EC933-G-AU – Lecture 5

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Asset Markets and Risk Sharing: Analytical Introduction of Uncertainty

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Plan of talk

- **introduction**

1. 2-period *2-state* **SOE** real model: *exo*genous prices
2. **Arrow-Debreu paradigm** and *complete* asset markets
3. **actuarially fair prices** and consumption *smoothing*
4. Arrow-Pratt **coefficient of relative risk aversion**
5. 2-period S-*state* **2-country** real model: *endo*genous prices
6. models with **financial market imperfections**

- **wrap-up**

Aim and learning outcomes

- **aim:** continue building up the microfoundations of open-economy macromodels by focusing on the *analytics* of **uncertainty (or risk)**
- **learning outcomes**
  - define and discuss
    - Arrow-Debreu *contingent claim securities*
    - *actuarially fair prices* and consumption smoothing across *states*
  - derive and interpret in *partial* and/or *general* equilibrium
    - standard *inter-state* Euler equations
    - coefficient of *relative risk aversion*
    - *equilibrium* prices and real interest rate
    - *equilibrium* consumption levels
  - analyse the basic set-up of microfounded *stochastic* OEMs

2-period 2-state SOE real model: assumptions

- assumptions **kept** from the nonstochastic 2-period SOE real model
  1. 2 countries, SOE (*H*) and RoW (*F*)
  2. that last for 2 periods
  3. a single, perishable /nonstorable/ and tradable good available to consume
  4. no production (function), i.e. an *endowment* model
  5. no investment
  6. no government spending
  7. no money, i.e. *real* model
  8. SOE takes RIR **and asset prices** as given, i.e. all prices are *exogenous*
  9. the *representative* individual
     - has *known* (thus, certain) income on date 1
     - starts out with *zero* net foreign assets
  10. a *constant* population size normalised at 1: *per capita* = *aggregate* quantities
- **additional** assumptions
  1. 2 states of nature *possible* on date 2, with *uncertain* actual realisation
     - occur *randomly*, according to a specified (i.e. *known*) probability distribution
     - differ *only* in their associated endowment (or output, or income) levels on date 2
  2. economic agents have sufficient foresight to *prearrange*, by explicit or implicit contracts, for trades in assets that protect them (partially) against contingencies

2-period 2-state SOE real model: utility

- **lifetime *expected* utility**

would be simply $u(c_2)$ under certainty (cf. lecture 4)

$$U_l \equiv u(c_1) + \overbrace{\underbrace{\beta\{\pi(1)u[c_2(1)] + \pi(2)u[c_2(2)]\}}_{\equiv \beta E_1[u(c_2)]}}$$

$\equiv \beta E_1[u(c_2)]$, i.e. *expected* (ex ante) utility of consumption on date 2

where

$$\beta \equiv \frac{1}{1+\delta} \qquad\qquad \pi(1) + \pi(2) = 1$$

- comprised of *time- and state-invariant*, *increasing* and *concave* **period utility**

2-period 2-state SOE real model: constraints

- net accumulation of assets

$$\overbrace{\underbrace{\frac{p(1)}{1+r}b_2(1)}_{\text{PV of insurance } \textit{if state } 1 \text{ on date } 2} + \underbrace{\frac{p(2)}{1+r}b_2(2)}_{\text{PV of insurance } \textit{if state } 2 \text{ on date } 2}}^{\text{would be simply } b_2 \text{ (with } b_1 \equiv 0\text{) under certainty (cf. lecture 4)}} \equiv \underbrace{y_1 - c_1}_{\text{date } 1 \text{ net saving}}$$

PV of *total* insurance for the uncertainty of date 2

- lifetime budget constraint

$$\underbrace{c_1 + \frac{\overbrace{p(1)c_2(1)+p(2)c_2(2)}^{\text{would be } c_2 \text{ under certainty (cf. lecture 4)}}}{1+r}}_{\text{PV of lifetime (state-)} \textit{contingent consumption}} = \underbrace{y_1 + \frac{\overbrace{p(1)y_2(1)+p(2)y_2(2)}^{\text{would be } y_2 \text{ under certainty (cf. lecture 4)}}}{1+r}}_{\text{PV of lifetime (state-)} \textit{contingent income}}$$

2-period 2-state SOE real model: consumer's problem and its FONCs

- objective function, as unconstrained optimisation problem

$$\max_{b_2(s)} U_1 = u\left[y_1 - \sum_{s=1}^{2} \frac{p(s)}{1+r} b_2(s) \right] + \sum_{s=1}^{2} \pi(s)\beta u[y_2(s) + b_2(s)]$$

