$(r, r+1)$-factorizations of $(d, d+1)$-graphs

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Abstract

A $(d, d+1)$-graph is a graph whose vertices all have degrees in the set $\{d, d+1\}$. Such a graph is *semiregular*. An $(r, r+1)$-factorization of a graph G is a decomposition of G into $(r, r+1)$-factors. For d-regular simple graphs G we say for which x and r G must have an $(r, r+1)$-factorization with exactly x $(r, r+1)$-factors. We give similar results for $(d, d+1)$-simple graphs and for $(d, d+1)$-pseudographs. We also show that if $d \geq 2r^2 + 3r - 1$, then any $(d, d+1)$-multigraph (without loops) has an $(r, r+1)$-factorization, and we give some information as to the number of $(r, r+1)$-factors which can be found in an $(r, r+1)$-factorization.

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Contents

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1 Introduction

We use the terminology *simple graph* for a graph with no loops or multiple edges, *multigraph* for a graph in which multiple edges are allowed but not loops, and *pseudograph* for when both loops and multiple edges are allowed.

An $(r, r+1)$-*pseudograph* is a graph whose degrees are all either r or $r+1$; in a pseudograph, a loop counts two towards the degree of the vertex it is on. An $(r, r+1)$-*factor* of a pseudograph G is an $(r, r+1)$-subpseudograph which spans G. An $(r, r+1)$-*factorization* of a pseudograph G is a decomposition of G into edge-disjoint $(r, r+1)$-factors of G. If there are no vertices of degree $r+1$ in an $(r, r+1)$-factor, we may refer to it as an r-*factor*. Similarly an r-*factorization* is an $(r, r+1)$-factorization in which none of the $(r, r+1)$-factors actually have any vertices of degree $r+1$.

Our main concern in this paper is to determine for which values of d, r and x, a $(d, d+1)$-graph may have an $(r, r+1)$-factorization with x $(r, r+1)$-factors.

Lovász [18] and Tutte [24] both gave proofs of the result that either was, or was equivalent to, the statement that any d-regular pseudograph has an $(r, r+1)$-factor for each $r \in \{0, 1, ..., d-1\}$. In both cases, their result was intended to be just an illustration of a deeper theorem. Thomassen [23] later gave a simple proof of the more general statement:

Theorem 1. (Thomassen). *Every $(d, d+1)$-pseudograph has an $(r, r+1)$-factor for each $r \in \{0, 1, ...d-1\}$.*

Lovász noted that, in the case of a simple graph, his result followed easily from Vizing's theorem [25]. In Section 2 we develop this connection further and show how some interesting facts about $(r, r+1)$-factorizations follow quite easily for d-regular simple graphs. To a certain extent, these facts provide the motivation for much of the discussion in this paper.

This discussion is also motivated by some further known results. On the question of the existence of $(r, r+1)$-factorizations of d-regular simple graphs, Era [6] and Egawa [5] proved the following result.

Theorem 2. *If G is a d-regular simple graph and*

$$d \geq \begin{cases} r^2 & if \quad r \quad is \ even, \\ r^2+1 & if \quad r \quad is \ odd, \end{cases}$$

then G has an $(r, r+1)$-factorization.

The result is best possible in the sense that the lower bound cannot be lowered even by 1. (However it should be noted that there are some lower values of d for which all d-regular simple graphs have an $(r, r+1)$-factorization; trivially if $d = r$ or $r+1$, but also $d = 2r+1$, for example, as shown in Lemma 30.)

For $(r, r+1)$-factorizations of $(d, d+1)$-multigraphs, Akiyama and Kano [2] and Cai [3] proved the following nice result (their results were actually more general than this).

Theorem 3. *If G is a $(d, d+1)$-multigraph and*

$$d \geq \begin{cases} 3r^2+r & if \quad r \quad is \ even, \\ 3(r+1)^2 & if \quad r \quad is \ odd, \end{cases}$$

then G has an $(r, r+1)$-factorization.

This result is not best possible, and, in Theorem 29, we shall give a better result (which is also not best possible).

Further known results are:

Theorem 4. (Petersen). *Let r be even and let $r|d$. Then any d-regular pseudograph has an r-factorization into $\frac{d}{r}$ r-factors.*

Another result for regular graphs is due to Hilton [14].

Theorem 5. (Hilton). *Let G be a d-regular simple graph with $|V(G|$ even, let $d \geq \frac{1}{2}|V(G)|$ and let $r|d$, $r > 1$. Then G has an r-factorization into $\frac{d}{r}$ r-factors.*

The requirement that $d \geq \frac{1}{2}|V(G)|$ cannot be removed if $r > 1$ is odd. The One-Factorization Conjecture [2] is that Theorem 5 is also true if $r = 1$.

An easy result which follows by colouring the edges of an Eulerian circuit alternately with two colours is:

Theorem 6. (Folklore). *Let $d = 2r$ and let G be a connected d-regular pseudograph with $|V(G)|$ even if r is odd. Then G has an r-factorization into $\frac{d}{r} = 2$ r-factors.*

2 Simple d-regular graphs

2.1 Initial discussion

In this section we make some preliminary observations about d-regular simple graphs; these observations serve partly to motivate further the main results in this paper.

The first result is a simple deduction from Vizing's theorem [25].

Theorem 7. *Let $r+1|d+1$. Then a d-regular simple graph G has an $(r, r+1)$-factorization into $\frac{d+1}{r+1}$ $(r, r+1)$-factors.*

Proof. By Vizing's theorem, the chromatic index $\chi'(G)$ of G satisfies $\chi'(G) \leq d+1$. Therefore we may colour the edges of G with $d+1$ colours so that no two edges incident with the same vertex have the same colour. Now collect the edges of the colours used into $(d+1)/(r+1)$ sets, each containing all the edges of $r+1$ colours. Form the union of each set of edges of $r+1$ colours. This union is an $(r, r+1)$-factor of G, and the set of all $(d+1)/(r+1)$ such unions is an $(r, r+1)$-factorization. □

There is a companion result to Vizing's theorem due to Gupta [9]. Recall that the *cover index* $\kappa'(G)$ of a graph G is the greatest number j of colours that one can colour the edges of G with so that, at each vertex, there is at least one edge of each colour. Gupta's theorem is that $\kappa'(G) \geq \delta(G) - 1$ for simple graphs G, where $\delta(G)$ is the minimum degree of G. For a deduction of Gupta's theorem from Vizing's theorem, via the inequality $\chi'(G) + \kappa'(G) \geq 2\delta(G)$, see [13].

We can use Gupta's theorem to obtain a theorem strikingly similar to Theorem 7, but not the same.

