MPhys Project

Optical angular momentum in dispersive, lossless media

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Abstract

Noether’s theorem is applied to the action for electromagnetism in dispersive, lossless and rotationally symmetric media. The angular momentum density and associated flux are exactly determined. In vacuo, the angular momentum tensor is related to the energy momentum tensor by a simple formula. I find that in media, the angular momentum tensor does not relate to the energy momentum tensor by such a simple formula. As a special case, the time averaged angular momentum density is found for monochromatic fields in both homogeneous and inhomogeneous media. As a consequence, spin angular momentum arises naturally from the theory as a dispersive contribution.
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1 Introduction

Albert Einstein wrote of Emmy Noether: ‘In the judgement of the most competent living mathematicians, Fräulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began’ [1].

I feel that Emmy Noether deserves more than the usual credit. She was born in Bavaria to Jewish parents, and attended lectures at the University of Erlangen at a time when female students were only allowed to do so unofficially. After attending for two years she moved on to the University of Göttingen where she saw classes given by Hilbert, Klein and Minkowski. She moved back to Erlangen and was granted a doctorate in 1907. However, the subsequent, typical academical route was not open to her. It was not until 1919 that - with support from Hilbert and Klein - she could lecture under her own name [2]. In the field of pure mathematics, she revolutionised the theories of rings, fields and algebras. In 1933 she lost her teaching position due to being a Jew and a woman [3]. Forced out of Germany by the Nazis, she moved to the USA, where she worked at Bryn Mawr College and Princeton. Her work in physics is of profound importance for many different fields and is briefly described below. In essence, she laid down the mathematics for the connection between symmetry in the action and conservation laws.

The conservation of optical angular momentum is a consequence of the invariance of the medium under rotations. Noether’s theorem shows how to use such a symmetry to construct the associated conserved quantity. Before detailing Noether’s theorem let’s have a quick review of Lagrangian Formalism.

1.1 Lagrangian formalism

The content of this subsection and the next (1.1−1.2) draws on research from S. Weinberg’s The Quantum Theory of Fields [4].

The Lagrangian is a key quantity of a system, one from which we can determine its dynamics. It is a functional of a set of fields $\Psi^i(x,t)$ and their time derivatives $\dot{\Psi}^i(x,t)$
such that

\[ L = L[\Psi(t), \dot{\Psi}(t)]. \]

Above, \( \Psi \) only has a \( t \) argument; the functional notation implies that the variables \( l \) and \( x \) run over all their values for a fixed \( t \). Informally, functionals are basically integrals of something and a Lagrangian is an integral over space of a Lagrangian density, hence \( x \) runs over all it’s values. We can define a functional called the action as

\[ S[\Psi] \equiv \int_{-\infty}^{\infty} dt L[\Psi(t), \dot{\Psi}(t)]. \]

Then, exploiting the fact that the Lagrangian is itself an integral over all space of something, we get that \( \delta S[\Psi] \) under an arbitrary variation in \( S[\Psi] \) is

\[ \delta S[\Psi] = \int_{-\infty}^{\infty} dt \int d^3 x \left[ \frac{\delta L}{\delta \Psi^l(x)} \delta \Psi^l(x) + \frac{\delta L}{\delta \dot{\Psi}^l(x)} \delta \dot{\Psi}^l(x) \right]. \]

Integrating by parts, whilst ignoring surface terms gives the field equations as

\[ \frac{\delta L}{\delta \Psi^l(x)} - \frac{d}{dt} \frac{\delta L}{\delta \dot{\Psi}^l(x)} = 0 \]

if the principle of stationary action is imposed (\( \delta S = 0 \)).

A comment should be made on ‘ignoring surface terms’ both here and throughout the report. It is not so much that they are ignored, rather that they vanish. The integration is over all space and the fields are taken to vanish at \( \pm \infty \).

1.2 Noether’s theorem

Consider an infinitesimal symmetry transformation of some arbitrary fields \( \psi^l \).

\[ \Psi^l(x) \rightarrow \Psi^l(x) + i\eta f^l(x) \]

where \( f^l(x) \) is an arbitrary continuous function and \( \eta \) is an infinitesimal constant. This is a type of transformation that leaves the action invariant (i.e. \( \delta S = 0 \)) even when the field’s dynamical equations are not satisfied. If we now consider a transformation of the form

\[ \Psi^l(x) \rightarrow \Psi^l(x) + i\eta(x) f^l(x) \]

then, in general \( \delta S \neq 0 \). But \( \delta S \) must be of the form

\[ \delta S = - \int J^\mu(x) \partial_\mu \eta \, d^4 x \quad (1) \]

so that \( \delta S \) does vanish when \( \eta \) is constant. Up to now we have assumed nothing of the dynamical field equations. If we now assume that the fields satisfy the dynamical
equations, then for all infinitesimal variations of the fields the variation in the action is zero. If we integrate (1) by parts we have (ignoring surface terms)

$$\delta S = \int \eta \partial_\mu J^\mu \, d^4 x = 0$$

so that

$$\partial_\mu J^\mu = 0.$$  \hspace{1cm} (2)

It becomes apparent that $\frac{dF}{dt} = 0$ with

$$F = \int J^0 \, d^3 x.$$  

So, there exists one constant of motion, $F$, for each independent symmetry transformation and one conserved current, $J^\mu$. This is referred to as Noether’s theorem; symmetries (in the action) imply conservation laws.

To summarise, the symmetry in the action allows us to write the variation of the action with the $\partial_\mu \eta$ term. If we are working in a scenario where the field equations apply, the variation in the action is zero and we get conservation in the form $\partial_\mu J^\mu$ upon integration by parts. So, Noether’s theorem is saying that if your action has a symmetry, then you can manipulate the variation in the action into a certain form and directly extract the conserved quantities.

As an illustrative example, if a particle travels freely through a homogeneous medium there is translational symmetry and hence an associated constant of motion; momentum.

1.3 Light in vacuo

1.3.1 Energy-momentum

Light can be used to transport energy, momentum and angular momentum. The mathematical expression for the conserved energy-momentum tensor of light in vacuo is well known from textbooks [4] as

$$\Theta^{\alpha\beta} = \varepsilon_0 (g^{\alpha\mu} F_{\mu\lambda} F_{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\lambda} F^{\mu\lambda}).$$

Here, $g^{\alpha\beta}$ is the Minkowski metric with signature $(+ - - -)$ and $F_{\mu\lambda}$ is the electromagnetic tensor, $F_{\mu\lambda} = \partial_\mu A^\lambda - \partial_\lambda A^\mu$.

$A^\lambda$ is the 4-potential

$$A^\lambda = (\phi, cA),$$

where $\phi$ is the scalar electric potential and $A$ is the magnetic vector potential. In schematic matrix form, the conserved energy-momentum tensor, $\Theta^{\alpha\beta}$, is

$$\begin{pmatrix}
    u & c g \\
    c g & T_{ij}
\end{pmatrix}$$
with SI units of \([Kgm^{-1}s^{-2}]\). The quantities

\[
u = \frac{1}{2}(\varepsilon_0 E^2 + \kappa_0 B^2),
\]

(3)

\[
g = \varepsilon_0 (E \times B),
\]

(4)

and

\[
T_{ij} = \frac{1}{2} g_{ij}(\varepsilon_0 E^2 + \kappa_0 B^2) - \varepsilon_0 E_i E_j - \kappa_0 B_i B_j
\]

(5)

are the energy density, momentum density and Maxwell stress tensor respectively. Note the symmetry of \(\Theta^{\alpha\beta}\). The fields \(E\) and \(B\) are defined by

\[
E = -\nabla \phi - \partial_t A, \quad B = \nabla \times A.
\]

1.3.2 Angular momentum

The conserved angular momentum tensor of light in vacuo is then described by the neat formula

\[
\mathcal{A}^{\alpha\beta\gamma} = 2\Theta^{\alpha[\beta x\gamma]},
\]

(6)

where the square brackets instruct us to take the antisymmetric parts. So for a general tensor \(\chi^{ij}\), we have

\[
\chi^{[ij]} = \frac{1}{2} (\chi^{ij} - \chi^{ji}).
\]

Hence \(\chi^{[ij]} = \chi^{ij}\) if \(\chi^{ij}\) is antisymmetric and \(\chi^{[ij]} = 0\) if \(\chi^{ij}\) is symmetric.

