On the number of $(r,r+1)$-factors in an $(r,r+1)$-factorization of a simple graph

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Abstract

For integers $d \geq 0$, $s \geq 0$, a $(d,d+s)$-*graph* is a graph in which the degrees of all the vertices lie in the set $\{d,d+1,\dots,d+s\}$. For an integer $r \geq 0$, an $(r,r+1)$-*factor* of a graph G is a spanning $(r,r+1)$-subgraph of G. An $(r,r+1)$-*factorization* of a graph G is the expression of G as the edge-disjoint union of $(r,r+1)$-factors.

For integers $r,s \geq 0$, $t \geq 1$, let $f(r,s,t)$ be the smallest integer such that, for each integer $d \geq f(r,s,t)$, each $(d,d+s)$-simple graph has an $(r,r+1)$-factorization with x $(r,r+1)$-factors for at least t different values of x. In this note we evaluate $f(r,s,t)$.

1 Introduction

For integers $d \geq 0$, $s \geq 0$, a $(d,d+s)$-*graph* is a graph in which the degrees of all the vertices lie in the set $\{d,d+1,\dots,d+s\}$. For an integer $r \geq 0$, an $(r,r+1)$-*factor* of a graph G is a spanning $(r,r+1)$-subgraph of G. An $(r,r+1)$-*factorization* of a graph G is the expression of G as the edge-disjoint union of $(r,r+1)$-factors.

A proper edge-colouring of a graph G corresponds to a $(0,1)$-factorization of the graph. Therefore, by Vizing's Theorem [11], a $(0,1)$-factorization of a simple graph G with x $(0,1)$-factors exists for each $x \geq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G (for large x, most of these $(0,1)$-factors would inevitably be the graph with vertex set $V(G)$ but with no edges). For $r \geq 1$, however, if a graph G has an $(r,r+1)$-factorization, then the number x of $(r,r+1)$-factors in the factorization is bounded above by $\lfloor \frac{1}{2}\Delta(G) \rfloor$. As an example of the type of phenomenon we are concerned with in this paper, consider a regular simple graph G of degree 29. All such graphs have $(2,3)$-factorizations into x $(2,3)$-factors for each $x \in \{10,11,12,13,14\}$, but for no other values of x.

For integers $r,s \geq 0$, $t \geq 1$, let $f(r,s,t)$ be the smallest integer such that, for each integer $d \geq f(r,s,t)$, each $(d,d+s)$-simple graph has an $(r,r+1)$-factorization into x $(r,r+1)$-factors for at least t different values of x. Although it is not obvious that the number $f(r,s,t)$ exists for all possible values of r,s,t, we shall in fact determine

$f(r,s,t)$ for all r,s,t. For the corresponding problem for pseudographs (where multiple edges and loops are permitted), the number corresponding to $f(r,s,t)$ does not always exist.

The remarks above about the correspondence between $(0,1)$-factorizations and proper edge-colourings of simple graphs lead quickly to the following conclusion.

Lemma 1. *For integers $s \geq 0$, $t \geq 1$, $f(0,s,t) = 0$.*

Proof. For any value of $d \geq 0$, by Vizing's Theorem, if $x \geq d+s+1$ then any $[d,d+s]$-simple graph has a proper edge-colouring with x colours. □

From now on we shall suppose that $r \geq 1$.

Most of the published results on this topic just consider the case when $t = 1$. The first noteworthy result was due to Era [5] and Egawa [4]. They proved in 1986:

Theorem 2. *For integers $r \geq 1$,*

$$f(r,0,1) = \begin{cases} r^2 & \text{if } r \text{ is even,} \\ r^2+1 & \text{if } r \text{ is odd.} \end{cases}$$

A much shorter proof of this was later provided by Hilton [7], who also proved:

Theorem 3. *For integers $r \geq 1$,*

$$f(r,1,1) = \begin{cases} r^2+r & \text{if } r \text{ is even,} \\ r^2+r+1 & \text{if } r \text{ is odd.} \end{cases}$$

Finally in 2005 in [9], Hilton and Wojciechowski evaluated $f(r,s,1)$:

Theorem 4. *For integers $r \geq 1$, $s \geq 0$,*

$$f(r,s,1) = \begin{cases} r^2+rs & \text{if } r \text{ is even and } 0 \leq s \leq 1, \\ r^2+rs+1 & \text{if } r \text{ is odd and } 0 \leq s \leq 1, \\ r^2+rs+r+1 & \text{if } s \geq 2. \end{cases}$$

Our main result here extends Theorem 4 to general values of t.

