y_2(2)}{1+r}\right] =$$

$$= \frac{\beta}{1+\beta}y_1 - \frac{1}{1+\beta}\left[\frac{p(1)y_2(1)}{1+r} + \frac{p(2)y_2(2)}{1+r}\right]. \tag{1.33}$$

Again, the expression above is parallel to the nonstochastic log-utility case.

In the two-country model of lecture 4 the current account was essentially interpreted as depending on comparative advantage in intERtemporal trade, i.e., in trade across time. This was done as an analogy with comparative advantage in usual intRAtemporal trade, i.e., in trade across different goods at the same point in time in classic international trade theory. Our conclusion was that the sign of the current account, CA_1, depended on the difference between the world and autarky real interest rates, r and r_A and r_A^*.[6] But the simple form of comparative advantage in intERtemporal trade does not carry over to trade across states of nature. The difficulty comes from the three "goods" involved in the two-period stochastic set up we developed, namely (sure) consumption on date 1 and (contingent) consumption in each state of date 2. In standard trade theory comparative advantage generally holds only in a weaker form when there are more than two goods.[7]

2. A Stochastic Two-Period Real Model of a Two-Country Global Economy: the CRRA Case

Several important implications of complete asset markets can only be understood if one passes from the partial equilibrium of a SOE to the setting of world general equilibrium with two large open economies. The present section sketches this extension, interpreting the essential results. In particular, the focus will be on the difference between global (or aggregate) and country-specific (or idiosyncratic) risk as well as on the conditions ensuring Arrow-Debreu prices to be actuarially fair.

[6] Recall our discussion in class of Figure 1.5, p. 24, in Obstfeld and Rogoff (1996).

[7] The interested student is referred to Obstfeld and Rogoff's textbook, sections 5.1.7 and and 5.1.8 as well as appendix 5B, if (s)he wishes to learn more on these more complicated aspects of our topic.

2.1. Assumptions.

(1) *CRRA* utility, for simplicity and to gain some initial intuition.
(2) $\mathcal{S} > 2$ states of nature.

2.2. Model Solution Algorithm.

Global general equilibrium requires that supply and demand balance in the $\mathcal{S}+1$ markets: for *date* 1 *output*,

$$c_1 + c_1^* = y_1 + y_1^*, \tag{2.1}$$

and for *date* 2 *output* delivered in each of the $\mathcal{S}$ possible states of nature,

$$c_2(s) + c_2^*(s) = y_2(s) + y_2^*(s), \qquad s = 1, 2, ..., \mathcal{S}. \tag{2.2}$$

2.2.1. *Equilibrium Prices. Equilibrium prices* are found by combining the above *market-clearing* conditions with the *Euler equations* for the national representative individuals.

We first define *total world output*:

$$y^W \equiv y + y^*.$$

With *CRRA* (period) utility now, $u(c) = \frac{c^{1-\rho}}{1-\rho}$, – intended to provide the insights of an *explicit* analytical solution, like in the case of *log* utility earlier – the Euler equations for state s securities in $H(ome)$ and $F(oreign)$ are:

$$\frac{p(s)}{1+r}\underbrace{[c_1]^{-\rho}}_{u'(c_1)} = \pi(s)\,\beta\underbrace{[c_2(s)]^{-\rho}}_{u'[c_2(1)]}, \qquad s = 1, 2, ..., \mathcal{S}, \tag{2.3}$$

$$\frac{p(s)}{1+r}\underbrace{[c_1^*]^{-\rho}}_{u'(c_1)} = \pi(s)\,\beta\underbrace{[c_2^*(s)]^{-\rho}}_{u'[c_2(1)]} \qquad s = 1, 2, ..., \mathcal{S}, \tag{2.4}$$

so that the date 2 (contingent) consumptions in H and F can be expressed, respectively, as:

$$\frac{p(s)}{1+r}\frac{1}{[c_1]^{\rho}} = \pi(s)\,\beta\frac{1}{[c_2(s)]^{\rho}}, \qquad s = 1, 2, ..., \mathcal{S},$$

$$[c_2(s)]^{\rho} = \frac{\pi(s)\,\beta\,(1+r)}{p(s)}[c_1]^{\rho}, \qquad s = 1, 2, ..., \mathcal{S},$$

$$c_2(s) = \left[\frac{\pi(s)\,\beta\,(1+r)}{p(s)}\right]^{\frac{1}{\rho}} c_1, \qquad s = 1, 2, ..., \mathcal{S}, \tag{2.5}$$

and

$$c_2^*(s) = \left[\frac{\pi(s)\,\beta\,(1+r)}{p(s)}\right]^{\frac{1}{\rho}} c_1^*, \qquad s = 1, 2, ..., \mathcal{S}. \tag{2.6}$$