Theorem 8. *Let $r|d-1$. A d-regular simple graph G has an $(r, r+1)$-factorization into $(d-1)/r$ $(r, r+1)$-factors.*

Proof. By Gupta's theorem, the cover index $\kappa'(G)$ satisfies $\kappa'(G) \geq d-1$. Thus the edges of G can be coloured with $d-1$ colours in such a way that, at each vertex, there is at least one edge of each colour. Now collect the edges into $(d-1)/r$ sets, each of all the edges of r colours. Form the union of each set of edges of r colours. This union is an $(r, r+1)$-factor, and the set all $(d-1)/r$ such unions is an $(r, r+1)$-factorization. □

Let us illustrate Theorems 7 and 8 with an example. If $d = 29$ and $r = 4$, then, by Theorem 7, G is the union of 6 (4, 5)-factors, and, by Theorem 8, G is the union of 7 (4, 5)-factors. Similarly if $d = 29$ and $r = 2$, then G is the union of 10 (2, 3)-factors and of 14 (2, 3)-factors.

If $r+1 \mid d+1$ then a d-regular graph cannot be the union of fewer than $(d+1)/(r+1)$ $(r,r+1)$-factors. For suppose that a graph G is the union of $x < (d+1)/(r+1)$ $(r,r+1)$-factors. Then $x \le (d+1)/(r+1) - 1 = (d-r)/(r+1)$, so the maximum degree is at most $x(r+1) \le d-r$, a contradiction. Similarly if $r|d-1$ and $r > 1$ then a d-regular graph cannot be the union of more than $(d-1)/r$ $(r,r+1)$-factors. For if a graph G were the union of $y > (d-1)/r$ $(r,r+1)$-factors then $y \ge (d-1)/r+1 = (d+r-1)/r$, and so the minimum degree of G is $ry \ge d+r-1 > d$.

We shall prove the following result about d-regular simple graphs when $r \mid d-1$ and $r+1 \mid d+1$.

Theorem 9. *Let $r, d, x \in \mathbb{Z}^+$, let G be a simple d-regular graph and let $r \ge 2$, $r \mid d-1$ and $r+1 \mid d+1$. Then G has an $(r,r+1)$-factorization with exactly x $(r,r+1)$-factors if and only if*

$$\frac{d+1}{r+1} \le x \le \frac{d-1}{r}.$$

In fact, this is a specialization of the next result, where we do not assume the divisibility conditions.

Theorem 10. *Let G be a simple d-regular graph and let $r \ge 2$.*

(i) G has an $(r,r+1)$-factorization with exactly x $(r,r+1)$-factors if

$$\frac{d}{r+1} < x < \frac{d}{r}$$

or if r is odd and $x = \frac{d}{r+1}$,
or if r is even and $x = \frac{d}{r}$.

(ii) If r is even and $r+1 \mid d$, then there are d-regular simple graphs G which are, and d-regular simple graphs G which are not $(r,r+1)$-factorizable into $x = \frac{d}{r+1}$ $(r,r+1)$-factors;

if r is odd and $r \mid d$, then there are d-regular simple graphs G which are, and d-regular simple graphs G which are not $(r,r+1)$-factorizable into $x = \frac{d}{r}$ $(r,r+1)$-factors.

(iii) If $x \notin [\frac{d}{r+1}, \frac{d}{r}]$ then no d-regular simple graph is $(r,r+1)$-factorizable into x $(r,r+1)$-factors.

The proof of Theorem 10 is placed after the proof of Theorem 15.

Proof of Theorem 9 (assuming Theorem 10).
If $\frac{d+1}{r+1} \le x \le \frac{d-1}{r}$ then $\frac{d}{r+1} < x < \frac{d}{r}$, so by Theorem 10 G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors. The remarks before Theorem 9 show that G has no $(r, r+1)$-factorization into x $(r, r+1)$-factors if $x > \frac{d-1}{r}$ or $x < \frac{d+1}{r+1}$. □

An additional remark concerning Theorem 10(ii). Suppose G is connected. Whether r is even and $r+1 \mid d$ or r is odd and $r \mid d$, if $x = 2$ then G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors if and only if $|V(G)|$ is even.

Proof of additional remark. Suppose first that r is even, $r+1 \mid d$ and $x = 2 = \frac{d}{r+1}$. If G has an $(r, r+1)$-factorization into two $(r, r+1)$-factors, then both $(r, r+1)$-factors must be $(r+1)$-factors. Since $|E(G)| = |V(G)|(r+1)$, $|E(G)|$ is even if and only if $|V(G)|$ is even. If $|E(G)|$ is even, then we can obtain an $(r+1)$-factorization by assigning alternate edges of an Eulerian circuit to the one $(r+1)$-factor, the remaining edges to the other $(r+1)$-factor. If $E(G)$ is odd then $|V(G)|$ is odd, and so G has no $(r+1)$-factor.

Next suppose that r is odd, $r \mid d$ and $x = 2 = \frac{d}{r}$. If G has an $(r, r+1)$-factorization into two $(r, r+1)$-factors, then both the $(r, r+1)$-factors must be r-factors. The argument is now very similar to that above. □

2.2 The Era-Egawa theorem

From Theorem 10 we can deduce easily the Era-Egawa result (Theorem 2). Let $\phi(r)$ be the smallest integer such that, if $d \ge \phi(r)$, then any d-regular simple graph has an $(r, r+1)$-factorization. Theorem 2 together with the assertion that Theorem 2 is best possible, can be expressed in the following way:

Theorem 11.

$$\phi(r) = \begin{cases} r^2 & \textit{if } r \textit{ is even }, \\ r^2+1 & \textit{if } r \textit{ is odd }. \end{cases}$$

Proof. Suppose first that r is even. From Theorem 10 it follows that each d-regular simple graph has an $(r, r+1)$-factorization if and only if there is an integer x such that

$$\frac{d}{r+1} < x \le \frac{d}{r} .$$

We first note that if $d = r^2 - 1$ there is no such integer x; for if there were such an integer x then $x \le \frac{r^2-1}{r}$, so $x \le r-1$. But on the other hand, $\frac{d}{r+1} =$

$\frac{r^2-1}{r+1} = r-1$ so $x \geq r$, a contradiction. Therefore $\phi(r) \geq r^2$. Now suppose that $d \geq r^2$. Let x be defined by $d = xr + s$, where $0 \leq s \leq r-1$. Then $\frac{d}{r} = x + \frac{s}{r} \geq x$. Since $d \geq r^2$, it follows that $x \geq r$. Since $d = x(r+1) - x + s$, we have $\frac{d}{r+1} = x - \left(\frac{x-s}{r+1}\right) < x$ since $x - s \geq r - s > 0$. Therefore $\phi(r) = r^2$.

