EC933-G-AU INTERNATIONAL FINANCE – LECTURE 4

THE INTERTEMPORAL APPROACH TO THE CURRENT ACCOUNT: ANALYTICAL INTRODUCTION OF TIME

ALEXANDER MIHAILOV

Abstract. The two preceding lectures outlined the earlier (or traditional) *flow*, *stock* and *stock-flow* approaches to balance of payments adjustment and nominal exchange rate determination. All open-economy models we summarised until now were *not* "microfounded", in the sense that economic behaviour was not explicitly derived from "first principles", that is, from utility and production functions and related *optimisation* problems on the part of households and firms. With the present lecture, we begin to introduce these missing microfoundations. As we shall see, apart from the consistency (or "discipline") of the analysis, these *microfounded* (or *optimising*) frameworks provide additional insights into the workings of various economic mechanisms. This lecture deals with the analytical introduction of *time* in models of the current account, to be followed by the analytical introduction of *uncertainty* in the next lecture. The presentation below focuses – in a quite simplified manner – on the concept of intERtemporal trade, to be distinguished from intRAtemporal trade, i.e., from the everyday meaning of the word "trade". Section 1 introduces a *basic* set-up whereas sections 2 and 3 extend it to *more realistic* versions. Still more realism will be added as we proceed further with the course.

Date: 7 February 2006 (First draft: 30 October 2004).

This set of lecture notes is preliminary and incomplete. It is based on parts of the four textbooks suggested as essential and supplementary reading for my graduate course in international finance at Essex as well as on the related literature (see the course outline and reading list at http://courses/essex.ac.uk/ec/ec933/). The notes are intended to be of some help to the students attending the course and, in this sense, many aspects of them will be clarified during lectures. The present second draft may be developed and completed in future revisions. The responsibility for any errors and misinterpretations is, of course, only mine. Comments are welcome, preferably by e-mail at mihailov@essex.ac.uk and/or a_mihailov@hotmail.com.

Contents

1. A Two-Period Small Open Economy Real Model: Partial Equilibrium

The standard two-period microeconomic model of *saving* due to Fisher (1930) is adapted in this section, following Obstfeld and Rogoff (1996: chapter 1), to the *small open economy* case.

1.1. **A Small Two-Period Endowment Economy.**

1.1.1. *Assumptions.*

(1) 2 countries, SOE (H) and RoW (F);
(2) that last for 2 periods, labelled 1 and 2;
(3) a *single, perishable (i.e., nonstorable) good* available for consumption;
(4) no production (function), i.e., an *endowment* (economy) model;
(5) no investment;
(6) no government spending;
(7) no money, i.e., a *real* (economy) model;
(8) SOE takes *the real interest rate (RIR), r, the only (relative) price* in the model, as given, i.e., the real interest rate is *exogenous.*

Such a set-up is obviously simple, or rather oversimplified. But that is one of the purposes of economic modelling: it abstracts away from details that do not constitute an essential part of the mechanism one wants to model. Such simplification tries to catch (only) those features and interactions that are the most important for a phenomenon. In the case here, our objective is to understand *how a country can gain from intERtemporal trade* (not intRAtemporal trade, trade between countries within a given period of time), i.e., from rearranging the timing of its consumption through international borrowing and lending. Much of the realism we abstract away from is not needed to clearly see the main point. In other words, the key lesson is more easily learnt in an environment stripped off of complexity, provided that it (or its essence) is also valid for more complex environments. Moreover, the simple analytical framework we begin describing now is useful as a building block of the more realistic models to be studied later. All these are good reasons to first handle simplicity, and then progressively move toward richer frameworks that resemble more the real world.

1.1.2. *The Consumer's Problem.* An individual j residing in the small open economy maximises *lifetime* (or *intertemporal*) utility U_l^j which depends on period consumption levels denoted c_1^j and c_2^j:

$$U_l^j \equiv u\left(c_1^j\right) + \beta u\left(c_2^j\right), \qquad 0 < \beta < 1. \tag{1.1}$$

β is a fixed preference parameter called the *subjective* (i.e., to individual j) discount factor or time-preference factor. It measures individual j's impatience to consume: if $\beta \to 1$, the individual is very patient, in the sense that he values future consumption nearly as much as current consumption: if $\beta \to 0$, the individual is very impatient. The subjective discount *factor*, β, is itself defined in terms of the subjective discount (or time-preference) *rate*, δ:

$$\beta \equiv \frac{1}{1+\delta}.$$

Hence

$$\delta \equiv \frac{1}{\beta} - 1 = \frac{1-\beta}{\beta}.$$

The *lifetime* utility function, U_l^j, is comprised of the *period* utility function, $u\left(c^j\right)$,[1] assumed to be *invariant across periods*, i.e.,

$$u_1(\cdot) = u_2(\cdot) = u(\cdot),$$

strictly increasing in its only argument, consumption,

[1] Also termed *momentary* or *instantaneous* utility function in *continuous*-time models, as opposed to the *discrete*-time model we are describing.

$$u'\left(c^j\right) \equiv \frac{du\left(c^j\right)}{dc^j} > 0,$$

and *strictly concave*:

$$u''\left(c^j\right) \equiv \frac{du'\left(c^j\right)}{dc^j} = \frac{d\frac{du\left(c^j\right)}{dc^j}}{dc^j} = \frac{d\left[du\left(c^j\right)\right]}{dc^j dc^j} = \underbrace{\frac{d^2u\left(c^j\right)}{d\left(c^j\right)^2}}_{\text{usual notation}} < 0.$$

Another assumption concerning the *marginal* utility of consumption, $u'\left(c^j\right)$, is:

$$\lim_{c^j \to 0} u'\left(c^j\right) = \infty.$$

This means that when consumption is too low (close to zero), the marginal utility of each additional unit of consumption is immense (close to infinity). The purpose of this assumption is to ensure that individuals desire at least a little consumption in every period, so that the *nonnegativity* constraint on consumption, $c^j \geq 0$, needs not be formally added.