Theorem 5. *For integers $s \geq 0$, $r,t \geq 1$,*

$$f(r,s,t) = \begin{cases} tr^2+tr+sr-r & \text{if } r \text{ is even and } 0 \leq s \leq 1, \\ tr^2+tr+sr-r+1 & \text{if } r \text{ is odd and } 0 \leq s \leq 1, \\ tr^2+tr+sr+1 & \text{if } s \geq 2. \end{cases}$$

For the corresponding problem for multigraphs (without loops) we make the following definition. For integers $r,s \geq 0$, $t \geq 1$, let $f_m(r,s,t)$ be the least integer, if there is one, such that, for each $d \geq f_m(r,s,t)$, all $[d,d+s]$-multigraphs have an $(r,r+1)$-factorization into x $(r,r+1)$-factors for at least t different values of x; if there is no such least integer, let $f_m(r,s,t) = \infty$. Here the m in $f_m(r,s,t)$ stands for 'multigraph'.

Specializations of a result of Akiyama and Kano [1] when r is even and of Cai [3] when r is odd yield the following result.

Theorem 6. *For positive integers r,s,*

$$f_m(r,s,1) \leq \begin{cases} (3r+1)(r+s-1) & \text{if } r \text{ is even,} \\ (3r+1)(r+s) & \text{if } r \text{ is odd.} \end{cases}$$

These bounds do not seem to be the best possible. Some improved bounds are given for the cases $s=0$ and $s=1$ in [7], where there is also a conjecture made about the value of $f_m(r,0,1)$. At present, multigraphs seem to be much more difficult to deal with than simple graphs.

2 Background information

The fundamental result which lies behind the proofs of Theorems 3 and 4, and of our main theorem, Theorem 5, is a result of Hilton and de Werra [8] on equitable colourings of simple graphs.

An *edge-colouring* of a pseudograph G (i.e. a graph in which loops and multiple edges are allowed) is a map $\lambda : E(G) \to \mathcal{C}$, where $\mathcal{C}$ is a set of colours (loops being counted as edges). An edge-colouring is *equitable* if for each vertex v of G and any two colours $C_1, C_2 \in \mathcal{C}$, the number of edges incident with v and coloured C_1 differs by at most one from the corresponding number of edges coloured C_2; here a loop on v coloured C_i counts as two edges on v. For an integer $k \geq 2$, the *k-core* of a pseudograph G is the subpseudograph induced by the vertices of G whose degrees are divisible by k. The theorem of Hilton and de Werra is:

Theorem 7. *Let k be an integer, $k \geq 2$, and let G be a simple graph. If the k-core of G contains no edges, then G has an equitable edge-colouring with k colours.*

The proof in [8] of Theorem 7 is an extensive elaboration of Vizing's argument introduced in [11]. Theorem 7 and a result of Fournier [6] possibly have a common generalization:

Conjecture 1. *Let k be an integer, $k \geq 2$, and let G be a simple graph. If the k-core of G is a forest, then G has an equitable colouring with k colours.*

Using Theorem 7, the following result about $(r,r+1)$-factorizations of d-regular simple graphs was proved in [7]. Theorem 7 was used in the proof of the "hard part", namely part 1.

Theorem 8. *Let G be a d-regular simple graph, and let x and r be integers with $r \geq 1$.*

1. *G has an $(r,r+1)$-factorization with exactly x $(r,r+1)$-factors if*

$$\frac{d}{r+1} < x < \frac{d}{r}$$

or if r is odd and $x = \frac{d}{r+1}$, or if r is even and $x = \frac{d}{r}$.