Next suppose that r is odd. By Theorem 10 it follows that each d-regular simple graph has an $(r, r+1)$-factorization if and only if there is an integer x such that

$$\frac{d}{r+1} \leq x < \frac{d}{r} .$$

We first note that if $d = r^2$ there is no such integer x; for if there were such an integer x then $x \geq \frac{r^2}{r+1} = r - 1 + \frac{1}{r+1}$, so $x \geq r$. But on the other hand $x < \frac{r^2}{r} = r$, a contradiction. Therefore $\phi(r) \geq r^2+1$. Now suppose $d \geq r^2+1$. Let x be defined by $d = xr + s$, where $1 \leq s \leq r$. Then $\frac{d}{r} = x + \frac{s}{r} > x$. Since $d \geq r^2+1$ it follows that $x \geq r$. Since $d = x(r+1) - x + s$, we then have that $\frac{d}{r+1} = x - \left(\frac{x-s}{r+1}\right) \leq x$, since $x - s \geq r - s \geq 0$. Therefore $\phi(r) = r^2 + 1$. $\square$

3 Edge-colourings

An *edge-colouring* of a pseudograph G is a map $\phi : E(G) \to \mathscr{C}$, where $\mathscr{C}$ is a set of colours; here loops are included as edges. Thus an edge-colouring is just a partition of the edge-set. An edge-colouring is *equitable* if

$$\Big| |\alpha(v)| - |\beta(v)| \Big| \leq 1$$

$\forall\, v \in V(G)$ and $\forall\, \alpha, \beta \in \mathscr{C}$, where $\alpha(r), \beta(r)$ denote the sets of edges of colours α, β incident with the vertex v; here a loop on v coloured α counts as two edges on v. We notice that an $(r, r+1)$-factorization of a $(d, d+1)$-graph G with x $(r, r+1)$-factors is just an equitable colouring of G with x colours.

3.1 Bipartite $(d, d+1)$-multigraphs

We can use the following simple result of McDiarmid [19] and de Werra [26] to bound the number of $(r, r+1)$-factors of bipartite $(d, d+1)$-multigraphs.

Theorem 12. *Let $k \geq 1$ be an integer and let G be a bipartite multigraph. Then G has an equitable edge-colouring with k colours.*

A corollary of this is:

Theorem 13. *Let G be a bipartite $(d, d+1)$-multigraph. Then G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors if and only if*

$$\tfrac{d}{r+1} < x < \tfrac{d+1}{r} \ ,$$

or

$$x = \tfrac{d}{r+1} \textit{ and } G \textit{ is } d\textit{-regular,}$$

or

$$x = \tfrac{d+1}{r} \textit{ and } G \textit{ is } (d+1)\textit{-regular.}$$

Proof. It is shown in an easy argument in Theorem 18(b) in Section 5 that if G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors, then the inequality $\frac{d}{r+1} \leq x \leq \frac{d+1}{r}$ is satisfied. If $x = \frac{d}{r+1}$, then clearly G has to be d-regular, and if $x = \frac{d+1}{r}$, then clearly G has to be $(d+1)$-regular. This proves the necessity.

For the sufficiency, notice that if G is d-regular and $x = \frac{d}{r+1}$, then G has an $(r+1)$-factorization with x $(r+1)$-factors, and that if G is $(d+1)$-regular and $x = \frac{d+1}{r}$, then G has an r-factorization with x r-factors. Thus, in both cases, G has an $(r, r+1)$-factorization with x $(r, r+1)$-factors. Now suppose that $\frac{d}{r+1} < x < \frac{d+1}{r}$. Then $r \leq \frac{d}{x}$ and $\frac{d+1}{x} \leq r+1$, so an equitable edge-colouring with x colours yields an $(r, r+1)$-factorization with x $(r, r+1)$-colours. $\square$

3.2 Equitable edge-colourings of simple $(d, d+1)$-graphs

The following quite difficult theorem of Hilton and de Werra [15] will be used to bound the number of $(r, r+1)$-factors of simple $(d, d+1)$-graphs. Let the *k-core* of a graph G be the subgraph induced by the vertices v such that $k \mid d_G(v)$. (The term *k-core* has been used in this sense in a number of papers, but one of the referees has pointed out that *k-core* has also been used for the largest induced subgraph of minimum degree at least k.) The theorem of Hilton and de Werra is:

Theorem 14. *Let k be a positive integer and let G be a simple graph. If the k-core of G contains no edges, then G has an equitable colouring with k colours.*

4 Simple $(d, d+1)$-graphs

4.1 The number of $(r, r+1)$-factors

We use Theorem 14 to prove the difficult part of Theorem 15, namely part (i).

Theorem 15. *Let $d, r \in \mathbb{Z}^+$.*

(i) *If $x \in \mathbb{Z}$ and $\frac{d+1}{r+1} < x < \frac{d}{r}$,*

$$\text{or} \quad x = \begin{cases} \frac{d}{r} & \text{if } r \text{ is even,} \\ \frac{d+1}{r+1} & \text{if } r \text{ is odd,} \end{cases}$$

then any simple $(d, d+1)$-graph G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors.

(ii) *If $x \in \mathbb{Z}$, $x \geq 2$ and*

$$x = \begin{cases} \frac{d}{r} & \text{if } r \text{ is odd,} \\ \frac{d+1}{r+1} & \text{if } r \text{ is even,} \end{cases}$$

then some simple $(d, d+1)$-graphs do and some do not have an $(r, r+1)$-factorization into x $(r, r+1)$-factors.

(iii) *If $x \in \mathbb{Z}$, $x \notin [\frac{d+1}{r+1}, \frac{d}{r}]$, then the only simple $(d, d+1)$-graphs G having an $(r, r+1)$-factorization into x $(r, r+1)$-factors occur when*

$$\begin{cases} x = \frac{d}{r+1} & \text{and } G \text{ is } d\text{-regular,} \\ x = \frac{d+1}{r} & \text{and } G \text{ is } (d+1)\text{-regular.} \end{cases}$$

Moreover, when these conditions pertain, some but not all such graphs have an $(r, r+1)$-factorization.

Note the following consequence of Theorem 15 (iii) (although it is easier to use Theorem 18(b)).

Corollary 16. *Let $r, d, x \in \mathbb{Z}^+$ and $x \notin [\frac{d}{r+1}, \frac{d+1}{r}]$. Then no $(d, d+1)$-graph is the union of x $(r, r+1)$-factors.*

Proof of Theorem 15.

(i) First suppose that $x \neq \frac{d}{r}$ or $\frac{d+1}{r+1}$. Then $\frac{d+1}{r+1} < x < \frac{d}{r}$, so $r < \frac{d}{x} < \frac{d+1}{x} < r+1$, and so $x \nmid d$ and $x \nmid d+1$. Therefore the x-core of G is empty, so, by Theorem 14, G has an equitable edge-colouring with x colours. Since $r < \frac{d}{x}$ and $\frac{d+1}{x} < r+1$, it follows that this equitable edge-colouring with x colours is the required $(r, r+1)$-factorization with x $(r, r+1)$-factors.

The remaining cases in part (i), when $x \in \left\{\frac{d}{r}, \frac{d+1}{r+1}\right\}$, are particular cases of Theorem 18(a)(i) or 18(a)(ii), which is stated and proved in Section 5.