The *objective* (function) of individual j is, by choosing consumption levels, to maximise utility over his lifetime, i.e., his lifetime utility:

$$\max_{c_1^j, c_2^j} \; U_l^j \equiv \max_{c_1^j, c_2^j} \; u\left(c_1^j\right) + \beta u\left(c_2^j\right). \tag{1.2}$$

Let y^j denote the individual's endowment (or, in a more general sense, "output") of the single perishable good in this model economy. r, as we mentioned, is the real interest rate for borrowing or lending in the world "capital" market on date 1. Then consumption must be chosen subject to the *lifetime* (or *intertemporal*) budget constraint:

$$c_1^j + \frac{c_2^j}{1+r} = y_1^j + \frac{y_2^j}{1+r}. \tag{1.3}$$

What the constraint says is that the present (period t) value of lifetime consumption spending must be equal to the present (period t) value of lifetime endowment (output or income). In this first model introducing the intertemporal approach to the current account we also adopt the extreme assumption that there is *no uncertainty* about the future endowment (output or income), so the set-up is *deterministic* (or one of *perfect foresight*).

Given the lifetime objective to maximise and the lifetime budget constraint, the optimisation problem of individual j could be written either by setting up the Lagrangian function, as we do below, or by first expressing c_2^j from the budget constraint and then substituting it in the utility and maximising with respect to c_1^j, as done in Obstfeld and Rogoff (1996).

The Lagrangian function for the problem we construct in the standard way is:

$$\mathcal{L}\left(c_1^j, c_2^j; \lambda\right) \equiv \underbrace{u\left(c_1^j\right) + \beta u\left(c_2^j\right)}_{\text{objective}} + \lambda \underbrace{\left(y_1^j + \frac{y_2^j}{1+r} - c_1^j - \frac{c_2^j}{1+r}\right)}_{\text{constraint}}. \tag{1.4}$$

The first order necessary conditions (FONCs) for optimal consumption with respect to the two choice variables are:

$$\frac{\partial \mathcal{L}\left(c_1^j, c_2^j; \lambda\right)}{\partial c_1^j} = 0 \Rightarrow u'\left(c_1^j\right) = \lambda,$$

$$\frac{\partial \mathcal{L}\left(c_1^j, c_2^j; \lambda\right)}{\partial c_2^j} = 0 \Rightarrow \beta u'\left(c_2^j\right) = \frac{\lambda}{1+r}.$$

Note that a third FONC should be taken, w.r.t. the Lagrange multiplier λ, which results in simply rewriting the equality constraint of the problem:[2]

$$\frac{\partial \mathcal{L}\left(c_1^j, c_2^j; \lambda\right)}{\partial \lambda} = 0 \Rightarrow c_1^j + \frac{c_2^j}{1+r} = y_1^j + \frac{y_2^j}{1+r}.$$

Dividing the second FONC by the first FONC above, one gets a sort of "compact" FONC which is called an *intertemporal* Euler equation, after the Swiss mathematician Leonhard Euler (1703-1783):

$$\underbrace{\frac{\beta u'\left(c_2^j\right)}{u'\left(c_1^j\right)}}_{\text{MRS in consumption}} = \underbrace{\frac{1}{1+r}}_{\text{market discount factor}} \equiv p. \tag{1.5}$$

The LHS is the *marginal rate of substitution (MRS)* of present (date 1) consumption for future (date 2) consumption; at a utility maximum, it should equal the *relative price* between present and future consumption, $p \equiv \frac{1}{1+r}$, in the RHS. Observe for future reference that in a more general context with many periods the real interest rate (RIR) may vary with time so what we have expressed above is, more precisely, the RIR for the particular period under analysis. In principle, the present value factor,

$$p_t \equiv \frac{1}{(1+r_1)(1+r_2)...(1+r_t)} = \frac{1}{\prod\limits_{i=1}^{t}(1+r_i)}, \tag{1.6}$$

applicable to problems with more than two periods is termed the *pricing kernel.*

An alternative writing of the intertemporal Euler equation (1.5) is:

$$u'\left(c_1^j\right) = (1+r)\,\beta u'\left(c_2^j\right), \tag{1.7}$$

Its interpretation is that, at a utility maximum, the consumer cannot gain from feasible shifts of consumption between periods: a one-unit reduction in period 1 consumption lowers U_l^j by $u'\left(c_1^j\right)$; the consumption unit thus saved can be converted (by lending it) into $1+r$ units of period 2 consumption that raise U_l^j by $(1+r)\,\beta u'\left(c_2^j\right)$. The Euler equation therefore states that at an optimum these two quantities are equal.