Since \(\mathcal{A}^{\alpha\beta\gamma}\) is antisymmetric in the last two indices, we can identify two distinct classes of elements; those of the form \(\mathcal{A}^{\alpha ij}\) and \(\mathcal{A}^{\alpha 0i}\). By use of the defining equation for the angular momentum tensor (6), and the expressions for momentum density (4) and the stress tensor (5) the \(\mathcal{A}^{\alpha ij}\) components are

\[
\mathcal{A}^{0ij} = (cg \times x)_p e^{bij},
\]

(7)

\[
\mathcal{A}^{kij} = T^{ki}x^j - T^{kj}x^i.
\]

(8)

These are \(c \times\) the vacuum angular momentum density and the vacuum angular momentum flux respectively. Evaluation of the \(\mathcal{A}^{\alpha 0i}\) components using (6) and components of the energy-momentum tensor leads to two distinct quantities. One is a statement on center of mass motion and they both explicitly depend on time [5].
1.4 Light in lossless media

There are complications when light propagates in a medium. If the light is composed of frequencies for which loss (net absorption) is significant then it is clear that there can be no conservation. If loss is negligible then a conserved energy-momentum tensor exists. When dispersion is accounted for, this tensor is more complicated than the vacuum expression [5]. The energy-momentum tensor for light in dispersive, lossless media can be written in schematic matrix form as

\[
\begin{pmatrix}
\rho & \mathbf{S} \\
\epsilon P_i & \sigma_{i}^j
\end{pmatrix},
\]

with SI units of \([Kg/m^2s^{-2}]\). The quantities

\[
\rho = \frac{\varepsilon_0}{2} \mathbf{E}[\varepsilon(r, i\partial_t)\mathbf{E}] + \frac{\kappa_0}{2} \mathbf{B}[\kappa(r, i\partial_t)\mathbf{B}] - \frac{\varepsilon_0}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{2n} (-1)^{n+m} \varepsilon_{2n}(r) \partial_{t}^{m-1} \mathbf{E} \cdot \mathbf{B}_{t}^{2n-m+1} \mathbf{E}
\]

\[
+ \frac{\kappa_0}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{2n} (-1)^{n+m} \kappa_{2n}(r) \partial_{t}^{m-1} \mathbf{B} \cdot \mathbf{B}_{t}^{2n-m+1} \mathbf{B},
\]

(9)

\[
\mathbf{S} = \kappa_0 \mathbf{E} \times [\kappa(r, i\partial_t)\mathbf{B}],
\]

(10)

\[
P_i = \varepsilon_0 \varepsilon_{ijk} [\varepsilon(r, i\partial_t)\mathbf{E}] B^j + \frac{\varepsilon_0}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{2n} (-1)^{n+m} \varepsilon_{2n}(r) \partial_{t}^{m-1} E_j \partial_{t}^{2n-m} \nabla_i E^j
\]

\[
- \frac{\kappa_0}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{2n} (-1)^{n+m} \kappa_{2n}(r) \partial_{t}^{m-1} B_j \partial_{t}^{2n-m} \nabla_i B^j.
\]

(11)

and

\[
\sigma_{i}^j = -\varepsilon_0 E_i \varepsilon(r, i\partial_t) E^j - \kappa_0 [\kappa(r, i\partial_t) B_i] B^j + \frac{1}{2} \delta_{i}^{j} [\varepsilon_0 E_k \varepsilon(r, i\partial_t) E^k + \kappa_0 B_k \kappa(r, i\partial_t) B^k]
\]

(12)

are the energy density, energy flux, momentum density and stress tensor respectively [6]. The constants \(\kappa_0\) and \(\varepsilon_0\) are related by \(c^2 = \frac{\kappa_0}{\varepsilon_0}\), with \(c\) the speed of light in vacuo and \(\mu_0 = \kappa_0^{-1}\). \(\varepsilon\) is the relative permittivity of the medium and \(\mu = \kappa^{-1}\) is the relative permeability. Here, and throughout this report \(\varepsilon\) and \(\kappa\) may have their arguments removed for brevity. If \(\varepsilon\), \(\kappa\) appear without a subscript 0, then take them to be have their appropriate argument.

It should be made clear that momentum is only conserved in homogeneous media, where \(\varepsilon\) and \(\kappa\) are independent of \(r\). Energy conservation only requires translational time symmetry, hence \(\varepsilon\) and \(\kappa\) can keep their spatial dependency.
A comment should be made on the physical significance of dispersive, lossless media. In some circumstances it is possible to neglect loss over significant frequency ranges. In fact, we can neglect loss in any medium with a small enough value of \( Im[\varepsilon(r, \omega)] \) over the required frequency range. Dispersion is dictated by \( Re(\varepsilon) \) and loss by \( Im(\varepsilon) \) such that if \( Re(\varepsilon) = 1 \) then there is no dispersion, and if \( Im(\varepsilon) = 0 \) then there is no loss. The Kramers-Kronig relations - which follow from the causal connection between polarisation and the electric field - relate the real part of \( \varepsilon \) to it’s imaginary part such that neither one can be zero in media. But, for very small \( Im(\varepsilon) \) we can neglect loss. It so happens that \( Re(\varepsilon) \) is even in \( \omega \) and \( Im(\varepsilon) \) is odd in \( \omega \) [5].

In conclusion, since we are assuming a lossless medium, \( \varepsilon \) and \( \kappa \) are real and even functions of \( \omega \), hence they can be expanded according to

\[
\varepsilon(r, i\partial_t)E(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varepsilon(r, \omega)E(r, \omega)e^{-i\omega t} d\omega,
\]

\[
\kappa(r, i\partial_t)B(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \kappa(r, \omega)B(r, \omega)e^{-i\omega t} d\omega,
\]

with

\[
\varepsilon(r, i\partial_t) = \sum_{n=0}^{\infty} \varepsilon_{2n}(r)(i\partial_t)^{2n}, \tag{13}
\]

\[
\kappa(r, i\partial_t) = \sum_{n=0}^{\infty} \kappa_{2n}(r)(i\partial_t)^{2n}. \tag{14}
\]

In a vacuum, \( \varepsilon \) and \( \kappa \) both equal unity, which implies that the infinite series in the components of the energy-momentum tensor in media (9)-(12), vanish and the vacuum results (3) - (5) are reproduced.

1.5 Outline of the problem

The aim of this project is to calculate the angular momentum density of light in a dispersive, lossless, rotationally symmetric medium. Here, the angular momentum cannot be simply written in terms of the energy-momentum tensor as in (6). It must be found by the same method as used for the energy momentum tensor, namely Noether’s theorem. In a vacuum the energy-momentum tensor is necessarily symmetric for angular momentum conservation [5]. However it is not symmetric in a medium [6], hence there is no clear relationship between the energy-momentum and angular momentum tensors.

To solve the problem, Noether’s theorem is applied to the action for light in a dispersive, lossless, rotationally symmetric medium [6]

\[
S = \int \left\{ \frac{\varepsilon_0}{2} E_r [\varepsilon(r, i\partial_t)E] - \frac{\kappa_0}{2} B_r [\kappa(r, i\partial_t)B] \right\} d^4x, \tag{15}
\]

The \( r \) in the argument for \( \varepsilon \) and \( \kappa \) has been replaced by \( r = \sqrt{r^2} \). This is the condition for rotational symmetry; spatial dependence must be of the distance from the origin \( r \)
only. The integrand in (15) is the Lagrangian density \( \mathcal{L} \). Variation of the action gives the Maxwell equations

\[
\varepsilon_0 \nabla . [\varepsilon \mathbf{E}] = 0, \tag{16}
\]
\[
\kappa_0 \nabla \times [\kappa \mathbf{B}] = \varepsilon_0 \varepsilon \partial_t \mathbf{E}, \tag{17}
\]
such that variation of the scalar potential \( \phi \) gives (16), and variation of the vector potential \( \mathbf{A} \) gives (17). I shall demonstrate this process below.