(ii) If G is a d-regular Class 1 simple graph (Class 1 means that $\chi'(G) = d$), and if $r|d$, then clearly G has an r-factorization into $\frac{d}{r}$ factors. Similarly if G is a $(d+1)$-regular Class 1 simple graph and $r+1 \,|\, d+1$ then clearly G has an $(r+1)$-factorization into $\frac{d+1}{r+1}$ factors. Thus in either case we have a $(d, d+1)$-simple graph with an $(r, r+1)$-factorization into x $(r, r+1)$-factors.

To provide examples where there is no such factorization, we consider two cases.

Case 1. *r is odd and $x = \frac{d}{r} \geq 2$.*

Suppose that d is odd. We shall construct a d-regular simple graph G which has no $(r, r+1)$-factorization into x $(r, r+1)$-factors. Let $A_1, \ldots, A_d$ be disjoint $(d-1, d)$-simple graphs of odd order, with each A_i having exactly one vertex, a_i, of degree $d-1$; this is possible since d is odd. Let a be a further vertex, and let G be the graph formed by joining a to each of $a_1, \ldots, a_d$. Then G is a d-regular simple graph. If G had an $(r, r+1)$-factorization into x $(r, r+1)$-factors, then each of the $(r, r+1)$-factors would be an r-factor. But as $x \geq 2$, one of these r-factors would not contain the edge $a\,a_1$, and so it would have a component of odd order as a subgraph of A_1. As r is odd, this is impossible.

[Note. We can vary this construction so that G is not d-regular.]

Next suppose that d is even. Let G be a d-regular simple graph of odd order. Since G is d-regular, if G were the union of $x = \frac{d}{r}$ $(r, r+1)$-factors, then all these $(r, r+1)$-factors would be r-factors. But as r is odd and $|V(G)|$ is odd, this is impossible.

Case 2. *r is even and $x = \frac{d+1}{r+1} \geq 2$.*

Suppose that d is even. We construct a $(d, d+1)$-simple graph G which has no $(r, r+1)$-factorization into x $(r, r+1)$-factors. The construction is similar to one in the previous case. Let $A_1, \ldots, A_d$ be disjoint $(d, d+1)$-simple graphs of odd order with each A_i having exactly one vertex, a_i, of degree d; since d is even, this is possible. Let a be a further vertex and form G by joining a to each of $a_1, \ldots, a_d$. Then G has exactly one vertex of degree d, namely a; also $|V(G)|$ is odd. If G had an $(r, r+1)$-factorization into x $(r, r+1)$-factors, then $x-1$ of these $(r, r+1)$-factors would be $(r+1)$-factors. But since $r+1$ is odd and $|V(G)|$ is odd, this is impossible.

Next suppose that d is odd. Let G be a $(d+1)$-regular graph of odd order. Since G is $(d+1)$-regular, if G had an $(r, r+1)$-factorization into x $(r, r+1)$-factors, then all these $(r, r+1)$-factors would be $(r+1)$-factors. Again this is impossible since $r+1$ is odd and $|V(G)|$ is odd.

(iii) If $x \notin \left[\frac{d+1}{r+1}, \frac{d}{r}\right]$, then either $x < \frac{d+1}{r+1}$ or $x > \frac{d}{r}$.

Suppose first that $x < \frac{d+1}{r+1}$. Then $r+1 < \frac{d+1}{x}$, so if a vertex has degree $d+1$, then it cannot be the union of x $(r, r+1)$-factors. But if G were d-regular and G were the union of x $(r, r+1)$-factors, then these factors would have to be $(r+1)$-factors, and x would have to satisfy $x = \frac{d}{r+1}$. If $(r+1)$ is even, then, by Theorem 4 (Petersen's theorem), G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors. If $(r+1)$ is odd and d is odd, we can construct a d-regular simple graph which does not have an $(r, r+1)$-factorization into x $(r, r+1)$-factors. The construction is the same as that in part (ii), Case 1, with very minor changes.

Now suppose that $x > \frac{d}{r}$. Then $r > \frac{d}{x}$, so if a vertex has degree d, then it cannot be the union of x $(r, r+1)$-factors. If G were $(d+1)$-regular and G were the union of x $(r, r+1)$-factors, then these factors would have to be r-factors, and x would have to satisfy $x = \frac{d+1}{r}$. If r is even, then, by Theorem 4, G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors. If r is odd and $d+1$ is odd, then there are d-regular simple graphs with no $(r, r+1)$-factorization into x $(r, r+1)$-factors. Their construction is similar to that in part (ii), Case 2. $\square$

4.2 Application to d-regular simple graphs

We use Theorem 15 to prove Theorem 10.

Proof of Theorem 10.
(i) By Theorem 15(i), G has an $(r, r+1)$-factorization if $\frac{d+1}{r+1} < x < \frac{d}{r}$, or if $x = \frac{d}{r}$ and r is even, or if $x = \frac{d+1}{r+1}$ and r is odd.

We may suppose that $x \neq 1$. If $x = \frac{d+1}{r+1}$ then $r+1 = \frac{d+1}{x} > \frac{d}{x} > r$, so $x \nmid d$. Therefore, by Theorem 14, G has an equitable edge-colouring with x colours. Each colour class is an $(r, r+1)$-factor, and the equitable colouring is an $(r, r+1)$-factorization with x $(r, r+1)$-factors.

It remains to consider the cases $\frac{d}{r+1} < x < \frac{d+1}{r+1}$ and $x = \frac{d}{r+1}$, r odd. The first of these is empty since we have $d < x(r+1) < d+1$. The second follows from Theorem 4 (Petersen's theorem), since $r+1$ is even.

In the proof of Theorem 15(ii), there are suitable examples of regular simple graphs which have $(r, r+1)$-factorizations into x $(r, r+1)$-factors [replace the occurrences of '$d+1$' in that proof by 'd' in this proof].

(ii) If r is odd, $r | d$ and $x = \frac{d}{r}$, the d-regular graphs considered in the proof of Theorem 15(ii), Case 1, provide suitable examples.

If r is even, $r+1 \mid d$ and $x = \frac{d}{r+1}$, then in any $(r, r+1)$-factorization of G into x $(r, r+1)$-factors, all the $(r, r+1)$-factors are $(r+1)$-regular. Then the d-regular graphs considered in the proof of Theorem 15(ii), Case 1, will again suffice (with r odd being replaced in the argument by $r+1$ odd).

(iii) If $x < \frac{d}{r+1}$, then the union of x $(r, r+1)$-graphs has maximum degree at most $x(r+1) < d$. Similarly if $x > \frac{d}{r}$, then the union of x $(r, r+1)$-graphs has minimum degree at least $xr > d$. □

4.3 An analogue for $(d, d+1)$-simple graphs of the Era-Egawa theorem

We can use the argument of Theorem 11 to give an analogue for $(d, d+1)$-simple graphs of the Era-Egawa result. Let $\psi(r)$ be the smallest integer such that, if $d \geq \psi(r)$, then any $(d, d+1)$-simple graph has an $(r, r+1)$-factorization. Clearly $\psi(r) \geq \phi(r)$.