Date 1 Prices of Contingent Securities. Adding (2.5) and (2.6) and applying world equilibrium conditions (2.1) and (2.2) yields

$$\underbrace{c_2(s) + c_2^*(s)}_{\equiv y_2^W(s)} = \left[\frac{\pi(s)\,\beta\,(1+r)}{p(s)}\right]^{\frac{1}{\rho}} \underbrace{(c_1 + c_1^*)}_{\equiv y_1^W}, \qquad s = 1, 2, ..., \mathcal{S},$$

which implies that the (general) equilibrium *date* 1 *price* of the state s contingent security is

$$\frac{p(s)}{1+r} = \pi(s)\,\beta\left[\frac{y_2^W(s)}{y_1^W}\right]^{-\rho}, \qquad s = 1, 2, ..., \mathcal{S}. \tag{2.7}$$

The assumption that H and F representative individuals share a common CRRA, ρ, which was used in the above derivations, simplifies them a lot. Without such a simplification (i.e., if

$\rho \neq \rho^*$), the model would not necessarily result in a closed-form solution in terms of aggregate output alone.

The preceding analytical expressions are insightful in understanding the conditions under which securities prices will be actuarially fair. This is easily seen when we write down equation (2.7) for any two different states belonging to the set $\mathcal{S}$, denoted s and s', and then divide the two respective equations:

$$\frac{p(s)}{p(s')} = \left[\frac{y_2^W(s)}{y_2^W(s')}\right]^{-\rho} \frac{\pi(s)}{\pi(s')} \tag{2.8}$$

For $\rho > 0$, all prices of contingent claims will be actuarially fair if and only if total world output is the *same* in all states of nature, i.e., $\frac{y_2^W(s)}{y_2^W(s')} = 1$. The requirement for actuarial fairness is therefore the absence of output uncertainty at the *aggregate* (that is, *world*, in the model context) level. If there is no aggregate uncertainty (i.e., if there are no world-wide shocks), it is feasible for both countries to have state-independent date 2 consumption levels. Consequently, there is no need that equilibrium prices provide incentives for consumption tilting in favour of states with relatively abundant world output. If, however, world output in state s' exceeds that in state s, *prices must be such as to induce* countries to consume relatively more in state s'. As it becomes clear from looking back to equation (2.8), *consumption in the state with relatively scarce world output*, s, will command a *premium over its actuarially fair price* while *consumption in the state with relatively abundant world output*, s', will sell at a *discount.*

Date 2 Prices of Contingent Securities. Having solved for the equilibrium *date* 1 Arrow-Debreu prices, we can now go on to find the equilibrium real interest rate (RIR), r, in the context of this global two-period stochastic model. Before doing that, an intermediate step in the algorithm we are following here is to solve for the equilibrium *date* 2 securities prices, $p(s)$, as well.

For any state s', the *arbitrage condition*

$$\sum_{s=1}^{\mathcal{S}} p(s) = 1$$

and expression (2.8) imply

$$p(s') = 1 - \sum_{s \neq s'} p(s) = 1 - p(s') \sum_{s \neq s'} \left[\frac{y_2^W(s)}{y_2^W(s')}\right]^{-\rho} \frac{\pi(s)}{\pi(s')},$$

an equation, which can be solved for $p(s')$:

$$p(s') = \frac{\pi(s') \left[y_2^W(s')\right]^{-\rho}}{\sum_{s=1}^{\mathcal{S}} \pi(s) \left[y_2^W(s)\right]^{-\rho}}. \tag{2.9}$$

Equilibrium Real Interest Rate. The above equation combined with (2.7) yields the model solution for the world (gross) RIR:

$$1 + r = \frac{\left[y_1^W\right]^{-\rho}}{\beta \sum_{s=1}^{\mathcal{S}} \pi(s) \left[y_2^W(s)\right]^{-\rho}}. \tag{2.10}$$

The intuition behind (2.10) is the following: *higher world output on date* 1 (i.e., higher *current-period* world output) implies a *lower RIR*, as it raises the price of date 2 consumption relative to date 1 consumption; by analogy, *higher world output on date* 2 in any state (i.e., higher *expected future* world output) implies a *higher RIR.*

2.2.2. *Equilibrium Consumption Levels.* The model we analyse under complete asset markets has strong implications concerning international risk sharing, and in particular the correlations in consumption levels across countries, time and states of nature. The reason for these strong model predictions is that complete asset markets allow all individuals in $Home$ and $Foreign$ to *equate* their marginal rates of substitution between certain current consumption and state-contingent future consumption to the *same* state-contingent security prices.