An important special case to consider in the context of the optimal consumption problem we have just solved is when the subjective discount *rate* δ equals the market (also called objective) discount *rate*, that is, the real interest *rate* r:

$$\text{if } \delta = r, \text{ then } \beta \equiv \frac{1}{1+\delta} = \frac{1}{1+r} \equiv p, \text{ i.e., } \beta = \frac{1}{1+r}. \tag{1.8}$$

To put it differently, in the case we now focus on the subjective discount *factor* of agent j, $\beta \equiv \frac{1}{1+\delta}$, equals the objective (or market) discount *factor*, $p = \frac{1}{1+r}$ (or the pricing kernel in the simple model here). Under (1.8), the Euler equation written as (1.7) or (1.5) reduces to:

$$u'\left(c_1^j\right) = u'\left(c_2^j\right),$$

which implies:

$$c_1^j = c_2^j = \overline{c}^j = const.$$

[2]For an arbitrary λ, there is no guarantee that the solutions to the system of three equations (that is, the FONCs) in three unknowns $\left(c_1^j, c_2^j, \lambda\right)$ will be optimal solutions to the original problem. Thus, the optimal (corresponding to the third FONC) λ, should be such that the equality constraint (which is usually interpreted as a *feasibility* condition) holds.

In the special case when an individual happens to have exactly the same subjective discount rate δ as the objective (or market) one, i.e., the real interest rate, r, the individual will prefer a *flat* lifetime consumption path. The budget constraint (1.3) then gives the constant consumption in both periods:

$$\overline{c}^j + \frac{\overline{c}^j}{1+r} = y_1^j + \frac{y_2^j}{1+r},$$

$$\frac{(1+r)\,\overline{c}^j + \overline{c}^j}{1+r} = \frac{(1+r)\,y_1^j + y_2^j}{1+r},$$

$$\overline{c}^j + r\overline{c}^j + \overline{c}^j = (1+r)\,y_1^j + y_2^j,$$

$$2\overline{c}^j + r\overline{c}^j = (1+r)\,y_1^j + y_2^j,$$

$$\overline{c}^j = \frac{(1+r)\,y_1^j + y_2^j}{2+r}. \tag{1.9}$$

Note that this is a *consumption smoothing* result: the individual with $\delta = r$ will desire (or, rather, it will be optimal for him/her) to smooth his/her lifetime consumption path.

1.1.3. *Equilibrium.* Two additional assumptions now are that:

(1) all individuals in the economy are *identical* (or *homogeneous*), i.e., this is a *representative* agent model;
(2) the *size* of the population is (normalised to) 1.

They allow us to drop the individual superscript j and to identify *per capita* quantity variables with national *aggregate* quantities, both denoted by the same letters as before but without the j-indexing. Hence, $c^j = c$ and $y^j = y$ for all identical individuals j.

With these two additional assumptions (simplifying the demographic structure of the SOE) the *representative individual*'s FONCs will also describe *aggregate* behaviour. The time path of aggregate consumption will therefore be flat. The model thus predicts – under the *special-case* assumption that $\delta = r$, do not forget – that countries would tend to *smooth* their consumption. In *general*, i.e., when $\delta \neq r$, there will be instead a motivation (coming out from the preferences embodied in the utility function) to *tilt* the consumption path. If, for example, at *equal* consumption levels, $c_1 = c_2$, the *subjective* discount *rate*, δ, is *lower* than the market (or objective) discount *rate*, i.e., the real interest *rate*, r, so that the *subjective* discount *factor* is *higher* than the market (or objective) discount *factor*, $\beta \equiv \frac{1}{1+\delta} > \frac{1}{1+r} \equiv p$, the world capital market offers a (real) rate of return that would more than compensate the (representative) individual for the postponement of a little more consumption. It follows from the Euler equation (1.7) then that $u'(c_1)$ should *exceed* $u'(c_2)$ in equilibrium; that is, the marginal utility of consumption in period 1 should be *higher* than the marginal utility of consumption in period 2, which implies that the (representative) individual will maximise utility by arranging for *consumption to rise* between dates 1 and 2. One would obtain the reverse consumption *tilting*, i.e., *an optimal fall in consumption*, in the opposite case of $\beta \equiv \frac{1}{1+\delta} < \frac{1}{1+r} \equiv p$, at $c_1 = c_2$. As for the effects of exogenous changes in the RIR, r, on (initial-period) consumption and saving, these are rather intricate. We shall say a little bit more on that later in this lecture.

1.1.4. *Back to the Current Account: Analytical Reinterpretation.* In an open economy, because *international borrowing and lending* are possible, consumption need not be closely tied to current endowment (output or income). In the special case of $\delta = r$ or, which is the same, $\beta = \frac{1}{1+r}$, $c_1 = c_2 = \overline{c}$, as in (1.9), but endowments need not be constant across periods. If $y_1 < y_2$, the SOE we examine borrows (the principal of) $\overline{c} - y_1$ from foreigners at date 1 and repays $(1+r)(\overline{c} - y_1) = (\overline{c} - y_1) + r(\overline{c} - y_1)$, i.e., the principal plus the due interest, on date 2. When $c_2 = y_2 - (1+r)(\overline{c} - y_1)$, the economy's intertemporal budget constraint (1.3) holds true.