Before I do this note that even if the expansions of \( \varepsilon \) and \( \kappa \) (13), (14), did contain odd order time derivatives, they would not contribute to the variation in the action. Consider a term of the form \( \mathbf{E} . \partial_t^{2n+1} \mathbf{E} \). Upon first order variation \( \delta \left( \mathbf{E} . \partial_t^{2n+1} \mathbf{E} \right) = \left( \delta \mathbf{E} \right) . \partial_t^{2n+1} \mathbf{E} + \mathbf{E} . \partial_t^{2n+1} \delta \mathbf{E} \) which equals zero upon integration by parts, ignoring surface terms. This shows that loss cannot be incorporated into an action of the form (15) since lossy media would have an \( \varepsilon \) and \( \kappa \) with odd order derivatives.

1.6 Derivation of Maxwell’s equations from action

1.6.1 Derivation of \( \nabla \cdot \varepsilon \mathbf{E} = 0 \)

If we let \( \phi \to \phi + \delta \phi \) in the action (15) such that \( \delta \phi = 0 \) at \( \pm \infty \), then

\[
\delta S = \frac{\varepsilon_0}{2} \int d^4x \left[ -\nabla \phi . \varepsilon (\nabla \delta \phi) - \nabla \delta \phi . \varepsilon (\nabla \phi) - \partial_t \mathbf{A} . \varepsilon (\nabla \delta \phi) - \nabla \delta \phi . \varepsilon (\partial_t \mathbf{A}) \right]
\]
to first order. Since \( \varepsilon \) and \( \kappa \) are even series, then integration by parts (ignoring surface terms) gives

\[
\delta S = \varepsilon_0 \int d^4x \left( -\nabla^i \varepsilon \nabla_i \phi - \nabla^i \varepsilon \partial_i A^i \right) \delta \phi.
\]

To find the equations of motion, we should follow the principle of stationary action and insist that \( \delta S = 0 \) for all \( \delta \phi \). Hence, the integrand should equal zero and we have that

\[
\varepsilon_0 \nabla \cdot \varepsilon \mathbf{E} = 0 = \varepsilon_0 \nabla \cdot \mathbf{D}
\]

1.6.2 Derivation of \( \varepsilon_0 \varepsilon \partial_t \mathbf{E} = \kappa_0 (\nabla \times (\kappa \mathbf{B})) \)

If we let \( \mathbf{A} \to \mathbf{A} + \delta \mathbf{A} \) in the action (15) such that \( \delta \mathbf{A} = 0 \) at \( \pm \infty \), then

\[
\delta S = \frac{1}{2} \int d^4x \left\{ \varepsilon_0 \left[ -\nabla^i \varepsilon (\nabla \delta A_i) + \partial_i A^i \varepsilon (\nabla \delta A_i) - \partial_i A^i \varepsilon E_i \right] - \kappa_0 \left[ \epsilon^{ijk} \nabla_j A_k \kappa (\epsilon_{ilm} \nabla^l \delta A^m) + \epsilon^{ijk} \nabla_j \delta A_k \kappa (\epsilon_{ilm} \nabla^l A^m) \right] \right\}
\]

Integration by parts as before gives

\[
\delta S = \int d^4x \left[ \varepsilon_0 \partial_t \varepsilon E_i \delta A^i - \kappa_0 (\nabla \times \kappa \mathbf{B})_i \delta A^i \right].
\]

Using the principle of stationary action then gives

\[
\varepsilon_0 \varepsilon \partial_t \mathbf{E} = \kappa_0 (\nabla \times (\kappa \mathbf{B})).
\]
2 Derivation of conserved quantities

Let us apply an active spatial rotation to the dynamical fields \( A^\mu \) such that

\[
\phi \rightarrow \phi(x - \delta x)
\]

\[
A^i \rightarrow A^i(x - \delta x) + \Omega^i_j A^j(x - \delta x)
\]

where \( \Omega_{ij} \) represents an antisymmetric rotation matrix of the form

\[
\begin{pmatrix}
0 & a & b \\
-a & 0 & c \\
-b & -c & 0
\end{pmatrix}
\]

such that

\[
\delta x_i = \Omega_{ij} x^j.
\]

An active rotation in this context is not just a co-ordinate transformation, but a real rotation of the fields relative to the media. Applying a Taylor expansion to the transformed fields gives, to first order in \( \Omega \):

\[
\delta \phi = -\Omega_{ij} x^j \nabla^i \phi
\]

\[
\delta A_k = \Omega_{kj} A^j - \Omega_{il} x^l \nabla^i A_k
\]

Insertion of these field variations into the first order variation in the action (15) gives

\[
\delta S = \int \frac{\varepsilon_0}{2} \left( \nabla_k \Omega_{ij} x^j \nabla^i \phi + \nabla_k \phi \nabla^i \phi + \nabla_k \phi \nabla^i \Omega_{ij} x^j - \Omega_{ij} \partial_i A^j - \partial_i \Omega_{kj} A^j + \partial_i \Omega_{il} x^l \nabla^i A_k + \Omega_{il} x^l \nabla^i \partial_i A_k \right) \varepsilon E^k
\]

\[
+ \frac{1}{\kappa_0} \epsilon_{kml} \left( \nabla^m \Omega_{ij} A_j - \nabla^m \Omega_{ij} x_j \nabla_i A^j - \Omega_{ij} x_j \nabla^m A^j \right) \kappa B^k
\]

\[
- \frac{\kappa_0}{2} \epsilon_{kml} \left[ \nabla^m \Omega_{ij} A_j + \Omega_{ij} \nabla^m A_j - \nabla^m \Omega_{ij} x_j \nabla_i A^j - \Omega_{ij} x_j \nabla^m A^j \right]
\]

(19)

where \( \epsilon_{kml} \) is the completely antisymmetric, 3-dimensional Levi-Civita tensor. All derivatives in (19) and indeed in this report act only on the quantity immediately to their right, unless brackets indicate otherwise. This lengthy and unenlightening equation has been included only to demonstrate the nature of the calculation, and to enable me to explain the next few steps.

The task is to rearrange \( \delta S \) into the form (1), or more explicitly,

\[
\delta S = -\frac{1}{2} \int \left( \tilde{L}^{ij} \partial_i \Omega_{ij} + \tilde{M}^{kij} \nabla_k \Omega \right) d^4 x.
\]

(20)

Note that the constant factor of \( \frac{1}{2} \) is present to ensure agreement with the conserved quantities in the vacuum case later on in the calculation. Hence, the objective in this
calculation is to arrange all terms such that they are proportional to first order derivatives of \( \Omega \). Then Noether’s theorem says that we can extract the angular momentum density \( \tilde{L}^{ij} \), and angular momentum flux \( \tilde{M}^{kij} \), and that the conservation law

\[
\partial_t \tilde{L}^{ij} + \nabla_k \tilde{M}^{kij} = 0
\]

is ensured.