Theorem 17.

$$\psi(r) = \begin{cases} r(r+1) & \textit{when } r \textit{ is even,} \\ r(r+1)+1 & \textit{when } r \textit{ is odd.} \end{cases}$$

Proof. Suppose first that r is even. From Theorem 15 it follows that each $(d, d+1)$-simple graph has an $(r, r+1)$-factorization if and only if there is an integer x such that

$$\frac{d+1}{r+1} < x \leq \frac{d}{r}.$$

We first note that if $d = r(r+1) - 1$ then there is no such integer x. For if there were such an x, then $x \leq \frac{r(r+1)-1}{r}$, so $x \leq r + 1 - \frac{1}{r}$, so, as x is an integer, $x \leq r$. But on the other hand $\frac{d+1}{r+1} = r$, so $x > r$, a contradiction. Therefore $\psi(r) \geq r(r+1)$. Now suppose that $d \geq r(r+1)$. Note that

$$\frac{d}{r} - \frac{d+1}{r+1} = \frac{d-r}{r(r+1)} > \frac{r^2-1}{r(r+1)} = \frac{r-1}{r}.$$

Hence

$$\frac{d+1}{r+1} < \frac{d}{r} - \frac{r-1}{r} \leq \left\lfloor \frac{d}{r} \right\rfloor < \frac{d}{r}.$$

So $x = \left\lfloor \frac{d}{r} \right\rfloor$ is the required integer. Therefore $\psi(r) = r(r+1)$.

Now suppose that r is odd. From Theorem 15, it follows that each $(d, d+1)$-simple graph has an $(r, r+1)$-factorization if and only if there is an integer x such that

$$\frac{d+1}{r+1} \leq x < \frac{d}{r}.$$

We first note that if $d = r(r+1)$ then there is no such integer x. For if there were such an x, then $x < \frac{r(r+1)}{r} = r+1$, so $x \le r$. But on the other hand, $x \ge \frac{d+1}{r+1} = \frac{r(r+1)+1}{r+1} = r + \frac{1}{r+1}$, so $x \ge r+1$, a contradiction. Therefore $\psi(r) \ge r(r+1)+1$. Now suppose that $d \ge r(r+1)+1$. Note that

$$\frac{d}{r} - \frac{d+1}{r+1} = \frac{d-r}{r(r+1)} > \frac{r^2}{r(r+1)} = \frac{r}{r+1}.$$

Hence

$$\frac{d+1}{r+1} \le \left\lceil \frac{d+1}{r+1} \right\rceil \le \frac{d+1}{r+1} + \frac{r}{r+1} < \frac{d}{r},$$

so $x = \left\lceil \frac{d+1}{r+1} \right\rceil$ is the required integer. Therefore $\psi(r) = r(r+1)+1$. $\square$

5 Pseudographs

5.1 $(d, d+1)$-pseudographs

The $(r, r+1)$-factorization properties for $(d, d+1)$-pseudographs contrast very noticeably with the $(r, r+1)$-factorization properties for $(d, d+1)$-simple graphs. The position is described in the following theorem.

Theorem 18.

(a) Let G be a $(d, d+1)$-pseudograph. Then

- *(i) If r is even, $r \mid d$ and $x = \frac{d}{r}$, then G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors;*
- *(ii) If r is odd, $r+1 \mid d+1$ and $x = \frac{d+1}{r+1}$, then G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors.*

(b) If $x \notin \left[\frac{d}{r+1}, \frac{d+1}{r}\right]$ then no $(d, d+1)$-pseudograph has an $(r, r+1)$-factorization into x $(r, r+1)$-factors.

(c) If $x \in \left[\frac{d}{r+1}, \frac{d+1}{r}\right]$ but x, r, d do not satisfy (a)(i) or (a)(ii) then there are examples of $(d, d+1)$-pseudographs which do and examples which do not have an $(r, r+1)$-factorization into x $(r, r+1)$-factors.

Thus condition (a) of Theorem 18 is very restrictive, contrasting with the comparatively unrestrictive conditions of Theorem 15(i).

For the proof of Theorem 18 we recall the following theorem of König [15].

Theorem 19. *Let G be a bipartite multigraph. Then $\chi'(G) = \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G.*

Proof of Theorem 18.

(a) We may assume that $x \geq 2$.

(a)(i) Let E be a maximal set of independent edges such that each edge of E joins a pair of vertices of degree $d+1$. Let A be the set of vertices of $G-E$ which have degree $d+1$. If $A = \phi$ then we can pass straight to the point marked (*) in the next paragraph. Otherwise A is a non-empty independent set. Let L be a set of loops such that no vertex has more than one loop of L on it, and each loop of L is incident with a vertex of degree (in $G-E$) $d+1$. Then $V(L) \subseteq A$. Now let B be the induced bipartite subgraph of $G-E$ with vertex sets $A - V(L)$ and $V(G) - A$. Each vertex of $A - V(L)$ has degree (in B) $d+1$, and each vertex of $V(G) - A$ has degree (in B) at most d. By Theorem 19 (König's theorem) $\chi'(B) = d+1$, so B can be properly edge-coloured with $d+1$ colours. Taking any colour class gives us a matching M from $A - V(L)$ into $V(G) - A$.

The graph $G - (E \cup L \cup M)$ is a $(d-1, d)$-pseudograph. Since r is even, it follows that d is also even. Pair off the vertices of degree $d-1$, and join each pair by a new edge; let N be the set of new edges. Then $(G - (E \cup L \cup M)) + N$ is regular of degree d. (*) By Theorem 4 (Petersen's theorem), $(G - (E \cup L \cup M)) + N$ is r-factorizable into $x = \frac{d}{r}$ r-factors. Removing the edges of N, we see that $G - (E \cup L \cup M)$ is $(r-1, r)$-factorizable into x $(r-1, r)$-factors. Each vertex which has degree $r-1$ in one of the $(r-1, r)$-factors has perforce degree r in all the other r-factors; also no loop of L is incident with a vertex which is incident with any edge of M. Therefore we can assign the edges of $L \cup M$ to the various $(r-1, r)$-factors in such a way that $G - E$ is $(r, r+1)$-factorized into x $(r, r+1)$-factors. For each edge $e \in E$, the vertices incident with e have degree r in all the $(r, r+1)$-factors; therefore they can be assigned arbitrarily to the $(r, r+1)$ factors of $G - E$, and produce an $(r, r+1)$-factorization of G into x $(r, r+1)$-factors.

(a)(ii) As r is odd, $r+1$ is even, so $d+1$ is even, and so d is odd. Therefore G has an even number of vertices of degree d. We pair these vertices off, and then join the vertices of each such pair by a further edge. Let E be the set of all such further edges. Then $G + E$ is a $(d+1)$-regular pseudograph. By Theorem 4 (Petersen's theorem), $G + E$ is the union of $x = \frac{d+1}{r+1}$ $(r+1)$-factors. Now remove the edges of E from each of these $(r+1)$-factors. Then G is expressed as the union of x $(r, r+1)$-factors.