As we have noted in our first lecture, the interpretation of the current account in analytical macroeconomics is as the change in a country's *net* foreign assets (NFA) for a given time period t, ΔNFA_t, i.e., as the change in the value of its net claims on the rest of the world. We have also seen that the current account is national saving less domestic investment (the latter being zero in the present model context). Now, to further clarify the concept of the current account, as reflecting intertemporal trade, let b_{t+1} denote the (real) value of the economy's net foreign assets at the end of period t (and the beginning of period $t+1$). The current account over period t – for a country with no *capital accumulation* or *government spending* – is then defined as:

$$CA_t \equiv \Delta NFA_t \equiv \underbrace{b_{t+1} - b_t}_{\text{national (dis)saving}} .$$

Also by definition, the current account is national income (or GNP) less domestic absorption (consisting of only consumption in this simple model) during period t:

$$CA_t \equiv \Delta NFA_t \equiv \underbrace{\overbrace{\underbrace{y_t}_{\equiv \text{GDP (or national } output\text{)}} + r_t b_t}^{\equiv \text{GNP (or national } income\text{)}} - \overbrace{c_t}^{\equiv \text{domestic absorption}}}_{\equiv \text{national (dis)saving}} ,$$

where $r_t b_t$ is the interest earned on the *net* foreign assets acquired previously: the timing convention used above therefore means that r_t is, in fact, the one-period interest rate that prevailed on date $t-1$ (since b_t is the stock of *net* foreign assets at the end of period $t-1$ and the beginning of period t). Note that thus both the rate of interest and the stock of NFA are indexed according to the *beginning* of the period one enters in.

Combining the two definitions, we can write:

$$CA_t \equiv \Delta NFA_t \equiv b_{t+1} - b_t = y_t + r_t b_t - c_t. \tag{1.10}$$

The intertemporal budget constraint (1.3) implicitly assumed $b_1 = 0$, so that $CA_1 = y_1 - c_1$. By writing (1.3) as a strict equality, we have also implicitly assumed that the economy ends period 2 holding no uncollected claims to foreigners, i.e., that $b_3 = 0$.

Thus:

$$CA_2 = y_2 + r b_2 - c_2 =$$

$$= y_2 + r \underbrace{\left(b_2 - \underbrace{b_1}_{=0} \right)}_{\equiv CA_1 = y_1 + r \underbrace{b_1}_{=0} - c_1} - c_2 =$$

$$= y_2 + r (y_1 - c_1) - \underbrace{c_2}_{= y_2 - (1+r)(c_1 - y_1)} =$$

$$= y_2 + r (y_1 - c_1) - [y_2 - (c_1 - y_1) - r (c_1 - y_1)] =$$

$$= y_2 + r (y_1 - c_1) - y_2 + (c_1 - y_1) \underbrace{+ r (c_1 - y_1)} =$$

$$= r (y_1 - c_1) + (c_1 - y_1) \overbrace{- r (y_1 - c_1)} =$$

$$= -(y_1 - c_1) = - \underbrace{\left(y_1 + r \underbrace{b_1}_{=0} - c_1 \right)}_{\equiv b_2 - \underbrace{b_1}_{=0}} =$$

$$= -\left(b_2 - \underbrace{b_1}_{=0}\right) \equiv -CA_1.$$

Or:

$$CA_2 = -CA_1.$$

In this *two-period* model with **zero** *initial* and *terminal* assets:

$$\underbrace{CA_1 + CA_2}_{cumulative\ \text{CA}} \equiv \underbrace{b_3 - b_1}_{change\ \text{in NFA}} = \underbrace{0}_{if\ b_1 = b_3 = 0}.$$

Over any stretch of time, as over a single period, a country's *cumulative* current account balance equals the *change* in its NFA for the *same* stretch of time.

Figure 1.1, p. 8, in Obstfeld and Rogoff (1996) [to be discussed in class] illustrates graphically the simple model we analysed (making no special assumption about the relation between δ and r) and clearly makes the point that an unbalanced current account is not necessarily a bad thing. IntERtemporal trade, made possible by international borrowing and lending, enables a country to achieve a *smoother* time profile of consumption relative to the case of no borrowing or lending opportunities available. The *gain* from intERtemporal trade arises when the *autarky* real interest rate (i.e., the one when there is no borrowing and lending), r_A, is *different* from (*above* or *below*) the *world* real interest rate, r. The *autarky* RIR is defined by (1.5) when endowments (or outputs) replace consumptions (in autarky, the representative individual simply consumes his/her endowment):

$$\frac{\beta u'(y_2)}{u'(y_1)} = \frac{1}{1+r_A} \equiv p_A, \tag{1.11}$$

$$\text{hence, } 1 + r_A = \frac{u'(y_1)}{\beta u'(y_2)} = \frac{1}{p_A}.$$

The above equation also gives the *autarky* relative price, p_A, of present consumption in terms of future consumption. See again Figure 1.1, p. 8, in Obstfeld and Rogoff (1996), for another interpretation in the context of gains from intERtemporal trade: what produces these gains is the chance to trade with somebody different from you, or with a country different from yours. Indeed, the greater is the difference, the bigger the gain. The only case of no gain would be if, by coincidence, $r_A = r$.