2. *If r is even and $(r+1) \mid d$, then there are d-regular simples graphs G which are, and d-regular simple graphs G which are not $(r,r+1)$-factorizable into $x = \frac{d}{r+1}$ $(r,r+1)$-factors; if r is odd and $r \mid d$, then there are d-regular simple graphs which are, and d-regular simple graphs which are not $(r,r+1)$-factorizable into $x = \frac{d}{r}$ $(r,r+1)$-factors.*

3. *If $x \notin \left[\frac{d}{r+1}, \frac{d}{r}\right]$, then no d-regular simple graph is $(r,r+1)$-factorizable into x $(r,r+1)$-factors.*

For simple $(d,d+1)$-graphs, the following similar result was also proved in [7].

Theorem 9. *Let d, r and x be integers with $d \geq r \geq 1$.*

1. *If*
$$\frac{d+1}{r+1} < x < \frac{d}{r}$$
or
$$x = \begin{cases} \frac{d}{r} & \text{if } r \text{ is even,} \\ \frac{d+1}{r+1} & \text{if } r \text{ is odd,} \end{cases}$$
then any simple $(d,d+1)$-graph G has an $(r,r+1)$-factorization into x $(r,r+1)$-factors.

2. *If $x \geq 2$ and*
$$x = \begin{cases} \frac{d}{r} & \text{if } r \text{ is even,} \\ \frac{d+1}{r+1} & \text{if } r \text{ is odd,} \end{cases}$$
then some simple $(d,d+1)$-graphs do and some do not have an $(r,r+1)$-factorization into x $(r,r+1)$-factors.

3. *If $x \notin \left[\frac{d+1}{r+1}, \frac{d}{r}\right]$, then the only simple $(d,d+1)$-graphs G having an $(r,r+1)$-factorization into x $(r,r+1)$-factors occur when*
$$\begin{cases} x = \frac{d}{r+1} & \text{and } G \text{ is } d\text{-regular, or} \\ x = \frac{d+1}{r} & \text{and } G \text{ is } (d+1)\text{-regular.} \end{cases}$$
Moreover, when these conditions pertain, some but not all such graphs have an $(r,r+1)$-factorization.

3 $(d,d+s)$-simple graphs with $(r,r+1)$-factorizations

In this section we give an analogue to Theorems 8 and 9 for $(d,d+s)$-simple graphs in the case when $s \geq 2$.

Theorem 10. *Let d,r,s,x be integers with $d \geq r \geq 1$ and $s \geq 2$.*

1. *If*
$$\frac{d+s}{r+1} < x < \frac{d}{r}$$
then any simple $(d,d+s)$-graph G has an $(r,r+1)$-factorization into x r-factors.

2. *If $x = \frac{d+s}{r+1}$ or $\frac{d}{r}$, then some simple $(d,d+s)$-graphs do and some do not have an $(r,r+1)$-factorization into x $(r,r+1)$-factors.*

Proof. 1. Let G be a $(d,d+s)$-simple graph and let $\frac{d+s}{r+1} < x < \frac{d}{r}$. Let $v \in V(G)$. Then
$$r < \frac{d}{x} \leq \frac{\mathrm{d}(v)}{x} \leq \frac{d+s}{x} < r+1$$
so $x \nmid \mathrm{d}(v)$. Thus the core of G contains no edges, and so, by Theorem 7, G has an equitable edge-colouring with x colours. But
$$\frac{\mathrm{d}(v)}{r+1} \leq \frac{d+s}{r+1} < x < \frac{d}{r} \leq \frac{\mathrm{d}(v)}{r},$$
so the degree of v in each colour class is either r or $r+1$. Thus the equitable edge-colouring corresponds to an $(r,r+1)$-factorization with x $(r,r+1)$-factors.

2. We show first that if $x = \frac{d+s}{r+1}$ or $\frac{d}{r}$, then some simple $(d,d+s)$-graphs do have an $(r,r+1)$-factorization into x $(r,r+1)$-factors.

Let $r \mid d$ and put $x = \frac{d}{r}$. Then any Class 1 d-regular simple graph is the union of d 1-factors, and so can be expressed as the union of x r-factors. Thus it can be expressed as the union of x $(r,r+1)$-factors.