(b) Suppose that G is a $(d, d+1)$-pseudograph and that G has an $(r, r+1)$-factorization with x $(r, r+1)$-factors. If $x < \frac{d}{r+1}$ then $x(r+1) < d$, so the maximum degree of G is less than d, a contradiction. Similarly if $x > \frac{d+1}{r}$ then $xr > d+1$, so the minimum degree of G is greater than $d+1$, a contradiction. Therefore $x \in \left[\frac{d}{r+1}, \frac{d+1}{r}\right]$.

(c) First we show that, whenever $x \in \left[\frac{d}{r+1}, \frac{d+1}{r}\right]$ but x, r, d do not satisfy (a)(i) or (a)(ii), then nonetheless there are examples of $(d, d+1)$-pseudographs which do have a factorization into x $(r, r+1)$-factors.

By Theorem 15(i), if $\frac{d+1}{r+1} < x < \frac{d}{r}$, or if $x = \frac{d}{r}$ when r is even, or $x = \frac{d+1}{r+1}$ when r is odd, then any $(d, d+1)$-simple graph is the union of x $(r, r+1)$-factors.

If $x = \frac{d}{r}$ and r is odd, then we can construct a suitable d-regular pseudograph G with $|V(G)| = 2m$ by letting G be the union of x r-regular pseudographs of order $2m$. A similar construction works if $x = \frac{d+1}{r}$. If $x = \frac{d+1}{r+1}$ and r is even, then we may let G be the union of x $(r+1)$-regular pseudographs of order $2m$, and if $x = \frac{d}{r+1}$ a similar construction also works.

There are no further values of x satisfying $\frac{d}{r+1} \leq x \leq \frac{d+1}{r}$ for if $\frac{d}{r+1} < x < \frac{d+1}{r+1}$ then $d < x(r+1) < d+1$, which is impossible, and it is similarly impossible for x to satisfy $\frac{d}{r} < x < \frac{d+1}{r}$.

Next we show that whenever $x \in \left[\frac{d}{r+1}, \frac{d+1}{r}\right]$ but x, r, d do not satisfy (a)(i) or (a)(ii), then there are examples of $(d, d+1)$-pseudographs which do not have an $(r, r+1)$-factorization into x $(r, r+1)$-factors.

We consider two cases. We may suppose that $x \geq 2$.

Case 1. *r even.* Then we have that $x \in \left[\frac{d}{r+1}, \frac{d+1}{r}\right]$ and either $r \nmid d$, or $r | d$ but $x \neq \frac{d}{r}$.

If d is even but $r \nmid d$, then let G have one vertex and $\frac{d}{2}$ loops. Clearly G has no $(r, r+1)$-factorization.

If d is odd, then $r \nmid d$. If also $r \nmid d-1$, let G have two vertices, a and b, let a and b be joined by exactly one edge, and let a and b both have $\frac{1}{2}(d-1)$ loops on them. Then G is d-regular. Suppose G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors. Since r is even, all except one of the $(r, r+1)$-factors are r-regular, and the remaining $(r, r+1)$-factor must be an $(r+1)$-factor. Therefore $r | d-1$, a contradiction.

If d is odd, so that $r \nmid d$, but $r | d-1$, and if $r \neq 2$, let G be a pseudograph with one vertex and $\frac{d+1}{2}$ loops on it. Then each $(r, r+1)$-factor must in fact be an r-factor, so $r | d+1$. But this is impossible since $r | d-1$ and $r \neq 2$.

Now suppose that d is odd and $r = 2$. Let G have two components, G_1 and G_2. Let G_1 consist of one vertex, a, with $\frac{d+1}{2}$ loops on it. Let G_2 consist of two vertices, b and c, with one edge joining b and c, and $\frac{d-1}{2}$ loops on each of b and c. The only $(r, r+1)$-factorization of G_1 consists of $\frac{d+1}{2}$ 2-factors (each a loop). Thus the only possibility for x is $x = \frac{d+1}{2}$. But then G_2 cannot be decomposed into $\frac{d+1}{2}$ $(2,3)$-factors as its highest degree is d.

Now suppose that $r | d$ but $x \neq \frac{d}{r}$. Since r is even, d is also even. Let G be a graph with one vertex and $\frac{d}{2}$ loops on it. Then the only $(r, r+1)$-factorization of G that can exist is one in which each factor is an r-factor, and there are exactly $\frac{d}{r}$ such r-factors. But $x \neq \frac{d}{r}$.

Case 2. *r odd.* Then we have that $x \in \left[\frac{d}{r+1}, \frac{d+1}{r}\right]$ and either $r+1 \nmid d+1$, or $r+1 | d+1$ but $x \neq \frac{d+1}{r+1}$.

If d is odd and $r+1 \nmid d+1$, let G have one vertex with $\frac{1}{2}(d+1)$ loops on it. Clearly G has no $(r, r+1)$-factorization.

If d is even, then $r+1 \nmid d+1$. If also $r+1 \nmid d+2$, let G have two vertices, a and b, each with $\frac{1}{2}d$ loops on them, and let a and b be joined by one edge. Then G is $(d+1)$-regular. Since $r+1$ is even, all except one of the $(r, r+1)$-factors is an $(r+1)$-factor, and the remaining $(r, r+1)$-factor must be an r-factor. Therefore $r+1 | d+2$, a contradiction.

If d is even, $r+1 \nmid d+1$, but $r+1 | d+2$ and $r \neq 1$, let G have one vertex with $\frac{d}{2}$ loops on it. Each $(r, r+1)$-factor of G must be an $(r+1)$-factor, so if G had an $(r, r+1)$-factorization then $(r+1) | d$. But this is impossible since $r+1 | d+2$ and $r \neq 1$.

If d is even and $r = 1$ (so that $r+1 = 2$), let G have two components G_1 and G_2. Let G_1 have one vertex, a, with $\frac{d}{2}$ loops on it, and let G_2 have two vertices, b and c, each with $\frac{d}{2}$ loops on them, and let G_2 have one edge joining b and c. The only $(r, r+1)$-factorization that G_1 could have is the $(r+1)$-factorization of G_1 into $\frac{d}{2}$ $(r+1)$-factors. But then the maximum degree of G_2 is $d+1$, which is too high to be contained in the union of $\frac{d}{2}$ $(1,2)$-factors. □

5.2 d-regular pseudographs

In the case of d-regular pseudographs, we can modify Theorem 18 to get the following analogue of Theorem 10.

Theorem 20.