2.1 Terms containing \( \partial_\mu \Omega \) in \( \delta S \)

Terms in the integrand of the variation in the action (19) that contain first order derivatives of \( \Omega \) without \( \varepsilon \) or \( \kappa \) acting on them are left alone, since they are in the desired form. Terms of the form \( f\varepsilon(\partial_\mu \Omega) \), \( f\kappa(\partial_\mu \Omega) \) are integrated by parts into the form \( g\partial_\mu \varepsilon(f) \), \( g\partial_\mu \kappa(f) \) with no overall sign change, ignoring surface terms. There is no resultant sign change after these integrations by parts since \( \varepsilon \) and \( \kappa \) are sums of even order derivatives. The major difficulty lies in dealing with terms in the integrand of the form

\[
f\varepsilon(\Omega g), \quad f\kappa(\Omega g).
\]

These can be rearranged by use of an integral identity, holding for arbitrary, well behaved functions \( f \) and \( g \),

\[
\int f\varepsilon(\Omega g) d^4x = \int (f\Omega\varepsilon g) - \sum_{n=1}^{\infty} \sum_{m=1}^{2n} (-1)^{n+m} \varepsilon g \partial_t^{-1} f \partial_t^{2n-m} g \partial_t \Omega) d^4x,
\]

with an analogous identity holding for \( \kappa \) in place of \( \varepsilon \) [6]. An ‘almost proof’ is attached in an appendix.

2.2 Terms containing \( \Omega \) in \( \delta S \)

Some of the terms proportional to \( \Omega \) are of the form \( \Omega_{ij} x^j \nabla^i \mathcal{L} \), which become, upon integration by parts \( x^j \mathcal{L} g^{ik} \nabla_k \Omega_{ij} \), ignoring surface terms. \( \mathcal{L} \) is the Lagrangian density from the action (15). The only terms left are rearranged by lengthy calculation. Use is made of the identities

\[
2\nabla^{[i} A^{m]} = \epsilon^{jim}(\nabla \times A)_j = \epsilon^{jim} B_j,
\]

\[
\epsilon_{mlk} \epsilon^{mij} = \delta_l^i \delta_k^j - \delta_l^j \delta_k^i,
\]

as well as integration by parts and the identity above (22), to turn such terms into infinite series proportional to \( \partial_\mu \Omega \). Equation (24) is known as the the Levi-Civita identity. Equations (23) and (24) are mentioned as they are used frequently throughout the project.
3 Resultant conserved quantities

The integrand is now of the desired form (20) and we identify

\[ -\tilde{L}^{ij} = -2\varepsilon_0\varepsilon E^i [A^j] + 2\varepsilon_0\varepsilon E^k \nabla^k [i A_j x^j] \]

\[ + x^j \left[ \varepsilon_0 \sum_{n=1,m=1}^{2n} (-1)^{n+m}\varepsilon_2 n \partial_t^{m-1} E^{[i} \partial_t^{2n-m} \nabla^j] E_k - \kappa_0 \sum_{n=1,m=1}^{2n} (-1)^{n+m}\kappa_2 n \partial_t^{m-1} B^{[i} \partial_t^{2n-m} \nabla^j] B_k \right] \]

\[ + \left[ \varepsilon_0 \sum_{n=1,m=1}^{2n} (-1)^{n+m}\varepsilon_2 n \partial_t^{m-1} E^{[i} \partial_t^{2n-m} E^j] - \kappa_0 \sum_{n=1,m=1}^{2n} (-1)^{n+m}\kappa_2 n \partial_t^{m-1} B^{[i} \partial_t^{2n-m} B^j] \right] \]

(25)

\[ -\tilde{M}^{kij} = 2\mathcal{L}_{k[i} x^{j]} + 2\varepsilon_0 \nabla^i [\phi x^j] \varepsilon E^k - 2\kappa_0 (\varepsilon^{mk[i} A^j] - \varepsilon^{mk} \nabla^i [A_i(x^j)] \kappa B_m, \] (26)

The straight brackets around an index in (25) indicate that we are not to take the antisymmetric parts with respect to that index. It is possible to show, making use of Maxwell’s equations in the absence of free charges and currents that

\[ \nabla_k \tilde{M}^{kij} + \partial_t \tilde{L}^{ij} = x^j \left\{ \varepsilon_0 \left[ \nabla^i (r, i \partial_t) \right] E^m \right\} E_m - \kappa_0 \left\{ \left[ \nabla^i \kappa (r, i \partial_t) B^m \right] B_m \right\}. \] (27)

It is clear that (27) is a conservation law if \( \varepsilon \) and \( \kappa \) have no spatial dependency. When \( \varepsilon \) and \( \kappa \) are spatially dependent only on the distance from the origin, then a conservation law (21) can be shown to hold. If we consider the combination \( x^j \nabla^i \varepsilon \) in spherical polar co-ordinates, then \( x^j \) is the vector \( \{ r, 0, 0 \} \). Since \( \varepsilon \) and \( \kappa \) depend only on \( r \), then \( \nabla \varepsilon \) and \( \nabla \kappa \) are themselves vectors proportional to the radial unit vector. Hence, \( x^j \nabla^i \varepsilon \), which is proportional to \( (x \times \nabla \varepsilon) \), becomes zero. This is Noether’s theorem in action; symmetries imply conservation laws.

3.1 Gauge invariance

\( \tilde{L}^{ij} \) and \( \tilde{M}^{ijk} \) are dynamically equivalent to gauge invariant quantities since they fail to be gauge invariant up to terms that can be shown to identically satisfy a conservation law. Consider the quantities

\[ F^k_{ij} := 2\varepsilon_0\varepsilon E^k A_{i[j}, \]

and

\[ G^{km}_{ij} := 2\kappa_0 A_{i[j} \kappa (\nabla^k A^m - \nabla^m A^k). \]

They can be shown to identically satisfy the conservation law,

\[ \partial_t \nabla_k F^k_{ij} + \nabla_k (-\partial_t F^k_{ij} + \nabla_m G^{km}_{ij}) = 0. \]
The quantities $F_{ij}^{k}$ and $G_{ij}^{km}$ are chosen by educated guesswork. The aim is to change terms in $\tilde{L}_{ij}$ and $\tilde{M}_{kij}$ involving $\nabla_i \phi$ or $\partial_t A_i$ into terms involving $E_i$; terms involving $\nabla_i A_i$ into terms involving $B_i$. Hence $\tilde{L}_{ij}$ and $\tilde{M}_{kij}$ have become the gauge invariant quantities $L_{ij}$, $M_{kij}$ according to

$$L_{ij} = \tilde{L}_{ij} + \nabla_k F_{ij}^{k},$$

and

$$M_{kj}^{k} = \tilde{M}_{kij} - \partial_t F_{ij}^{k} + \nabla_m G_{ij}^{km}.$$ 

A calculation using the identities (23), (24) and Maxwell’s equations give the gauge invariant angular momentum density and associated flux in dispersive, lossless media as

$$L_{ij} = \varepsilon_0 \varepsilon_{ij} \left[ x \times (\varepsilon E \times B) \right]^{mn}$$

$$- x_{ij} \left[ \varepsilon_0 \sum_{n=1}^{\infty} \sum_{m=1}^{2n} (-1)^{n+m} \varepsilon_{2n} \partial_t^{m-1} E_k \partial_t^{2n-m} \nabla_j \nabla_i E_k - \kappa_0 \sum_{n=1}^{\infty} \sum_{m=1}^{2n} (-1)^{n+m} \kappa_{2n} \partial_t^{m-1} B_k \partial_t^{2n-m} \nabla_i B_k \right]$$

and

$$M_{kj}^{k} = - (\varepsilon_0 E_m \varepsilon E^m + \kappa_0 B_m \kappa B^m) \delta_{[i}^{k} x_{j]} + 2 \varepsilon_0 \varepsilon E_k \partial_t x_{i} + 2 \kappa_0 B_k \kappa B_{i} x_j,$$ 

(28)

respectively.