Now let $(r+1) \mid (d+s)$ and put $x = \frac{d+s}{r+1}$. Then any Class 1 $(d+s)$-regular simple graph can be expressed as the union of x $(r+1)$-factors, and so is the union of x $(r,r+1)$-factors.

Next we show that is $x = \frac{d+s}{r+1}$ or $\frac{d}{r}$, then some simple $(d,d+s)$-graphs do not have an $(r,r+1)$-factorization into x $(r,r+1)$-factors.

Let
$$d = r^2 + rs + r = r(r+s+1)$$
so that
$$d+s = r^2 + rs + r + s = (r+1)(r+s).$$
Then
$$\frac{d+s}{r+1} = r+s < r+s+1 = \frac{d}{r}.$$

We first construct a $(d,d+s)$-simple graph G_1 which has the property that if its $(r,r+1)$-factorizations have x_1 $(r,r+1)$-factors then $x_1 \leq r+s$. Then we construct a $(d,d+s)$-simple graph G_2 which has the property that if its $(r,r+1)$-factorizations have x_2 $(r,r+1)$-factors then $x_2 \geq r+s+1$. Thus the disjoint union of G_1 and G_2 is a $(d,d+s)$-graph with no $(r,r+1)$-factorizations.

Construction of G_1.

(i) *r odd, d even.* Then s is even. Let G_1 be a regular simple graph of degree d and odd order. Then the union of $r+s+2$ or more $(r,r+1)$-factors has degree at least $r(r+s+2) = d+r > d$, which is too large. Moreover any $(r,r+s)$-factorization of G_1 into $r+s+1$ $(r,r+1)$-factors would have all its factors regular of degree r, which is impossible since G_1 has odd order.

(ii) *r odd, d odd.* Then s is odd. Let G_1 have one vertex of degree $d+1$, the remainder having degree d. Then G_1 has odd order. An example of such a graph can be made by removing $\frac{1}{2}(d+1)$ independent edges from K_{d+2}. As above, there is no $(r,r+1)$-factorization of G_1 with more than $r+s+1$ $(r,r+1)$-factors. Moreover, any $(r,r+1)$-factorization of G_1 with $r+s+1$ $(r,r+1)$-factors would have all bar one factor regular of degree r. But this is impossible since G_1 has odd order.

(iii) *r even.* Let G_1 have one vertex of degree $d+2$, the rest having degree d. Such a graph can be made by removing a cycle of length $d+2$ from K_{d+3}. As in (i), there is no $(r,r+1)$-factorization of G_1 with more than $r+s+1$ $(r,r+1)$-factors. Moreover any $(r,r+1)$-factorization of G_1 with $r+s+1$ $(r,r+1)$-factors would have two $(r,r+1)$-factors each with one vertex of degree $r+1$, the rest having degree r. But since r is even, this is impossible.

Thus in each case $x_1 \leq r+s$.

Construction of G_2. Note that if r is even, then d is even.

(i) *r even, s even.* Then $d+s$ is even. Let G_2 be a simple regular graph of degree $d+s$ and odd order. The union of $r+s-1$ or fewer $(r,r+1)$-factors has degree at most $(r+1)(r+s-1) = d+s-r-1 < d+s$, and so it is not large enough. Moreover we cannot factorize G_2 into $r+s$ $(r,r+1)$-factors, for then all $(r,r+1)$-factors would be $(r+1)$-regular; this is impossible as $r+1$ and the order of G_2 are both odd.

(ii) *r even, s odd.* The $d+s$ is odd. Let G_2 be a simple graph with one vertex of degree $d+s-1$ and the remaining vertices of degree $d+s$. Since $d+s$ is odd, G_2 has odd order. Such a graph can be obtained from K_{d+s+2} by deleting $\frac{1}{2}(d+s-1)$ independent edges and a $K_{1,2}$ which is disjoint from all these independent edges. As above, there is no $(r,r+1)$-factorization with fewer than $r+s$ $(r,r+1)$-factors. Moreover in any $(r,r+1)$-factorization of G_2 with $r+s$ $(r,r+1)$-factors, all bar one of the $(r,r+1)$-factors would be regular of degree $r+1$, which is impossible since $r+1$ and the order of G_2 are both odd.