(a) Let G be a d-regular pseudograph. Then G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors in the following cases:

(i) r *even,* $r|\,d$ *and* $x = \frac{d}{r}$ *,*

(ii) r *even,* $r|\,d-1$ *and* $x = \frac{d-1}{r}$ *,*

(iii) r *odd,* $r+1|\,d$ *and* $x = \frac{d}{r+1}$ *,*

(iv) r *odd,* $r+1|\,d+1$ *and* $x = \frac{d+1}{r+1}$ *.*

(b) If $x \notin \left[\frac{d}{r+1}, \frac{d}{r}\right]$*, then no* d*-regular pseudograph has an* $(r, r+1)$*-factorization into* x r*-factors.*

(c) If $x \in \left[\frac{d}{r+1}, \frac{d}{r}\right]$*, but* x*,* r*,* d *do not satisfy any of (a)(i)-(a)(iv), then there are examples of* d*-regular pseudographs which do and examples of* d*-regular pseudographs which do not have an* $(r, r+1)$*-factorization into* x $(r, r+1)$*-factors.*

Proof. (of Theorem 20)

(a) (i) and (iii) follow from Theorem 18(a)(i) and (a)(ii) respectively. In fact in (i) the factors are r-regular, and in (iii) the factors are $(r+1)$-regular.

In the case of (ii) we have that r is even, $r|\,d-1$ and $x = \frac{d-1}{r}$. By Theorem 1, the pseudograph G has an $(r, r+1)$-factor F. The pseudograph $G - E(F)$ is a $(d-r-1, d-r)$-pseudograph. Note that $r|\,d-r-1$, so, by Theorem 18, $G - E(F)$ has an $(r, r+1)$-factorization into $\frac{d-r-1}{r}$ $(r, r+1)$-factors. Therefore G has an $(r, r+1)$-factorization into $\frac{d-r-1}{r} + 1 = \frac{d-1}{r}$ factors.

In the case of (iv) we have that r is odd, $r+1|\,d+1$ and $x = \frac{d+1}{r+1}$. Since r is odd, $r+1$ is even, so $d+1$ is even, and so d is odd. Therefore $|V(G)|$ is even, so we can pair off all the vertices. We join each of these pairs of vertices by an extra edge. Let E be the set of all the extra edges. Then $G+E$ is regular of degree $d+1$. By Theorem 4 (Petersen's theorem), $G+E$ has an $(r+1)$-factorization into $\frac{d+1}{r+1}$ $(r+1)$-factors. Removing the edges of E leaves an $(r, r+1)$-factorization of G into $(r, r+1)$-factors.

(b) This is the same as the proof of Theorem 10(iii).

(c) We show first that if $x \in \left[\frac{d}{r+1}, \frac{d}{r}\right]$, but if x, r, d do not satisfy any of (a)(i)-(a)(iv), then there are examples of d-regular pseudographs which have an $(r, r+1)$-factorization into x r-factors. By Theorem 10(i), if G is a simple graph and $\frac{d}{r+1} < x < \frac{d}{r}$, or if r is odd and $x = \frac{d}{r+1}$, or if r is even and $x = \frac{d}{r}$, then G has an $(r, r+1)$-factorization into

x $(r, r+1)$-factors. If r is even and $x = \frac{d}{r+1}$, then we may form a d-regular pseudograph of even order from x $(r+1)$-regular pseudographs on the same vertex set. Similarly if r is odd and $x = \frac{d}{r}$, we form a d-regular pseudograph of even order from x r-regular pseudographs on the same vertex set.

Next we show that whenever $x \in \left[\frac{d}{r+1}, \frac{d}{r}\right]$ but x, r, d do not satisfy any of (a)(i)-(a)(iv), then there are examples of d-regular pseudographs which do not have an $(r, r+1)$-factorization into x $(r, r+1)$-factors. We consider two cases.

Case 1. *r even.* We consider the possibilities that $r \nmid d-1$ and $r \nmid d$, or $r \mid d-1$ and $x \neq \frac{d-1}{r}$ or $r \mid d$ and $x \neq \frac{d}{r}$.

If d is even, then a suitable example is provided by one vertex with $\frac{d}{2}$ loops on it. This has only one $(r, r+1)$-factorization, and this is when $r \mid d$ and $x = \frac{d}{r}$.

If d is odd, a suitable example is provided by a pseudograph with two vertices a and b, each with $\frac{d-1}{2}$ loops on it, and with a and b joined by one edge. This has only one $(r, r+1)$-factorization, and this is when $r \mid d-1$ and $x = \frac{d-1}{r}$; there are $x-1$ r-factors and one $(r+1)$-factor.

Case 2. *r odd.* We consider the possibilities that $r+1 \nmid d$ and $r+1 \nmid d+1$, or $r+1 \mid d$ and $x \neq \frac{d}{r+1}$, or $r+1 \mid d+1$ and $x \neq \frac{d+1}{r+1}$.

If d is even, then a suitable example is provided by one vertex with $\frac{d}{2}$ loops on it. This has only the $(r, r+1)$-factorization with $\frac{d}{r+1}$ $(r+1)$-factors.

If d is odd, a suitable example is provided by a pseudograph with two vertices, a and b, each with $\frac{d-1}{2}$ loops, and with one edge joining a and b. This has only the $(r, r+1)$-factorization with $\left(\frac{d+1}{r+1} - 1\right)$ $(r+1)$-factors and one r-factor, making $\frac{d+1}{r+1}$ factors in all. $\square$

6 d-regular multigraphs (without loops)

6.1 Bounds on the number of $(r, r+1)$-factors

As each multigraph is also a pseudograph (a pseudograph without loops). Theorem 20(a) gives some cases in which a d-regular multigraph has an $(r, r+1)$-factorization into x $(r, r+1)$-factors. However much more is true when no loops are present.

First we note the following additional bounds for the number of $(r, r+1)$-factors in an $(r, r+1)$-factorization of a d-regular multigraph.

Lemma 21. *Let G be a d-regular multigraph of odd order, $2n+1$, let $1 \leq r < d$, and suppose that G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors. Then*

(a) $x \leq \frac{d(2n+1)}{r(2n+1)+1}$ *if r is odd,*

(b) $x \geq \frac{d(2n+1)}{(r+1)(2n+1)-1}$ *if r is even.*

Proof.

(a) Since $|V(G)| = 2n+1$, which is odd, we have that when r is odd, each $(r, r+1)$-factor has at least one vertex of degree $r+1$ so the number of edges in an $(r, r+1)$-factor is at least $\frac{(2n+1)r+1}{2}$. The total number of edges is $\frac{d(2n+1)}{2}$, so $x \leq \frac{d(2n+1)}{(2n+1)r+1}$.

(b) Similarly, when r is even, each $(r, r+1)$-factor has at least one vertex of degree r so the number of edges in an $(r, r+1)$-factor is at most $\frac{(2n+1)(r+1)-1}{2}$. The total number of edges is $\frac{d(2n+1)}{2}$, so $x \geq \frac{d(2n+1)}{(r+1)(2n+1)-1}$.