1.1.5. *Temporary vs Permanent Endowment Changes, RIR and CA.* Another result of the simple model we are discussing here that carries over to more realistic environments concerns the effects on the current account of temporary vs permanent shocks on endowments (or output or income). A natural benchmark to judge about such effects is the special case of $\delta = r$ or, which is the same, $\beta = \frac{1}{1+r}$. In this case, multiplying both sides of equation (1.11) by $1 + r$ yields:

$$\frac{u'(y_2)}{u'(y_1)} = \frac{1+r}{1+r_A} \tag{1.12}$$

Now it becomes clear from (1.12) that the only reason for the *world* and *autarky* real interest rates to differ are the different endowment (or output) levels in the two periods considered. That is, $r_A \neq r \Leftrightarrow y_1 \neq y_2$.

Imagine that an economy expects its output (endowment) to be constant over time, i.e., that $y_1 = y_2 = \overline{y} = const.$ Such an economy will plan on a balanced current account, as under our assumption here of $\delta = r$ its homogeneous population prefers a smooth consumption path. What will happen to this economy if it is hit by a temporary vs permanent output shock?

To answer this question in the simple framework we developed, let us define – in a very stylised fashion, it is true – what a temporary output (endowment) change and a permanent output (endowment) change would mean. A positive *temporary* output shock would increase output only in the *first* period, relative to the constant output expected, i.e., $y_1 > \overline{y}$ but $y_2 = \overline{y}$.

A positive *permanent* output shock would increase output in *both* periods by the *same* amount, relative to what was expected, i.e., $y_1 = y_2 > \overline{y}$.

A positive *temporary* output (endowments or income) shock would, given the wish to smooth consumption over one's lifetime, lead the country into a current account *surplus*, $y_1 - c_1 > 0$, with initial assets $b_1 = 0$ (as assumed). The autarky interest rate, r_A, will fall (see (1.11)), below the world interest rate, r (see (1.12)), and people will be willing to lend some of their temporarily high output to foreigners. This results in the CA surplus in period 1. Then, by the end of the second period, the loans will have to be repaid, with terminal assets $b_3 = 0$ (as also assumed). A *negative* temporary shock on output (endowments or income) would, symmetrically, give rise to a current account deficit in the first period, and the borrowing to smooth consumption should have to be repaid in the second period.

A positive (or negative) *permanent* shock on output (endowments or income), by contrast, will *not* have an effect on the current account. The reason is that the autarky interest rate, r_A, does *not* change in this case, as could be seen in (1.11).

Even though these results were derived in an oversimplified analytical framework, they have been confirmed to be valid as well in (i) multi-period models and in (ii) large economy models. Large economies are important players in the world capital market so – unlike a SOE – by lending and borrowing they also affect the world price of capital, r.

1.2. **Introducing Production, Investment and Government Spending.** Now let us generalise the model of the preceding section by adding into it more realism. Keeping for now the small open economy assumption, we introduce:

- a *production* function, with a sole argument, or factor of production, capital:

$$y \equiv F(k); \tag{1.13}$$

alternatively, one could think of a production function with constant labour, as a second productive factor, $y \equiv F(k, \overline{n})$;

- capital accumulates through *investment* (and for simplicity, we abstract from capital depreciation):

$$k_{t+1} = k_t + i_t; \tag{1.14}$$

- we finally allow for *government spending*, g.

1.2.1. *The Current Account Again, Now as Saving Less Investment.* First of all, the introduction of production changes the analysis in Figure 1.1 in Obstfeld and Rogoff (1996) in the sense that in addition to the *budget line*, now (with production, investment and government spending) defined by:

$$\text{budget line:} \quad c_2 = y_2 - i_2 - g_2 - (1+r)(c_1 + i_1 + g_1 - y_1), \tag{1.15}$$

there is also a *production possibility frontier (PPF) curve*, defined by:

$$\text{PPF:} \quad c_2 = \underbrace{F\left[\overbrace{k_1 + \overbrace{F(k_1) - c_1 - g_1}^{=i_1}}^{=k_2}\right]}_{=y_2} + \overbrace{\overbrace{k_1}^{\text{inherited capital}} + \overbrace{\underbrace{F(k_1)}_{=y_1} - c_1 - g_1}^{=i_1}}^{=k_2}. \tag{1.16}$$

It is easily seen that the slope of the budget line (1.15) is:

$$\text{slope of budget line:} \quad \frac{\partial c_2}{\partial c_1} = -(1+r);$$

and the slope of the PPF curve (1.16) is:

$$\text{slope of PPF:} \quad \frac{\partial c_2}{\partial c_1} = F' \left[k_1 + F(k_1) - c_1 - g_1 \right] \left[k_1 + F(k_1) - c_1 - g_1 \right]' - 1 =$$

$$= F' \underbrace{\left[k_1 + F(k_1) - c_1 - g_1 \right]}_{=k_2} (-1) - 1 = - \left[1 + F'(k_2) \right].$$

[Analysis of Figure 1.3, p. 20, in Obstfeld and Rogoff (1996) to be done in class.]

The transition equation for wealth, including financial assets (b) plus physical assets (k), in the enriched model here takes into account these new features:

$$\Delta \mathcal{W}_{t+1} \equiv b_{t+1} + k_{t+1} - (b_t + k_t) = y_t + rb_t - c_t - g_t,$$

$$CA_t \equiv b_{t+1} - b_t = \underbrace{\underbrace{\underbrace{y_t}_{\equiv GDP_t} + rb_t}_{\equiv GNP_t} - c_t - g_t}_{\equiv s_t} - \underbrace{(k_{t+1} - k_t)}_{\equiv i_t},$$

$$CA_t \equiv b_{t+1} - b_t = s_t - i_t.$$

The interpretation of the current account as the difference between saving and investment emphasises that it is fundamentally an intertemporal phenomenon.