(iii) *r odd*. Then $d+s-2$ is even. Let G_2 have one vertex of degree $d+s-2$ and the remainder have degree $d+s$. Such a graph may be obtained from

K_{d+s+2} by removing $\frac{1}{2}(d+s-2)$ independent edges and a $K_{1,3}$ which is disjoint from these independent edges. As in (i) above, there is no $(r,r+1)$-factorization of G_2 with fewer than $r+s$ $(r,r+1)$-factors. Moreover in any $(r,r+1)$-factorization into $r+s$ $(r,r+1)$-factors, two of the $(r,r+1)$ factors would have one vertex of degree r and the remainder of degree $r+1$. But this is impossible since r is odd.

Thus in each case, $x_2 \geq r+s+1$. □

For integers $s \geq 0$, $d \geq r \geq 1$, let $S(d,s,r)$ be the set of integers x such that, for all simple $(d,d+s)$-graphs G, G has an $(r,r+1)$-factorization into x $(r,r+1)$-factors. From Theorems 8, 9 and 10, we can now show:

Theorem 11. *Let $d \geq r \geq 1$. For $s \in \{0,1\}$*

$$S(d,s,r) = \begin{cases} \left(\dfrac{d+s}{r+1}, \dfrac{d}{r}\right] & \text{if } r \text{ is even,} \\ \left[\dfrac{d+s}{r+1}, \dfrac{d}{r}\right) & \text{if } r \text{ is odd,} \end{cases}$$

and for $s \geq 2$

$$S(d,s,r) = \left(\frac{d+s}{r+1}, \frac{d}{r}\right).$$

Proof. Suppose first that $s=0$. By Theorem 8(1), it follows that

$$\begin{cases} \left(\dfrac{d}{r+1}, \dfrac{d}{r}\right] \subseteq S(d,0,r) & \text{if } r \text{ is even,} \\ \left[\dfrac{d}{r+1}, \dfrac{d}{r}\right) \subseteq S(d,0,r) & \text{if } r \text{ is odd.} \end{cases}$$

By Theorem 8(2), the end values $x = \frac{d}{r+1}$ if r is even and $x = \frac{d}{r}$ if r is odd do not lie in $S(d,0,r)$. By Theorem 8(3), values of x smaller than $\frac{d}{r+1}$ or larger than $\frac{d}{r}$ are not in $S(d,0,r)$. Therefore

$$S(d,0,r) = \begin{cases} \left(\dfrac{d}{r+1}, \dfrac{d}{r}\right] & \text{if } r \text{ is even,} \\ \left[\dfrac{d}{r+1}, \dfrac{d}{r}\right) & \text{if } r \text{ is odd.} \end{cases}$$

The argument if $s=1$ is identical, using Theorem 9 instead of Theorem 8.

Next suppose that $s=2$. Since any $(d,d+1)$-graph or any $(d+1,d+2)$-graph is also a $(d,d+2)$-graph, it follows from the $s=1$ case that

$$\begin{aligned} S(d,2,r) &\subseteq S(d,1,r) \cap S(d+1,1,r) \\ &= \begin{cases} \left(\dfrac{d+1}{r+1}, \dfrac{d}{r}\right] \cap \left(\dfrac{d+2}{r+1}, \dfrac{d+1}{r}\right] = \left(\dfrac{d+2}{r+1}, \dfrac{d}{r}\right] & \text{if } r \text{ is even,} \\ \left[\dfrac{d+1}{r+1}, \dfrac{d}{r}\right) \cap \left[\dfrac{d+2}{r+1}, \dfrac{d+1}{r}\right) = \left[\dfrac{d+2}{r+1}, \dfrac{d}{r}\right) & \text{if } r \text{ is odd.} \end{cases} \end{aligned}$$