### 3.2 Dualisation

The angular momentum density $L_{ij}$, is a 2nd rank tensor and is antisymmetric in $ij$ which, in our case of angular momentum, amounts to saying that

$$x \times p = -p \times x$$

and (clearly)

$$|x \times p| = |p \times x|.$$ 

Regardless, the $ij$ and $ji$ components are not independent, they have the same modulus. So we are being a little inefficient with our labelling; $L_{ij}$ only really contains 3 independent components. It is possible to create a 1$^{st}$ rank tensor $L^p$, from the 2$^{nd}$ rank tensor $L_{ij}$. We can use the same logic and mathematics to create a 2$^{nd}$ rank tensor $M^{kp}$, from the third rank tensor $M_{ij}^{k}$,

$$L^p = \frac{\epsilon^{p} \epsilon^{ij}}{2} L_{ij}, \
M^{kp} = \frac{\epsilon^{p} \epsilon^{ij}}{2} M_{ij}^{k}.$$  

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Use is also made of the fact that
\[ \delta^i_1 = 3 \]
since \( \delta^i_1 \) is the trace of the \( 3 \times 3 \) identity matrix. In explicit form, the gauge invariant, dualised forms of angular momentum density and associated flux in dispersive, lossless media are

\[
L^p = \varepsilon_0 |\mathbf{x} \times (\varepsilon \mathbf{E} \times \mathbf{B})|^p \\
+ \frac{1}{2} \varepsilon^{pji} x_i \left[ \varepsilon_0 \sum_{n=1}^{\infty} \sum_{m=1}^{2n} (-1)^{n+m} \varepsilon_2 n \partial_t^{n-1} \mathbf{E}^k \partial_t^{2n-m} \nabla_j \mathbf{E}_k - \kappa_0 \sum_{n=1}^{\infty} \sum_{m=1}^{2n} (-1)^{n+m} \kappa_2 n \partial_t^{m-1} \mathbf{B}^k \partial_t^{2n-m} \nabla_j \mathbf{B}_k \right] \\
+ \frac{1}{2} \left[ \varepsilon_0 \sum_{n=1}^{\infty} \sum_{m=1}^{2n} (-1)^{n+m} \varepsilon_2 n \partial_t^{m-1} \mathbf{E} \times \partial_t^{2n-m} \mathbf{E}^p - \kappa_0 \sum_{n=1}^{\infty} \sum_{m=1}^{2n} (-1)^{n+m} \kappa_2 n \partial_t^{m-1} \mathbf{B} \times \partial_t^{2n-m} \mathbf{B}^p \right],
\] (30)

and

\[
M^{kp} = \frac{1}{2} \varepsilon^{pji} x_i [\varepsilon_0 \varepsilon \mathbf{E} + \kappa_0 \mathbf{B} \times \mathbf{B}] - \varepsilon_0 \varepsilon \mathbf{E} \times (\mathbf{x} \times \mathbf{E})^p - \kappa_0 \mathbf{B} \times (\mathbf{x} \times \mathbf{B})^p
\] (31)

respectively. The square brackets are no longer needed around \( ij \) since the Levi-Civita tensor is itself antisymmetric in \( ij \).

### 3.3 Consistency in vacuo

In vacuo where \( \varepsilon = \kappa = 1 \), the infinite series disappear and \( L^p \) reduces to

\[
L^p|_{\varepsilon,\kappa=1} = \varepsilon_0 |\mathbf{x} \times (\mathbf{E} \times \mathbf{B})|^p,
\] (32)

Hence the angular momentum density in vacuo can be written as

\[
L^p|_{\varepsilon,\kappa=1} = \varepsilon^{pji} x_i g_j = \mathbf{x} \times \mathbf{g} = (\mathbf{x} \times \mathbf{P}|_{\varepsilon,\kappa=1})
\]

where \( \mathbf{g} \) is the vacuum momentum density (4) and \( \mathbf{P}|_{\varepsilon,\kappa=1} \) is the momentum density in dispersive, lossless media (11) (for the special case of a vacuum). The angular momentum density has reduced in vacuo to the expected expression, i.e. the vector product of position and momentum density.

The vacuum form of the angular momentum flux, \( M^{kp} \) is

\[
M^{kp}|_{\varepsilon,\kappa=1} = \frac{1}{2} \varepsilon^{pji} x_j \delta^i_k (\varepsilon_0 \varepsilon E^2 + \kappa_0 B^2) - \varepsilon_0 \varepsilon E^k (\mathbf{x} \times \mathbf{E})^p - \kappa_0 \mathbf{B} \times (\mathbf{x} \times \mathbf{B})^p
\] (33)

This can then be written as

\[
M^{kp}|_{\varepsilon,\kappa=1} = \varepsilon^{pji} x_j (T^k_j) = \varepsilon^{pji} x_j \sigma^k_j|_{\varepsilon,\kappa=1}
\]

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where $T^k_j$ is the Maxwell stress tensor (5), akin to a momentum flux. The quantity $\sigma^k_j|_{\varepsilon,\kappa=1}$ is the momentum flux in dispersive, lossless media (12) for the special case of a vacuum. Now, $\epsilon^{pji}x_i T^k_j$ and $\epsilon^{pji}x_i \sigma^k_j|_{\varepsilon,\kappa=1}$ can be written as
\[-(T^k \times x)^p, \quad (-\sigma^k|_{\varepsilon,\kappa=1} \times x)^p\]
respectively. Thus
\[M^{kp}|_{\varepsilon,\kappa=1} = -(T^k \times x)^p = -(-\sigma^k|_{\varepsilon,\kappa=1} \times x)^p.\]
The double underlining indicates the matrix representing the 2nd rank tensor $T^k_j$. For example, $(T^k \times x)^m$ is the $m^{th}$ component of the cross product over the 2\textsuperscript{nd} index of the ‘matrix’ $T^k_j$ with position. It can be thought of as the cross product over all columns $j$, for a particular row $k$, of the ‘matrix’ $T^k_j$. The angular momentum flux has reduced in vacuo to the expected expression, i.e. the vector product of position and momentum flux.
4 Interpretation

The vacuum angular momentum tensor \( \mathcal{A}^{\alpha\beta\gamma} \) is 4-dimensional in all three of its indices. So the question arises, which components of an angular momentum tensor in media \( M \), are the (pre-dualisation) quantities \( L^{ij} \) and \( M^{kij} \)? The angular momentum flux \( M^{kij} \) has the three spatial indices \( ijk \). So it is clear that \( M^{kij} \) should be identified as the space-space-space component of \( M \) up to a constant factor \( K \).

The space-space-space component of the angular momentum tensor in vacuum, (7) is

\[
\mathcal{A}^{kij} = 2\Theta [i x j] = T^{kij} - T^{kji}x^i
\]

where \( T^{kij} \) is the Maxwell stress tensor (5). When evaluated this is

\[
\frac{1}{2}g^{kij}(\varepsilon_0 E^2 + \kappa_0 B^2) - \varepsilon_0 E^k E^{[i}x^{j]} - \kappa_0 B^k B^{[i}x^{j]} = -M^{kij}|_{\varepsilon,\kappa=1},
\]

where \( M^{kij}|_{\varepsilon,\kappa=1} \) is the media angular momentum flux evaluated in vacuo. We take the constant of proportionality \( K \) to be \(-1\).

The angular momentum density \( L^{ij} \) has two spatial indices, \( ij \). The time-space-space component of the angular momentum tensor in vacuum, (7) is

\[
\mathcal{A}^{0ij} = 2\Theta^{0[i x j]} = cg^{i}x^{j} - cg^{j}x^{i}
\]

where \( g_j \) is the momentum density in vacuum, (4). \( \mathcal{A}^{0ij} \) becomes

\[
\mathcal{A}^{0ij} = \epsilon^{0ij}(cg \times x)_p = (-c)L^{ij}|_{\varepsilon,\kappa=1}
\]

where \( L^{ij}|_{\varepsilon,\kappa=1} \) is the media angular momentum density evaluated in vacuo. Hence, we should take this hint from the vacuum and equate the angular momentum density in media with the time-space-space component of the angular momentum tensor in media \( M \), with a constant numerical factor.

\[
(-c)L^{ij} = M^{0ij}.
\]

Thus far, we have a schematic matrix form for \( M^{\mu\lambda\nu} \) of

\[
\begin{pmatrix}
M^{00j} =? & M^{0ij} = (-c)L^{ij} \\
M^{i0j} =? & M^{kij} = -M^{kij}
\end{pmatrix}
\]

It turns out that there are no \( M^{00j} \) or \( M^{i0j} \) components. The 2\(^{nd}\) and 3\(^{rd}\) indices refer to a rotation; \( M \) is antisymmetric in those indices. Hence, \( M^{00j} \) and \( M^{i0j} \) would (if they existed) be derived from invariance in the action under \([0j]\) rotations, i.e time-space rotations. A time-space rotation is a Lorentz Boost [7]. In contrast to a vacuum, media have no Lorentz Boost symmetry. This is because a medium has a ‘preferred frame.’ All
inertial frames in a vacuum are equivalent whereas all inertial frames are not equivalent in a medium. In media you can tell that you are moving by ‘looking out of the car window.’ The non-equivalence of inertial frames implies non-invariance under Lorentz Boosts, therefore there are no $M^{(0)i}$ or $M^{(ij)}$ components. We only have the $M^{\mu ij}$ components.