1.2.2. *The Optimisation Problem, Its FONCs and Interpretation.* To derive the intertemporal budget constraint, analogous to equation (1.3) in our initial endowment model, but now with production, investment and government spending already accounted for in the present extended version, we write successively the current account in periods 1 and 2:

$$CA_1 \equiv b_2 - \underbrace{b_1}_{=0} = y_1 + r \underbrace{b_1}_{=0} - c_1 - g_1 - i_1 = y_1 - c_1 - g_1 - i_1;$$

$$CA_2 \equiv \underbrace{b_3}_{=0} - b_2 = y_2 + rb_2 - c_2 - g_2 - i_2.$$

Solving the second equation for b_2, and substituting back into the first equation, one obtains the intertemporal budget constraint for the extended model:

$$\underbrace{\frac{-y_2 + c_2 + g_2 + i_2}{1 + r}}_{b_2} = y_1 - c_1 - g_1 - i_1,$$

$$\frac{-y_2 + c_2 + g_2 + i_2}{1 + r} = y_1 - c_1 - g_1 - i_1,$$

$$c_1 + i_1 + \frac{c_2 + i_2}{1 + r} = y_1 - g_1 + \frac{y_2 - g_2}{1 + r}. \tag{1.17}$$

The representative agent thus maximises the same lifetime utility, (1.1), under the intertemporal budget constraint (1.17), in which equation (1.13) is used to replace y with $F(k)$ and equation (1.14) to replace i with the change in k.

To simplify further, it is natural to assume that people will never wish to carry capital past the terminal period 2. Thus, similarly to our previous assumption that $b_3 = 0$, we now also impose $k_3 = 0$, which implies:

$$i_2 = \underbrace{k_3}_{=0} - k_2 = -k_2.$$

Using equation (1.17) to eliminate c_2 from U_l transforms the individual problem into:

$$\max_{c_1, i_1} \quad u(c_1) + \beta u \left\{ \underbrace{(1+r)[F(k_1) - c_1 - g_1 - i_1] + F(\underbrace{i_1 + k_1}_{=k_2}) - g_2 + \underbrace{i_1 + k_1}_{=k_2 = -i_2}}_{c_2} \right\}. \tag{1.18}$$

k_1 is given, by history, so it is not subject to choice on date 1. The two corresponding FONCs are then the Euler equation, with respect to c_1,

$$\frac{\partial U_l}{\partial c_1} = u'(c_1) + \beta u'(c_2) \cdot [(1+r) \cdot (-1)] = 0, \text{ hence}$$

$$u'(c_1) = (1+r)\beta u'(c_2),$$

which is exactly the same as (1.7) in the simpler model we started the lecture with, and now a second optimality (or efficiency) condition, with respect to i_1,

$$\frac{\partial U_l}{\partial i_1} = \beta u'(c_2) \cdot [(1+r) \cdot (-1) + F'(k_2) \cdot 1 + 1] = 0, \text{ hence}$$

$$F'(k_2) = r. \tag{1.19}$$

In a closed economy with competitive factor markets equation (1.19) has the usual interpretation that the *marginal product of capital*, $F'(k_2)$, is in equilibrium the same as the *real interest rate*, r. In the particular context of our extended SOE model here with investment and government spending, (1.19) says that period 1 investment should continue to the point at which its marginal return, $F'(k_2)$, equalises that of the foreign loan, r. A critical feature of equation (1.19) is its implication that the desired capital stock is independent of domestic consumption preferences. This *separation of investment from consumption decisions* in the economy would be the case under several assumptions:

(1) SOE: so that the *investment decisions* of the small economy do *not* change the *interest rate* at which investment projects can be financed in the world capital market;
(2) the SOE produces and consumes a *single tradable good*: if *nontraded goods* are allowed for, consumption shifts can affect investment;
(3) capital markets are *free of imperfections* that might act to limit borrowing: if *default risk* restricts access to international borrowing, national saving can influence domestic investment.

In the set-up we considered, investment is also independent of *government* consumption. In particular, government spending *does not crowd out* investment in a SOE facing a *perfect world capital market.*

2. A Two-Period Two-Region World Economy Real Model: General Equilibrium

This section sketches a two-country model similar to the SOE one of the preceding section. The crucial difference is that we now look at a world economy consisting of two *large* countries, and focus on *how the world real interest rate*, taken as exogenous in the previous section, *is endogenously determined* here.

2.1. A Global Two-Period Endowment Economy.

Abstracting again from production, investment and government spending to simplify, we call the two countries $H(ome)$ and $F(oreign)$ and impose on them a parallel, or symmetric structure. As is the tradition, $Foreign$ variables are distinguished by a superscript asterisk $(*)$.