By Theorem 10(2), the end-values $\frac{d+2}{r+1}$ and $\frac{d}{r}$ do not lie in $S(d,2,r)$. Thus

$$S(d,2,r) \subseteq \left(\frac{d+2}{r+1}, \frac{d}{r}\right).$$

But by Theorem 10(1)

$$\left(\frac{d+2}{r+1}, \frac{d}{r}\right) \subseteq S(d,2,r).$$

Therefore

$$S(d,2,r) = \left(\frac{d+2}{r+1}, \frac{d}{r}\right).$$

Now let $s \geq 2$. As an induction hypothesis, suppose that, for $2 \leq \sigma \leq s$,

$$S(d,\sigma,r) = \left(\frac{d+\sigma}{r+1}, \frac{d}{r}\right).$$

Since any $(d,d+s)$-graph or any $(d+1,d+s+1)$-graph is also a $(d,d+s+1)$-graph, it follows that

$$\begin{aligned} S(d,s+1,r) &\subseteq S(d,s,r) \cap S(d+1,s,r) \\ &= \left(\frac{d+s}{r+1}, \frac{d}{r}\right) \cap \left(\frac{d+s+1}{r+1}, \frac{d+1}{r}\right) \\ &= \left(\frac{d+s+1}{r+1}, \frac{d}{r}\right). \end{aligned}$$

But by Theorem 10(1),

$$\left(\frac{d+s+1}{r+1}, \frac{d}{r}\right) \subseteq S(d,s+1,r),$$

so that

$$S(d,s+1,r) = \left(\frac{d+s+1}{r+1}, \frac{d}{r}\right).$$

Therefore, by induction, $S(d,s,r) = \left(\frac{d+s}{r+1}, \frac{d}{r}\right)$ for all $s \geq 2$. □

4 Proof of Theorem 5

We now turn to the proof of Theorem 5.

Proof. There are t distinct values of x such that any simple $(d,d+s)$-graph has an $(r,r+1)$-factorization with x $(r,r+1)$-factors if and only if $S(d,s,r)$ contains t distinct integers. By Theorem 11, this occurs if and only if, for some integer x,

$\frac{d+s}{r+1} < x$ and $x+t-1 \leq \frac{d}{r}$ if $s \in \{0,1\}$ and r is even,

$\frac{d+s}{r+1} \leq x$ and $x+t-1 < \frac{d}{r}$ if $s \in \{0,1\}$ and r is odd,

$\frac{d+s}{r+1} < x$ and $x+t-1 < \frac{d}{r}$ if $s \geq 2$.

Suppose $s \in \{0,1\}$ and r is even. If $d = tr^2 + tr + sr - r - 1$ and $S(d,s,r)$ contains t distinct integers, then let $x \in \mathbb{Z}$ be as large as possible such that

$$x + t - 1 \leq \frac{d}{r} = tr + t - 1 + s - \frac{1}{r};$$

then $x = tr + s - 1$. But $\frac{d+s}{r+1} = tr + s - 1 < x$, a contradiction. Therefore $f(r,s,t) \geq tr^2 + tr + sr - r$.

Next suppose $s \in \{0,1\}$ and r is odd. If $d = tr^2 + tr + sr - r$ and $S(d,s,r)$ contains t distinct integers, then let $x \in \mathbb{Z}$ be as small as possible such that $x \geq \frac{d+s}{r+1}$. Then $x \geq tr + s - 1 + \frac{1}{r+1}$. Therefore $x = tr + s$. But $\frac{d}{r} = tr + t + s - 1 > x + t - 1$, so $tr + s > x$, a contradiction. Therefore $f(r,s,t) \geq tr^2 + tr + sr - r + 1$.