To summarise, Noether’s theorem provides a conserved current $J^\mu_A$ where $A$ are the parameters of a symmetry group. In our case of spatial rotations, the symmetry (in the action) group is the rotation group, $ij$. The zeroth component $J^0_A$ is the ‘Noether Charge'; in our case the angular momentum density. The $J^k_A$ components are the associated flux, in our case the angular momentum flux. Since our rotations were antisymmetric in $ij$ we could dualise to reduce the rank of the tensor. If we had been working in vacuum, we would have had Lorentz Boost symmetry, i.e. rotational symmetry over all space-time.

4.1 Interpretation in media

The vacuum angular momentum tensor, $A^{\alpha\beta\gamma}$ (6) is a nice cross product type relation of the energy-momentum tensor $\Theta^{\alpha\beta}$ with position. This poses the question, what is the relationship between the angular momentum and the energy-momentum in media? It can be shown that the angular momentum density (30) is related to the momentum density (11), according to

$$L^p = \varepsilon^{pji} x_j P_i + \mathcal{F}^p,$$

where

$$\mathcal{F}^p = \frac{1}{2} \varepsilon_0 \sum_{n=1}^{\infty} \sum_{m=1}^{2n} (-1)^{n+m} \varepsilon_{2n} (\partial_l^{m-1} E \times \partial_l^{2n-m} E)^p$$

$$- \frac{1}{2} \kappa_0 \sum_{n=1}^{\infty} \sum_{m=1}^{2n} (-1)^{n+m} \kappa_{2n} (\partial_l^{m-1} B \times \partial_l^{2n-m} B)^p.$$

However, the angular momentum flux is related to the momentum flux (12), according to

$$M^{kp} = \varepsilon^{pji} x_j \sigma^k = (\mathbf{x} \times \mathbf{\sigma})^p.$$

Whilst the angular momentum density has not reduced to a cross product type expression, the angular momentum flux has. In media, the angular momentum flux is just the cross product of the momentum flux with position.

This is categorical evidence Noether’s theorem had to be used to derive the conserved quantities. Had we guessed that vector-product type expressions for angular momentum were correct, we would have one correct answer; the flux, and one incorrect answer; the density. The ability to write the vacuum angular momentum tensor in the form (6) is reliant on the symmetry of the vacuum energy-momentum tensor (3)-(5). However, the energy-momentum tensor in media is not symmetric (9)-(12), so it should not be a surprise that we do not get such a nice formula. We get some insight into the meaning of $\mathcal{F}^p$ when considering monochromatic waves.
5 Special case: time-averaged monochromatic waves

A monochromatic wave has an electric field of the form
\[ E(r, t) = \frac{1}{2} [E_0(r)e^{-i\omega_0 t} + E^*_0(r, t)e^{i\omega_0 t}] \] (35)
and a magnetic field of the form
\[ B(r, t) = \frac{1}{2} [B_0(r)e^{-i\omega_0 t} + B^*_0(r, t)e^{i\omega_0 t}], \] (36)
where a superscript * indicates the complex conjugate. These fields are substituted into the angular momentum density (30) and a time average is taken. The oscillatory terms vanish upon time-averaging. After a significant calculation, again using (23), (24) and application of the infinite expansion for \( \varepsilon \) (13), the time averaged angular momentum density for monochromatic fields is
\[ L_{\text{mono}}^p = \varepsilon_0 \varepsilon(r, \omega_0) 2 \left\{ x \times \left[ \text{Re}(E_0 \times B^*_0) \right] \right\}^p 
- \frac{\varepsilon_0}{4} \frac{d\varepsilon(r, \omega_0)}{d\omega_0} \left\{ \varepsilon^{pji} x_i [\text{Im}(E_0^k \nabla_j E^*_0)] + \text{Im}(E_0 \times E^*_0)^p \right\} 
+ \frac{\varepsilon_0}{4} \frac{d\varepsilon(r, \omega_0)}{d\omega_0} \left\{ \varepsilon^{pji} x_i [\text{Im}(B_0^k \nabla_j B^*_0)] + \text{Im}(B_0 \times B^*_0)^p \right\}, \] (37)
where \( \varepsilon(r, \omega_0), \kappa(r, \omega_0) \) are the implicit infinite series
\[ \sum_{n=0}^{\infty} \varepsilon_{2n}(r)\omega_0^{2n}, \quad \sum_{n=0}^{\infty} \kappa_{2n}(r)\omega_0^{2n} \]
respectively. \( L_{\text{mono}}^p \) is a real quantity with the bar over \( L_{\text{mono}}^p \) signifying a time average. If \( z = x + iy \), I am using the convention that \( \text{Im}(z) = y \).

Equation (37) shows that even for time-averaged monochromatic waves, we still get a dispersive contribution to the angular momentum density. This dispersive contribution has two components. The first is of the form of a cross product of position with terms from the time averaged momentum density for monochromatic waves; these are the \( \varepsilon^{pji} x_i [\text{Im}(E_0^k \nabla_j E^*_0)] \) and \( \varepsilon^{pji} x_i [\text{Im}(B_0^k \nabla_j B^*_0)] \) terms. Although, for a proper comparison with the momentum case, we should be working in the realm of homogeneous media, since that is where momentum is conserved. The second dispersive contribution, i.e the terms of the form \( \text{Im}(E_0 \times E^*_0)^p \) and \( \text{Im}(B_0 \times B^*_0)^p \), are a little unexpected. They may be related to spin angular momentum; more to follow.

5.1 Monochromatic, plane waves in homogeneous media

This subsection (5.1) draws on research from L.D. Landau and E.M. Lifshitz’s The Classical Theory of Fields [7].

In a homogeneous medium, a single plane wave electric field is of the form \( \mathbf{E}(r, t) = \text{Re}(\mathbf{E}_0 e^{i(kr - \omega t)}) \). The magnetic field is of the same form. The frequency \( \omega \) and \( k \) are related by
where $n = \sqrt{\varepsilon \mu}$ is the refractive index of the medium. So, the value of $\omega$ only fixes the magnitude of $k$. If we write the argument of the complex vector $\vec{E}_0$ as $-2\alpha$ then the vector $b$, defined according to

$$\vec{E}_0 = be^{-i\alpha},$$

has a real square. This criterion imposes, by a simple calculation, that the real and imaginary components of $b$ are orthogonal. Let’s call these $b_1$ and $b_2$ respectively. Our plane wave is now of the form

$$\mathbf{E}(r, t) = \text{Re}(be^{i(k\cdot r - \omega t - \alpha)}).$$

Since plane waves are transverse we can choose that the wave propagates along the $x$-axis and that - without loss of generality - $b_1$ is parallel to the $y$-axis and $b_2$ is parallel to the $z$-axis. We now have electric fields according to

$$E_y = b_1 \cos(\omega t - k\cdot r + \alpha), \quad E_z = \pm b_2 \sin(\omega t - k\cdot r + \alpha).$$

The plus sign corresponds to taking $b_2$ positive along the positive $z$-axis. It is clear that

$$\frac{E_y^2}{b_1^2} + \frac{E_z^2}{b_2^2} = 1$$

and we have elliptical polarisation in the plane perpendicular to the direction of propagation of the wave, namely $x$. This means that the endpoint of the rotating electric field vector describes an ellipse in the $yz$ plane. So, monochromatic plane waves are elliptically polarised. If $b_1 = b_2$ then we have circular polarisation. If $b_1 = 0$ or $b_2 = 0$ then we have linear polarisation. Linear and circular polarisation are special cases of elliptical polarisation. We can run through the same process for $\mathbf{B}$.