Equilibrium in the global output (or rather endowment) market requires equal supply and demand on each date t:

$$\underbrace{y_t + y_t^*}_{\text{supply}} = \underbrace{c_t + c_t^*}_{\text{demand}}.$$

Subtracting world consumption from both sides:

$$y_t + y_t^* - c_t - c_t^* = 0,$$

$$s_t + s_t^* = 0,$$

i.e., world saving must be zero. Since there is no investment and government spending in the model, the current account simply equals national saving in both countries, so we can also write the above equation as:

$$CA_t + CA_t^* = 0.$$

Using Walras law, we can reduce the two interdependent markets – for output today and output in the future – to one market. Figure 1.5, p. 24, in Obstfeld and Rogoff (1996) [to be discussed in class] shows how the *equilibrium* real interest rate is determined for *given* present and future endowments. The key lesson is that the equilibrium world interest rate, r, must lie *between* the two autarky rates:

$$r_A < r < r_A^*. \tag{2.1}$$

2.2. **Saving and the Interest Rate.** To justify the shapes of the saving schedules in Figure 1.5 in Obstfeld-Rogoff (1996), we need to introduce the concept of the elasticity of intertemporal substitution (in consumption), EIS, also called intertemporal substitutability (in consumption), which measures the sensitivity of the intertemporal consumption allocation to a change in the (real) interest rate.

2.2.1. *Elasticity of Intertemporal Substitution in Consumption.* We go back to the across-date FONC (1.5) and *take natural logarithms* from both sides of it:

$$\ln\left[\frac{\beta u'(c_2)}{u'(c_1)}\right] = \ln\left[\frac{1}{1+r}\right],$$

$$\underbrace{\ln\beta}_{=const} + \ln u'(c_2) - \ln u'(c_1) = \underbrace{\ln 1}_{=0} - \ln(1+r),$$

$$\ln(1+r) = \ln u'(c_1) - \ln u'(c_2) - \underbrace{\ln\beta}_{=const}.$$

We now *totally differentiate* the above equation:

$$\frac{d\ln(1+r)}{d(1+r)}d(1+r) = \frac{d\ln u'(c_1)}{dc_1}dc_1 - \frac{d\ln u'(c_2)}{dc_2}dc_2,$$

$$\frac{1}{1+r}d(1+r) = \frac{1}{u'(c_1)}u''(c_1)\,dc_1 - \frac{1}{u'(c_2)}u''(c_2)\,dc_2,$$

$$\frac{1}{1+r}\underbrace{\frac{1+r}{1+r}}_{=1}d(1+r) = \frac{u''(c_1)}{u'(c_1)}\underbrace{\frac{c_1}{c_1}}_{=1}dc_1 - \frac{u''(c_2)}{u'(c_2)}\underbrace{\frac{c_2}{c_2}}_{=1}dc_2,$$

$$\frac{1+r}{1+r}\frac{d(1+r)}{1+r} = c_1\frac{u''(c_1)}{u'(c_1)}\frac{dc_1}{c_1} - c_2\frac{u''(c_2)}{u'(c_2)}\frac{dc_2}{c_2},$$

$$\underbrace{\frac{d(1+r)}{1+r}}_{\equiv d\ln(1+r)} = \frac{c_1u''(c_1)}{u'(c_1)}\underbrace{\frac{dc_1}{c_1}}_{\equiv d\ln c_1} - \frac{c_2u''(c_2)}{u'(c_2)}\underbrace{\frac{dc_2}{c_2}}_{\equiv d\ln c_2},$$

$$d\ln(1+r) = \frac{c_1u''(c_1)}{u'(c_1)}d\ln c_1 - \frac{c_2u''(c_2)}{u'(c_2)}d\ln c_2. \tag{2.2}$$

In accordance with the general definition of elasticity, one may define the elasticity, $\varepsilon_{u'(c)}$, of the marginal utility of consumption, $u'(c)$, to be:

$$\varepsilon_{u'(c)} \equiv -\frac{\frac{du'(c)}{u'(c)}}{\frac{dc}{c}} = -\frac{du'(c)}{u'(c)}\frac{c}{dc} = -\underbrace{\frac{du'(c)}{dc}}_{\equiv u''(c)}\frac{c}{u'(c)} = -\frac{cu''(c)}{u'(c)}.$$

We next define the *inverse* of the above elasticity of the marginal utility of consumption by:

$$\sigma(c) \equiv \frac{1}{\varepsilon_{u'(c)}} \equiv -\frac{u'(c)}{cu''(c)}. \tag{2.3}$$

$\sigma(c)$, as defined by (2.3), is called the elasticity of intertemporal substitution (in consumption). When $\sigma(c) = \sigma = const$, equation (2.2) becomes:

$$d\ln(1+r) = \underbrace{\frac{c_1 u''(c_1)}{u'(c_1)}}_{\equiv -\frac{1}{\sigma}} d\ln c_1 - \underbrace{\frac{c_2 u''(c_2)}{u'(c_2)}}_{\equiv \frac{1}{\sigma}} d\ln c_2,$$

$$d\ln(1+r) = -\frac{1}{\sigma}d\ln c_1 + \frac{1}{\sigma}d\ln c_2,$$

$$d\ln(1+r) = \frac{1}{\sigma}(-d\ln c_1 + d\ln c_2),$$

$$d\ln\left(\frac{c_2}{c_1}\right) = \sigma d\ln(1+r).$$

Thus, a *high* σ (a usual parameter of the *period* utility function) – which corresponds to a *gently curving* period utility – implies a *sensitive* response of relative consumption to a change in the interest rate.