- FONCs, as inter-state Euler equations

$$\frac{\partial U_1}{\partial b_2(s)} = 0 \qquad s = 1,2 \Leftrightarrow \frac{p(s)}{1+r} u'(c_1) = \pi(s)\beta u'[c_2(s)], \qquad s = 1,2$$

- FONCs, as MRS equal to relative price

$$\frac{\pi(s)\beta u'[c_2(s)]}{u'(c_1)} = \frac{p(s)}{1+r}, \qquad s = 1,2$$

2-period 2-state SOE real model: actuarially fair prices and consumption smoothing

- creating **synthetic assets** form **primal A-D securities =>** $p(1)+p(2)=1$

$$\underbrace{(1+r)}_{\text{units of state 1 A-D}}\underbrace{\frac{p(1)}{1+r}}_{\text{unit price of state 1 A-D}}+\underbrace{(1+r)}_{\text{units of state 2 A-D}}\underbrace{\frac{p(2)}{1+r}}_{\text{unit price of state 2 A-D}}=\underbrace{1}_{\text{cost in terms of date 1 output units to buy 1 bond}}$$

- adding FONCs => stochastic Euler eq for a riskless bond

$$\underbrace{[p(1)+p(2)]}_{=1}u'(c_1)=(1+r)\beta\underbrace{\{\pi(1)u'[c_2(1)]+\pi(2)u'[c_2(2)]\}}_{\equiv E_1[u'(c_2)],\text{ by definition}}$$

- compact FONC => actuarially fair prices of A-D securities

$$\frac{\pi(1)u'[c_2(1)]}{\pi(2)u'[c_2(2)]}=\frac{p(1)}{p(2)} \qquad \frac{\pi(1)}{\pi(2)}=\frac{p(1)}{p(2)}$$

$$u'[c_2(1)]=u'[c_2(2)],\text{ hence } c_2(1)=c_2(2)=c_2=const$$

- *special* case: **consumption smoothing** *optimal* only if **A-D prices actuarially fair**

2-period 2-state SOE real model: Arrow-Pratt coefficient of relative risk aversion (I)

- start from *across-state Euler equation* and **take natural logs**

$$\ln\left[\frac{\pi(1)u'[c_2(1)]}{\pi(2)u'[c_2(2)]}\right] = \ln\left[\frac{p(1)}{p(2)}\right]$$

$$\ln p(1) - \ln p(2) = \ln u'[c_2(1)] - \ln u'[c_2(2)] + \underbrace{\ln \pi(1)}_{=const} - \underbrace{\ln \pi(2)}_{=const}$$

- **totally differentiate** result

$$\frac{d\ln p(1)}{dp(1)}dp(1) - \frac{d\ln p(2)}{dp(2)}dp(2) = \frac{d\ln u'[c_2(1)]}{dc_2(1)}dc_2(1) - \frac{d\ln u'[c_2(2)]}{dc_2(2)}dc_2(2)$$

$$\frac{dp(1)}{p(1)} - \frac{dp(2)}{p(2)} = \frac{1}{u'[c_2(1)]}u''[c_2(1)]\underbrace{\frac{c_2(1)}{c_2(1)}}_{=1}dc_2(1) - \frac{1}{u'[c_2(2)]}u''[c_2(2)]\underbrace{\frac{c_2(2)}{c_2(2)}}_{=1}dc_2(2)$$

$$d\ln\left[\frac{p(1)}{p(2)}\right] = \frac{c_2(1)u''[c_2(1)]}{u'[c_2(1)]}d\ln c_2(1) - \frac{c_2(2)u''[c_2(2)]}{u'[c_2(2)]}d\ln c_2(2)$$

2-period 2-state SOE real model: Arrow-Pratt coefficient of relative risk aversion (II)

- define the Arrow-Pratt **coefficient of relative risk aversion** as

$$\rho(c) \equiv -\frac{cu''(c)}{u'(c)}$$

- assume it to be **constant: C(onstant)RRA** ⇔ $\rho(c) = \rho = const$
- then, the last equation on previous slide simplifies to

$$d\ln\left[\frac{p(1)}{p(2)}\right] = \underbrace{\frac{c_2(1)u''[c_2(1)]}{u'[c_2(1)]}}_{=-\rho} d\ln c_2(1) - \underbrace{\frac{c_2(2)u''[c_2(2)]}{u'[c_2(2)]}}_{\equiv\rho} d\ln c_2(2)$$

$$d\ln\left[\frac{p(1)}{p(2)}\right] = \rho d\ln\left[\frac{c_2(2)}{c_2(1)}\right] \qquad d\ln\left[\frac{c_2(2)}{c_2(1)}\right] = \frac{1}{\rho}d\ln\left[\frac{p(1)}{p(2)}\right]$$