### 5.2 Angular momentum density in homogeneous media

Let’s work in homogeneous media from now on. Assume a single plane wave and the spatial dependence of $\mathbf{E}_0(\mathbf{r})$, $\mathbf{B}_0(\mathbf{r})$ will become $\vec{E}_0 e^{i\mathbf{k}\cdot \mathbf{r}}$, $\vec{B}_0 e^{i\mathbf{k}\cdot \mathbf{r}}$ respectively. Also, $\kappa$ and $\varepsilon$ will no longer have any spatial dependence. The time-averaged angular momentum density for monochromatic waves in homogeneous media (37) then becomes by simple calculation

$$\overline{\mathcal{L}^p_{\text{mono,homog}}} = \frac{\varepsilon_0 \varepsilon(\omega_0)}{2} \left\{ \mathbf{x} \times \left[ \text{Re}(\vec{E}_0 \times \vec{E}_0^*) \right]^p \right\} - \frac{\varepsilon_0}{4} \frac{d \varepsilon(\omega_0)}{d\omega_0} \left\{ (\mathbf{k} \times \mathbf{x})^p |\vec{E}_0|^2 + \text{Im}(\vec{E}_0^* \times \vec{E}_0)^p \right\} + \frac{\kappa_0}{4} \frac{d \kappa(\omega_0)}{d\omega_0} \left\{ (\mathbf{k} \times \mathbf{x})^p |\vec{B}_0|^2 + \text{Im}(\vec{B}_0^* \times \vec{B}_0)^p \right\}. \quad (39)$$
I should make clear the distinction between the terms \( \{ \mathbf{x} \times [\Re(\mathbf{E}_0 \times \mathbf{B}_0^*)] \}^p \), \( \mathbf{x} \times k \Re(\mathbf{E}_0 \times \mathbf{B}_0^*) \), \( |\mathbf{E}_0|^2 \) and the unexpected dispersive terms, \( \Im(\mathbf{E}_0 \times \mathbf{E}_0^*)^p \) and \( \Im(\mathbf{B}_0 \times \mathbf{B}_0^*)^p \). It is simple to see that the former terms cannot contribute any angular momentum in the direction of propagation, given by \( k \). The quantity \( \mathbf{E}_0 \times \mathbf{E}_0^* \) is parallel to \( k \) since for plane waves \( \mathbf{E} \hat{k} = 0 \) and \( \mathbf{B} = \frac{1}{c} \mathbf{k} \times \mathbf{E} \). Hence the cross product of \( \mathbf{E}_0 \times \mathbf{E}_0^* \) with \( \mathbf{x} \) will give zero contribution in the \( k \) direction. So, any contribution to angular momentum in the direction of propagation will have to come from the unexpected terms. Before proceeding, I should make a comment on this axial component of angular momentum. It is intrinsic in the sense that it arises from the elliptical polarisation, i.e. the nature of the wave itself; the ‘corkscrewing’ motion. Any components of angular momentum in other directions depend on the location of the origin and hence are of less importance.

By analogy with the above discussion, taking the direction of propagation to be parallel to the \( x \)-axis,

\[
\Im(\mathbf{E}_0 \times \mathbf{E}_0^*)^p = \Im \left[ (b_1 + i b_2) \times (b_1 - i b_2) \right]^p
\]

which after a small calculation gives

\[
\Im(\mathbf{E}_0 \times \mathbf{E}_0^*)^p = -2\{b_1 b_2, 0, 0\}
\]

in \( xyz \) co-ordinates. There will be a similar contribution from the \( \Im(\mathbf{B}_0 \times \mathbf{B}_0^*)^p \) term and hence our angular momentum density has a component in the direction of propagation of the form

\[
\left\{ \frac{\varepsilon_0 b_1 b_2}{2} \left[ \frac{d\varepsilon(\omega_0)}{d\omega_0} - \frac{d\kappa(\omega_0)}{d\omega_0} \right], 0, 0 \right\}.
\]

(40)

Hence, for a specific \( k \), elliptically polarised monochromatic plane waves in dispersive, lossless and homogeneous media have a non-zero angular momentum component in the direction of propagation, upon time averaging, of the form (40). Equation (40) shows that this is not the case in vacuum since \( \frac{d\varepsilon(\omega_0)}{d\omega_0} \) and \( \frac{d\kappa(\omega_0)}{d\omega_0} \) vanish. These findings relate to a well-known paradox of axial angular momentum in vacuum [8]. One would expect that a circularly polarised plane wave should carry angular momentum in the direction of propagation, since the \( \mathbf{E} \) and \( \mathbf{B} \) fields are perpendicular to the direction of propagation and ‘corkscrewing’ around. But, as we can see from (40), and directly from consideration of (39) in vacuo, theory states that there is no component in the axial direction. This paradox is resolved by the realisation that theoretical plane waves have an infinite cross section, which is physically unrealisable. A calculation involving a finite cross section shows that in fact, circularly polarised light in a vacuum does carry spin angular momentum in the direction of propagation of \( \pm \hbar \) per photon [9].

The axial angular momentum associated with circularly polarised light in vacuo is called spin angular momentum [8]. Returning to the media case, if \( b_1 = b_2 \), we have circularly polarised light and an axial contribution to the angular momentum even for an idealised infinite plane wave. This evidence suggests that we have extra spin angular
momentum in media due to dispersion. This extra spin angular momentum comes out directly from the terms $Im(\vec{E}_0 \times \vec{E}_0^*)^p$ and $Im(\vec{B}_0 \times \vec{B}_0^*)^p$, without us having to perform more complex calculations to compensate for the unrealistic nature of perfect plane waves. This spin term is a consequence of dispersion, since the spin term is zero when there is no dispersion, i.e. $\varepsilon$ and $\kappa$ are constant.

6 Conclusion

I have found the conserved components of the angular momentum tensor in lossless, dispersive, rotationally symmetric and not necessarily homogeneous media. Both the angular momentum density and flux reduce to the expected cross product relation of the energy-momentum tensor with position in vacuo. In media the relation between the angular momentum and momentum densities is non-trivial. However the angular momentum flux and momentum flux are related by the expected cross product type formula.

As a special case, monochromatic, time averaged waves were investigated for both inhomogeneous and homogenous media. It was found that neither of these reduce to a Minkowski type $[x \times (D \times B)]$ expression, but rather there are non-trivial dispersive contributions. In the homogeneous case, single elliptically polarised plane waves were shown to carry an intrinsic spin angular momentum. This showed itself as a dispersive contribution. In contrast, in vacuum a single elliptically polarised plane wave has no spin angular momentum.