Now suppose that $s \geq 2$. If $d = tr^2 + tr + sr$ and $S(d,s,r)$ conatins t distinct integers then let $x \in \mathbb{Z}$ be as large as possible such that $x + t - 1 < \frac{d}{r} = tr + t + s$. Then $x = tr + s$. But $\frac{d+s}{r+1} = tr + s < x$, a contradiction. Therefore

$$f(r,s,t) \geq tr^2 + tr + sr + 1.$$

Return now to the case when $s \in \{0,1\}$ and r is even. Let $d \geq tr^2 + tr + sr - r$, say $d = tr^2 + tr + sr - r + \varepsilon$. Then $\frac{d}{r} = tr + t + s - 1 + \frac{\varepsilon}{r}$. Choose an integer x as large as possible so that $x + t - 1 \leq \frac{d}{r}$. Then

$$x + t - 1 = tr + t + s - 1 + \left\lfloor \frac{\varepsilon}{r} \right\rfloor,$$

so $x = tr + s + \left\lfloor \frac{\varepsilon}{r} \right\rfloor$. Now

$$\frac{d+s}{r+1} = \frac{tr^2 + tr + sr - r + s + \varepsilon}{r+1},$$

so $\frac{d+s}{r+1} = tr + s - 1 + \frac{1+\varepsilon}{r+1}$. Put $\varepsilon = pr + q$ where $0 \leq q < r$. Then $x = tr + s + p$. Also

$$\begin{aligned}
\tfrac{d+1}{r+1} &= tr + s - 1 + \tfrac{pr+q+1}{r+1} = tr + s - 1 + p + \tfrac{q+1-p}{r+1} \\
&= tr + s + p + \tfrac{q-r-p}{r+1} = x + \tfrac{q-r-p}{r+1} \\
&< x,
\end{aligned}$$

since $r > q$ and $p \geq 0$. Therefore $f(r,s,t) \leq tr^2 + tr + sr - r$. Therefore $f(r,s,t) = tr^2 + tr + sr - r$.

Next suppose that $s \in \{0,1\}$ and r is odd. Let $d \geq tr^2 + tr + sr - r + 1$, say $d = tr^2 + tr + sr - r + 1 + \varepsilon$. Then $\frac{d}{r} = tr + t + s - 1 + \frac{\varepsilon+1}{r}$. Choose $x \in \mathbb{Z}$ as large as possible so that $x + t - 1 < \frac{d}{r} = tr + t + s - 1 + \frac{\varepsilon+1}{r}$. Then $x + t - 1 = tr + t + s - 1 + \left\lfloor \frac{\varepsilon}{r} \right\rfloor$, so $x = tr + s + \left\lfloor \frac{\varepsilon}{r} \right\rfloor$. Now $\frac{d+s}{r+1} = tr + s - 1 + \frac{2+\varepsilon}{r+1}$. Put $\varepsilon = pr + q$ where $0 \leq q < r$. Then $x = tr + s + p$ and $\frac{d+s}{r+1} = tr + s - 1 + \frac{2+pr+q}{r+1}$, so

$$\frac{d+s}{r+1} = tr + p + s + \frac{q+1-r-p}{r+1} = x + \frac{q+1-r-p}{r+1} \leq x$$

since $r \geq q+1$ and $p \geq 0$. Therefore $f(r,s,t) \geq tr^2 + tr + sr - r + 1$. Therefore $f(r,s,t) = tr^2 + tr + sr - r + 1$.

Finally suppose that $x \geq 2$. Let $d \geq tr^2 + tr + sr + 1$, say $d = tr^2 + tr + sr + 1 + \varepsilon$. Then $\frac{d}{r} = tr + t + s + \frac{1+\varepsilon}{r}$. Choose an integer x as large as possible so that $x + t - 1 < \frac{d}{r} = tr + t + s + \frac{1+\varepsilon}{r}$. Then $x - t - 1 = tr + t + s + \lfloor \frac{\varepsilon}{r} \rfloor$, so $x = tr + s + 1 + \lfloor \frac{\varepsilon}{r} \rfloor$. Now $\frac{d+s}{r+1} = tr + s + \frac{1+\varepsilon}{r+1}$. Put $\varepsilon = pr + q$, where $0 \leq q < r$. Then $x = tr + s + 1 + p$ and

$$\frac{d+s}{r+1} = tr + s + \frac{1+\varepsilon}{r+1} = x + \frac{q-r-p}{r+1} < x,$$

since $r > q$ and $p \geq 0$. Therefore $f(r,s,t) \geq tr^2 + tr + sr + 1$, so it follows that $f(r,s,t) = tr^2 + tr + sr + 1$, as asserted. □

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