- the **inverse** of CRRA, $1/\rho$, is, by definition, the **elasticity of substitution** b/n *state-contingent* consumption levels with respect to relative *A-D* prices
- high ρ ⇔ $0 < 1/\rho < 1$ ⇔ **inelastic** response of relative consumption to change in relative price of insurance

2-period 2-state SOE real model: pros and cons of CRRA utility and log utility

- RRA *constant* if special **CRRA** (cf. isoelastic) *period* **utility** (class of functions)

$$u(c) = \frac{c^{1-\rho}}{1-\rho}, \quad \rho > 0, \rho \neq 1 \qquad u(c) = \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \quad \sigma > 0$$

- for **log utility**, **Euler equations** reduce to

$$\frac{p(1)}{1+r} \underbrace{\frac{1}{c_1}}_{u'(c_1)} = \pi(1)\beta \underbrace{\frac{1}{c_2(1)}}_{u'[c_2(1)]} \qquad \frac{p(2)}{1+r} \underbrace{\frac{1}{c_1}}_{u'(c_1)} = \pi(2)\beta \underbrace{\frac{1}{c_2(2)}}_{u'[c_2(2)]}$$

- **optimal consumption demands** with **log utility** are shown in lecture to be

$$c_1 = \frac{1}{1+\beta}\mathcal{W}_l = \frac{1}{1+\beta}\left[y_1 + \frac{p(1)y_2(1)+p(2)y_2(2)}{1+r}\right]$$

$$\frac{p(1)}{1+r}c_2(1) = \frac{\pi(1)\beta}{1+\beta}\mathcal{W}_l = \frac{\pi(1)\beta}{1+\beta}\left[y_1 + \frac{p(1)y_2(1)+p(2)y_2(2)}{1+r}\right]$$

$$\frac{p(2)}{1+r}c_2(2) = \frac{\pi(2)\beta}{1+\beta}\mathcal{W}_l = \frac{\pi(2)\beta}{1+\beta}\left[y_1 + \frac{p(1)y_2(1)+p(2)y_2(2)}{1+r}\right]$$

2-period 2-state SOE real model: consumption demands and CA under log utility

- expression **parallel** to the nonstochastic *log*-utility case

$$CA_1 \equiv y_1 - c_1 = y_1 - \frac{1}{1+\beta}\left[y_1 + \frac{p(1)y_2(1)+p(2)y_2(2)}{1+r}\right] =$$

$$= \frac{\beta}{1+\beta}y_1 - \frac{1}{1+\beta}\left[\frac{p(1)y_2(1)}{1+r} + \frac{p(2)y_2(2)}{1+r}\right]$$

- but **cannot** be directly interpreted by analogy with comparative advantage, as we did for intertemporal trade
- because now **three**, not two, **"goods"**
  1. certain consumption on date 1
  2. contingent consumption on date 2 state 1
  3. contingent consumption on date 2 state 2

2-period $\mathcal{S}$-state 2-country global real model: general equilibrium under CRRA utility

- **assumptions**
  - now 2 **large** economies, *H* and *F* (instead of SOE-RoW)
  - **CRRA** utility, for simplicity and to gain some initial intuition
  - **more** than 2 states of nature
- **market-clearing (GE) conditions**

$$c_1 + c_1^* = y_1 + y_1^* \qquad c_2(s) + c_2^*(s) = y_2(s) + y_2^*(s), \qquad s = 1,2,\dots,\mathcal{S} \qquad y^W \equiv y + y^*$$

- **Euler equations under CRRA utility**

$$\frac{p(s)}{1+r}\underbrace{[c_1]^{-\rho}}_{u'(c_1)} = \pi(s)\beta\underbrace{[c_2(s)]^{-\rho}}_{u'[c_2(1)]}, \qquad s = 1,2,\dots,\mathcal{S} \qquad \frac{p(s)}{1+r}\underbrace{[c_1^*]^{-\rho}}_{u'(c_1)} = \pi(s)\beta\underbrace{[c_2^*(s)]^{-\rho}}_{u'[c_2(1)]}$$

$$c_2(s) = \left[\frac{\pi(s)\beta(1+r)}{p(s)}\right]^{\frac{1}{\rho}} c_1, \qquad s = 1,2,\dots,\mathcal{S}, \qquad c_2^*(s) = \left[\frac{\pi(s)\beta(1+r)}{p(s)}\right]^{\frac{1}{\rho}} c_1^*$$

2-period $\mathcal{S}$-state 2-country global real model: date 1 A-D prices in GE under CRRA utility