The *multi*state analogues of equations (1.7) and (1.13) imply

$$\frac{\pi(s)\,\beta u'[c_2(s)]}{u'[c_1]} = \frac{p(s)}{1+r} = \frac{\pi(s)\,\beta u'[c_2^*(s)]}{u'[c_1^*]} \tag{2.11}$$

and

$$\frac{\pi(s)\,u'[c_2(s)]}{\pi(s')\,u'[c_2(s')]} = \frac{p(s)}{p(s')} = \frac{\pi(s)\,u'[c_2^*(s)]}{\pi(s')\,u'[c_2^*(s')]} \tag{2.12}$$

for all states s and s'. These two equations are usually interpreted as *fundamental necessary conditions for efficient resource allocation*: all individuals' marginal rates of substitution in consumption – over time and across states – are equal, so no potential gains from trade remain to be exploited.[8] The fundamental optimality FONCs (2.11) and (2.12) are general, in the sense that they are not limited to specific utility functions: the marginal utilities involved in them are written generally. But at the cost of restricting attention to special-case utilities, such as the CRRA utility here, we could gain additional clarity in uncovering some interesting economic mechanisms at work.

The last two equations, (2.11) and (2.12), combined with (2.7) and the assumption of CRRA preferences over consumption imply

$$\frac{c_2(s)}{c_2(s')} = \frac{c_2^*(s)}{c_2^*(s')} = \frac{y_2^W(s)}{y_2^W(s')} \tag{2.13}$$

and

$$\frac{c_2(s)}{c_1} = \frac{c_2^*(s)}{c_1^*} = \frac{y_2^W(s)}{y_1^W} \tag{2.14}$$

for all states. (2.13) then implies the equalities

$$\frac{c_2(s)}{y_2^W(s)} = \frac{c_2(s')}{y_2^W(s')}, \qquad \frac{c_2^*(s)}{y_2^W(s)} = \frac{c_2^*(s')}{y_2^W(s')},$$

which mean that H consumption is a constant fraction, ϕ, of world date 2 output regardless of the realised state. Symmetrically, F state-invariant consumption share of world output is $1-\phi$. Equation (2.14) states that consumption growth rates are the same across countries in every state of nature and equal to the growth rate of world output.

(2.14) implies, in turn, the equalities (for all $s \in \mathcal{S}$)

$$\frac{c_2(s)}{y_2^W(s)} = \phi = \frac{c_1}{y_1^W}, \qquad \frac{c_2^*(s)}{y_2^W(s)} = 1-\phi = \frac{c_1^*}{y_1^W},$$

meaning that the countries' date 1 consumption shares in world output are the *same* as their date 2 shares. Obstfeld and Rogoff show in their textbook, p. 289, footnote 16, that a country's share in world consumption is, in fact, the country's share of the world present discounted output on date 1, evaluated at equilibrium Arrow-Debreu prices. They also stress that when date 2 consumption is uncertain, at equilibrium prices neither H nor F arranges for constant consumption across states. Each country's consumption is internationally diversified,

[8] Obstfeld and Rogoff's book also duly points out to the equivalence between efficiency and *Pareto optimality* in the present context as well as to the *key assumptions* under which the claimed results hold intact – see footnote 15 on p. 288.

however, in the sense that any consumption risk it does absorb is entirely due to *systematic* output uncertainty, that is, uncertainty in *global* output.[9]

2.3. **Model Interpretation: Efficient Risk Pooling.** Date 2 equilibrium is illustrated graphically in Figure 5.1 on p. 290 in Obstfeld and Rogoff (1996). We shall discuss this Edgeworth box diagram in class in the context of the global stochastic model section 2 considered here.

3. Models with Capital Market Imperfections

We would not have time to highlight the complications in the idealised complete asset markets world summarised thus far which are introduced by relaxing some of its assumptions. The curious student is left to explore these aspects alone in case they are of interest to him/her. We only provide below initial (that is, graduate textbook) reference, namely, chapter 6 in Obstfeld and Rogoff (1996). It is an excellent starting point on the *imperfections* of international financial markets in general, as well as by key subtopics.

3.1. **Sovereign Risk.** Section 6.1 in Obstfeld and Rogoff (1996) deals with sovereign risk as a problem arising from the difficulties in *enforcing financial contracts* outside the jurisdiction of the nation states.

3.2. **Risk Sharing with Hidden Information.** *Asymmetric information* is another crucial type of imperfection in real-world financial markets. It is usually considered as arising either from hidden information often resulting in adverse selection problems or from hidden actions termed in the literature moral hazard. Section 6.3 in Obstfeld and Rogoff (1996) summarises the open-economy theory on *risk sharing under hidden information.*

3.3. **Moral Hazard in International Lending.** Section 6.4 in Obstfeld and Rogoff (1996) presents, in turn, the literature on *moral hazard in international lending.*

[9]Obstfeld and Rogoff (1996), p. 289, footnote 17, point out as well that the equilibrium resource allocation under complete asset markets we studied corresponds to that to be chosen by a benevolent social planner.

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