6.1 Future work?

A sensible next step would be to try to interpret the meaning of the dispersive spin contribution to the angular momentum density. Another project would be to take solutions of Maxwell’s equations in rotationally symmetric media and calculate, using the derived expressions, some angular momentum densities.
Appendix I - an ‘almost proof’ of integral identity (22)

Here, I attempt an independently constructed proof of the integral identity that has been extremely useful in the calculation.

\[
\int f\epsilon(\Omega g)d^4x = \int (f\Omega g - \sum_{n=1}^{\infty} \sum_{m=1}^{2n} (-1)^{n+m}\epsilon f\partial_l^{2n-m}g\partial_l\Omega)d^4x
\]

Consider the left hand side (LHS) of the identity. It has \(\epsilon\) acting on a product. So, we need to know that

\[
\partial_l^{2n}(\Omega g) = \sum_{m=0}^{2n} \binom{2n}{m} \partial_l^m \Omega \partial_l^{2n-m}g,
\]

i.e differentials of products follow a binomial type expansion. Inserting this into the LHS gives

\[
\int f\epsilon(\Omega g)d^4x = \int f \sum_{n=0}^{\infty} \epsilon f\partial_l^{2n}g d^4x + \sum_{n=1}^{\infty} \sum_{m=1}^{2n} \epsilon f\partial_l^{2n-m}g d^4x.
\]

When \(m = 0\), all the derivatives act on \(g\). So

\[
\int f\epsilon(\Omega g)d^4x = \int f\Omega g d^4x + \sum_{n=1}^{\infty} \sum_{m=1}^{2n} \epsilon f\partial_l^{2n-m}g d^4x.
\]

The first term on the right hand side (RHS) is one to keep. It is the first term on the RHS of the identity we are trying to prove. Let’s now consider the second term on the RHS. If we integrate by parts \((m - 1)\) times and ignore surface terms we get

\[
- \int \sum_{n=1}^{\infty} \sum_{m=1}^{2n} \epsilon f\partial_l^{2n-m}g d^4x
\]

which becomes, upon expanding the differential of the product

\[
- \int \sum_{n=1}^{\infty} \sum_{m=1}^{2n} \sum_{p=0}^{m-1} \epsilon f\partial_l^{2n-1-p}g d^4x.
\]

Let’s now consider a fixed \(n\). Then we can try to arrange each term in the sum with respect to \(n\) into the appropriate form. Hence, we are just considering the expression

\[
- \int \sum_{m=1}^{2n} \sum_{p=0}^{m-1} (-1)^{n+m}\epsilon f\partial_l^{2n-1-p}g d^4x.
\]

We can rearrange the terms in this double series to be derivatives of \(f\) of order \(m - 1\). Since the \(p\)-series sums up to \(p = m - 1\), derivatives of \(f\) of order 0 exist for \(m = [1, 2n]\);
derivatives of \( f \) of order 1 exist for \( m = [2, 2n] \); …… derivatives of \( f \) of order \( (m' - 1) \) exist for \( m = [m', 2n] \).

So, if we want derivatives of \( f \) of order \( m - 1 \) for some given \( m \), then we have to take the \( p = m - 1 \) contributions from the \( p \)-series with maxima \( m - 1, m, m + 1, \ldots, 2n - 1, 2n \). These contributions will be

\[
- \sum_{k=m}^{2n} (-1)^{n+k} \binom{2n}{k} \left[ \sum_{p=0}^{k-1} \binom{k-1}{p} \partial^p_t f \partial^{2n-1-p}_t g \right]_{p=m-1}
\]

where \( k \) is a dummy counting index. Upon imposing \( p = m - 1 \) this becomes

\[
-(-1)^n \partial^{m-1}_t f \partial^{2n-m}_t g \sum_{k=m}^{2n} (-1)^k \binom{2n}{k} \binom{k-1}{m-1}.
\]

MAPLE software seems to suggest that the series

\[
\sum_{k=m}^{2n} (-1)^k \binom{2n}{k} \binom{k-1}{m-1} = (-1)^m,
\]

I have checked this using MAPLE for values of \( n \) and \( m \), large and small. I think that what is happening here is that we have an even row of Pascal’s triangle and are summing it’s elements. But we are summing the elements of the row with a \((-1)^k\) oscillating sign. Not only that, we are not summing from the first row element, hence there is cancellation of elements in the row. Since we are not summing from the first row element, the final element, namely 1, will not cancel with it’s partner on the opposite side of the row. This 1 then picks up the sign from the \((-1)^k\) and hence the series sums to \( 1(-1)^k = (-1)^k \). I have not been able to prove this by hand.

So, if we say that the checking of various examples on MAPLE is sufficient (hence the ‘almost proof’), we have that the RHS of the previous equation,

\[
-(-1)^n \partial^{m-1}_t f \partial^{2n-m}_t g \sum_{k=m}^{2n} (-1)^k \binom{2n}{k} \binom{k-1}{m-1} = -(-1)^{n+m} \partial^{m-1}_t f \partial^{2n-m}_t g.
\]

So, as we go through all \( m \) values for a fixed \( n \), with our terms collected such that derivatives of \( f \) are of order \( (m - 1) \), we have the series

\[
- \sum_{m=1}^{2n} (-1)^{n+m} \partial^{m-1}_t f \partial^{2n-m}_t g.
\]

Now that we have arranged each term in the series with respect to \( n \) into the desired form, the ‘almost proof’ is complete.
B Appendix II - Work not elaborated upon in report

I spent a substantial amount of time working on material upon which I have not elaborated in my report. Before calculating the conserved quantities associated with angular momentum, I ‘trained’ myself by performing the calculations necessary to reproduce the results for the energy-momentum tensor in media [6]. I have not included the details of these calculations. This is for the sake of both brevity and sanity (for the reader and author), but also because they follow a very similar pattern to the calculation for angular momentum. I shall however outline the starting point for these calculations.

B.1 Energy

To extract the conserved energy density and flux, we make an infinitesimal variation in the time dependency of the fields \( \phi \) and \( A \) of the form

\[
\phi(r, t) \to \phi(r, t + \eta(r, t)) \quad A(r, t) \to A(r, t + \eta(r, t)).
\]

Then, we Taylor expand to first order;

\[
\phi(r, t + \eta) = \phi(r, t) + \eta \partial_t \phi(r, t) \quad A(r, t + \eta) = A(r, t) + \eta \partial_t A(r, t)
\]

such that

\[
\delta \phi = \eta \partial_t \phi, \quad \delta A = \eta \partial_t A,
\]

where \( \eta \) has had it’s argument removed. These variations give us the variation in the action and then we rearrange by a similarly lengthy method as for angular momentum into the form

\[
\delta S = - \int d^4x [\tilde{\rho} \partial_t \eta + \tilde{S} \nabla \eta].
\]

We then identify \( \tilde{\rho} \) and \( \tilde{S} \) as the conserved energy density and flux, and proceed to make them gauge invariant, check conservation etc. We arrive at the results of [6].

B.2 Momentum

To extract the conserved energy density and flux, we make an infinitesimal variation in the time dependency of the fields \( \phi \) and \( A \) of the form

\[
\phi(r, t) \to \phi(r + \eta(r, t), t) \quad A(r, t) \to A(r + \eta(r, t), t).
\]

We also have to assume that the medium is homogeneous; since translational spatial invariance gives conservation of linear momentum by Noether’s theorem. This amounts to restricting \( \varepsilon \) and \( \kappa \) to have no spatial dependency.

Then, we Taylor expand to first order;

\[
\phi(r + \eta, t) \to \phi(r, t) + (\eta \nabla) \phi(r, t)
\]
\[ \mathbf{A}(r + \eta, t) \rightarrow \mathbf{A}(r, t) + (\eta \nabla)\mathbf{A}(r, t) \]

such that

\[ \delta\phi = (\eta \nabla)\phi, \quad \delta \mathbf{A} = (\eta \nabla)\mathbf{A}, \]

where \( \eta \) has had it’s argument removed. Once again we insert these into the variation of the action and rearrange into the form

\[ \delta S = - \int (\tilde{P}_i \partial_t \eta^i + \tilde{\sigma}^i_j \nabla_j \eta^i) \, d^4x. \]

We then identify \( \tilde{P}_i \) and \( \tilde{\sigma}^i_j \) as the conserved momentum density and flux, and proceed to make them gauge invariant, check conservation etc. We arrive at the results of [6].

\section*{C Appendix III - Record of work}

Here, I detail my progress throughout the project from my records.

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References


2. University of St Andrews History of Maths, (http://www-history.mcs.st-andrews.ac.uk/history/).


Acknowledgements

I would like very much to thank Tom Philbin for his - from my perspective - brilliant supervision during my project. Whenever I have had problems or queries he has not hesitated to meet with me and mull things over, or even over and over and over as was sometimes necessary.