- equilibrium **date 1 prices**: *summing up* consumption

$$\underbrace{c_2(s) + c_2^*(s)}_{\equiv y_2^W(s)} = \left[\frac{\pi(s)\beta(1+r)}{p(s)} \right]^{\frac{1}{\rho}} \underbrace{(c_1 + c_1^*)}_{\equiv y_1^W}, \qquad s = 1, 2, \ldots, \mathcal{S}$$

$$\frac{p(s)}{1+r} = \pi(s)\beta \left[\frac{y_2^W(s)}{y_1^W} \right]^{-\rho}, \qquad s = 1, 2, \ldots, \mathcal{S}$$

- *dividing* through => *actuarially fair* prices **iff** *world output invariant* across states $\dfrac{p(s)}{p(s')} = \left[\dfrac{y_2^W(s)}{y_2^W(s')} \right]^{-\rho} \dfrac{\pi(s)}{\pi(s')}$

2-period $\mathcal{S}$-state 2-country global real model: date 2 A-D prices and RIR in GE under CRRA utility

- equilibrium **date 2 prices**: for any state *s'*, *arbitrage condition*
- and *last* expression on previous slide imply $\sum_{s=1}^{\mathcal{S}} p(s) = 1$

$$p(s') = 1 - \sum_{s \neq s'} p(s) = 1 - p(s') \sum_{s \neq s'} \left[\frac{y_2^W(s)}{y_2^W(s')} \right]^{-\rho} \frac{\pi(s)}{\pi(s')}$$

- an equation which can be *solved for p(s')*

$$p(s') = \frac{\pi(s')\left[y_2^W(s')\right]^{-\rho}}{\sum_{s=1}^{\mathcal{S}} \pi(s)\left[y_2^W(s)\right]^{-\rho}}$$

- equilibrium **RIR**: above eq and date 1 *price* eqs => $1 + r = \dfrac{\left[y_1^W\right]^{-\rho}}{\beta \sum_{s=1}^{\mathcal{S}} \pi(s)\left[y_2^W(s)\right]^{-\rho}}$

2-period S-state 2-country global real model: GE consumption levels under CRRA utility

- multi-state analogues of two-state Euler equations

$$\frac{\pi(s)\beta u'[c_2(s)]}{u'[c_1]} = \frac{p(s)}{1+r} = \frac{\pi(s)\beta u'[c_2^*(s)]}{u'[c_1^*]} \qquad \frac{\pi(s)u'[c_2(s)]}{\pi(s')u'[c_2(s')]} = \frac{p(s)}{p(s')} = \frac{\pi(s)u'[c_2^*(s)]}{\pi(s')u'[c_2^*(s')]}$$

- combined with last equation two slides ago and CRRA utility

$$\frac{c_2(s)}{c_2(s')} = \frac{c_2^*(s)}{c_2^*(s')} = \frac{y_2^W(s)}{y_2^W(s')} \qquad \frac{c_2(s)}{c_1} = \frac{c_2^*(s)}{c_1^*} = \frac{y_2^W(s)}{y_1^W}$$

- constant fractions of world date 2 output and its growth rate

$$\frac{c_2(s)}{y_2^W(s)} = \frac{c_2(s')}{y_2^W(s')}, \qquad \frac{c_2^*(s)}{y_2^W(s)} = \frac{c_2^*(s')}{y_2^W(s')} \qquad \frac{c_2(s)}{y_2^W(s)} = \phi = \frac{c_1}{y_1^W}, \qquad \frac{c_2^*(s)}{y_2^W(s)} = 1-\phi = \frac{c_1^*}{y_1^W}$$

- graphical interpretation: O-R(96), Fig. 5.1, p. 290

Models with capital market imperfections

- up to here: *idealised* situation of **complete** asset markets => international *risk sharing*
- modelling various types of *realistic* **imperfections** of world *financial* markets
  - difficulty in *enforcing* financial contracts outside national jurisdiction: **sovereign risk** => O-R(96), section 6.1
  - problem of *asymmetric* information
    - **hidden *information*** and risk sharing: adverse selection => O-R(96), section 6.3
    - hidden *actions*: **moral hazard** in international lending => O-R(96), section 6.4

Concluding wrap-up

- **What have we learnt?**
  - define and analyse the implications of
    - Arrow-Debreu securities and complete asset markets
    - actuarially fair contingent claim prices and consumption smoothing
  - derive and interpret
    - standard *inter-state* Euler equations
    - Arrow-Pratt coefficient of relative risk aversion
    - risk sharing in theory and in practice
  - summarise the *baseline* set-up of microfounded *stochastic* OEMs
- **Where we go next**: to *applications/extensions* of the analytical framework introduced in a series of important models/papers