The class of period utility functions characterised by a *constant elasticity* of intertemporal substitution is called *isoelastic* utility and is represented by:

$$u(c) = \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \quad \sigma > 0. \tag{2.4}$$

For $\sigma = 1$, the right-hand side of equation (2.4) should be replaced by its *limit*, $\ln c$ (found by application of l'Hôpital's rule). This limiting case with *logarithmic consumption* in the utility function, $u(c) = \ln c$ (implying $\sigma = 1$), has been quite popular in the literature because, although simple and special, it allows the insights of arriving at an explicit *analytical* solution to many problems.

2.2.2. *Substitution, Income and Wealth Effects.* Let us finally use the computational simplification the isoelastic utility class brings to illustrate how the response of relative consumption to a change in the interest rate depends on three effects, known from microeconomic theory, namely the substitution, income and wealth effects.

The representative agent maximises (1.1) subject to (1.3) with isoelastic period utility as in (2.4). Since utility is isoelastic,

$$u'(c) = \frac{d\left(\frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}\right)}{dc} = \frac{1}{1-\frac{1}{\sigma}}\left(1-\frac{1}{\sigma}\right)c^{1-\frac{1}{\sigma}-1} = c^{-\frac{1}{\sigma}}.$$

The Euler equation (1.7),

$$u'(c_1) = (1+r)\beta u'(c_2)$$

now implies

$$c_1^{-\frac{1}{\sigma}} = (1+r)\beta c_2^{-\frac{1}{\sigma}},$$

and, raising both sides to the power of $-\sigma$,

$$\left(c_1^{-\frac{1}{\sigma}}\right)^{-\sigma} = \left[(1+r)\,\beta c_2^{-\frac{1}{\sigma}}\right]^{-\sigma},$$

$$c_1 = (1+r)^{-\sigma}\,\beta^{-\sigma} c_2,$$

$$(1+r)^{\sigma}\,\beta^{\sigma} c_1 = c_2.$$

Using the budget constraint, we find that consumption in period 1 is:

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r},$$

$$c_1 + \frac{(1+r)^{\sigma}\,\beta^{\sigma} c_1}{1+r} = y_1 + \frac{y_2}{1+r},$$

$$c_1 + (1+r)^{\sigma-1}\,\beta^{\sigma} c_1 = y_1 + \frac{y_2}{1+r},$$

$$c_1\left[1 + (1+r)^{\sigma-1}\,\beta^{\sigma}\right] = y_1 + \frac{y_2}{1+r},$$

$$c_1 = \frac{1}{1+(1+r)^{\sigma-1}\,\beta^{\sigma}}\left(y_1 + \frac{y_2}{1+r}\right). \tag{2.5}$$

(1) *substitution effect*: a rise in the interest rate makes saving more attractive and induces people to reduce consumption today, i.e., to *substitute present consumption with saving, hence future consumption*; the rise in r thus means a rise in the price of present consumption in terms of future consumption;

(2) *income effect*: a rise in the interest rate also allows higher consumption in the future given the present value of lifetime resources; this expansion of the feasible consumption set is a positive income effect that leads people to *raise present consumption and curtail saving*;

the *tension* between the substitution and income effect is reflected in the term $(1+r)^{\sigma-1}$ in the consumption equation (2.5):

(a) when $\sigma > 1$, the *substitution effect* dominates because consumers are *relatively willing* to substitute present consumption for future consumption;

(b) when $\sigma < 1$, the *income effect* dominates instead;

(c) when $\sigma = 1$ (the *log*-consumption case in period utility), $(1+r)^{1-\sigma} = (1+r)^{1-1} = (1+r)^0 = 1$ and the expression $\frac{1}{1+(1+r)^{\sigma-1}\beta^{\sigma}}$ in (2.5) collapses to $\frac{1}{1+\beta}$: the fraction of lifetime income, $y_1 + \frac{y_2}{1+r}$, spent on present consumption, c_1, does not depend on the interest rate but it is simply $\frac{1}{1+\beta}$ (thus depending inversely /negatively/ on the subjective discount factor β only);

(3) *wealth effect*: the previous two effects both refer to the fraction of lifetime income, $y_1 + \frac{y_2}{1+r}$, devoted to present consumption, c_1; by contrast, the wealth effect comes from the change in lifetime income, $y_1 + \frac{y_2}{1+r}$, caused by a change in the interest rate, measured in date 1 consumption units (visible in the denominator of the last expression).

As just seen, the conflict among the three effects can be resolved in favour of each of the effects: thus, theory offers no definite prediction about how a change in interest rates will affect consumption and saving.

3. A Dynamic Real Model of a Small Open Economy: Finite and Infinite Horizons

[... The topic will not be presented in class, due to lack of time; those interested may see Obstfeld and Rogoff (1996), section 2.1. Moreover, we shall consider similar frameworks, essentially extending the basic set-up introduced in this lecture to *many* periods, in the models to be studied further in the course. ...]

References

[1] Fisher, Irving (1930), *The Theory of Interest*, New York: Macmillan.
[2] Obstfeld, Maurice and Kenneth Rogoff (1996), *Foundations of International Macroeconomics*, MIT Press.

Department of Economics, University of Essex, Wivenhoe Park, Colchester CO4 3SQ, UK
E-mail address: mihailov@essex.ac.uk
URL: http://www.essex.ac.uk/economics/people/staff/mihailov.shtm