□

The argument which establishes the bounds for the number of $(r, r+1)$-factors in an $(r, r+1)$-factorization of a simple graph G given in Theorem 10 also gives the same bounds if G is a multigraph. Combining this with Lemma 21 we obtain:

Theorem 22. *Let G be a d-regular multigraph, let $1 \leq r < d$, and suppose that G has an $(r, r+1)$-factorization into x $(r, r+1)$ factors. Then*

(i) If $|V(G)|$ is even and r is even then

$$\frac{d}{r+1} < x \leq \frac{d}{r} .$$

(ii) If $|V(G)|$ is even and r is odd, then

$$\frac{d}{r+1} \leq x < \frac{d}{r} .$$

(iii) If $|V(G)| = 2n+1$ and r is even, then

$$\frac{d(2n+1)}{(r+1)(2n+1)-1} \leq x \leq \frac{d}{r} .$$

(iv) If $|V(G)| = 2n+1$ *and* r *is odd, then*

$$\frac{d}{r+1} \le x \le \frac{d(2n+1)}{r(2n+1)+1} .$$

Notice that the bounds of Lemma 21 really are stronger than those of $\frac{d}{r+1} \le x \le \frac{d}{r}$. For example, suppose that G has three vertices and 29 edges between each pair of vertices; then $d = 58$. If $r = 3$ then the bound $x \le \frac{d}{r}$ tells us only that $x \le 19$, whereas the bound of Lemma 21 tells us that $x \le 17$. Similarly, if $r = 2$ then the bound $x \ge \frac{d}{r+1}$ tells us only that $x \ge 20$, whereas the bound of Lemma 21 tells us that $x \ge 22$.

The converse of Theorem 22 is not true in the sense that there are values of x satisfying the various inequalities and d-regular graphs G for which G does not have an $(r, r+1)$-factorization into x $(r, r+1)$-factors.

Theorem 23. *Let* x *satisfy the inequalities of Theorem 22, but let*

(i) $x < \frac{3d}{3r+2}$ *if* d *and* r *are even;*

(ii) $x > \frac{3d}{3r+1}$ *if* d *is even and* r *is odd;*

(iii) $x < \frac{3d-1}{3r+2}$ *if* d *is odd and* r *is even;*

(iv) $x > \frac{3d+1}{3r+1}$ *if* d *and* r *are odd.*

Then there are d*-regular multigraphs* G *which do not have an* $(r, r+1)$*-factorization into* x $(r, r+1)$*-factors.*

Proof. First suppose that d is even. Consider a d-regular multigraph G with a component C of order 3 in which each pair of vertices is joined by $\frac{d}{2}$ edges. By Theorem 22(iii) if r is even, or by Theorem 22(iv) if r is odd, C does not have an $(r, r+1)$-factorization into x $(r, r+1)$-factors, and so neither does G.

Next suppose that d is odd. Consider a d-regular multigraph with a component C of order 6 with vertices a_{i1}, a_{i2}, a_{i3} $(i = 1, 2)$. Suppose that, for $i = 1, 2$, a_{i1} is joined to each of a_{i2} and a_{i3} by $\frac{d-1}{2}$ edges, and that a_{i2} and a_{i3} are joined by $\frac{d+1}{2}$ edges. Also suppose that a_{11} and a_{21} are joined by an edge.

If r is odd, then in any $(r, r+1)$-factorization of G, there are two possibilities for the factor F of G containing the bridge $a_{11}\,a_{21}$. One possibility is that F consists of the bridge $a_{11}\,a_{21}$, $\frac{r-1}{2}$ edges joining a_{11} to each of a_{12} and a_{13}, and $\frac{r+1}{2}$ edges joining a_{12} to a_{13}. The other possibility is that F

contains the bridge $a_{11}\,a_{21}$, $\frac{r+1}{2}$ edges joining a_{12} to each of a_{11} and a_{13}, and $\frac{r-1}{2}$ edges joining a_{11} and a_{13}, or an isomorphic graph with $\frac{r+1}{2}$ edges joining a_{13} to each of a_{11} and a_{12}, and $\frac{r-1}{2}$ edges joining a_{12} and a_{13}. The graph $G-F$ contains in the one case a component C' which is $(d-r)$-regular of order 3 on the vertices a_{11}, a_{12} and a_{13}, and in the other case a component C'' in which a_{13} is joined to each of a_{11} and a_{12} by $\frac{d-r}{2}$ edges, and a_{11} and a_{12} are joined by $\frac{d-r}{2}-1$ edges, or a multigraph isomorphic to this; C'' is thus a $(d-r-1,\, d-r)$-multigraph with one vertex of degree $d-r$ and two of degree $d-r-1$.

In the first case, if G has an $(r,r+1)$-factorization into x $(r,r+1)$-factors in which one of the factors is F, then by applying Theorem 22(iv) to C' we see that $x-1 \leq \frac{3(d-r)}{3r+1}$ so that $x \leq \frac{3d+1}{3r+1}$. In the second case, from Theorem 27(iv) applied to C'', it follows that $x-1 \leq \frac{3(d-r)}{3r+4}$ so $x \leq \frac{3d-2}{3r+4}$. Theorem 27 does not, of course, depend on this theorem. It now follows that if $x > \frac{3d+1}{3r+1}$ then there are d-regular multigraphs with no $(r,r+1)$-factorization into x $(r,r+1)$-factors.

Now suppose that r is even (and that d is still odd). There are three possibilities for the factor F of G containing the bridge $a_{11}\,a_{21}$. One possibility is that F contains the bridge $a_{11}\,a_{21}$, $\frac{r}{2}-1$ edges joining a_{11} to a_{12}, $\frac{r}{2}$ edges joining a_{11} to a_{13}, and $\frac{r}{2}+1$ edges joining a_{12} to a_{13}. The second possibility is that F contains the bridge $a_{11}\,a_{21}$, and $\frac{r}{2}$ edges joining each of the pairs $\{a_{11},a_{12}\}$, $\{a_{11},a_{13}\}$, $\{a_{12},a_{13}\}$. The third possibility is that F contains the bridge $a_{11}a_{21}$, $\frac{r}{2}$ edges joining a_{11} to each of a_{12} and a_{13}, and $\frac{r}{2}+1$ edges joining a_{12} to a_{13}. In the first two cases, the graph $G-F$ contains a component C'' which is a $(d-r-1,\, d-r)$-multigraph with one vertex of degree $d-r-1$ and two of degree $d-r$. In the third case $G-F$ contains a component C' which is $(d-r-1)$-regular of order 3 on the vertices a_{11}, a_{12} and a_{13}.

In the first and second cases, if G has an $(r,r+1)$-factorization into x $(r,r+1)$-factors in which one of the factors is F, then by applying Theorem 27(iii) to C'' it follows that $x-1 \geq \frac{3(d-r-1)}{3r+2}$ so that $x \geq \frac{3d-1}{3r+2}$. In the third case, by applying Theorem 22(iii) to C' it also follows that $x \geq \frac{3d-1}{3r+2}$. $\square$

In the light of Theorems 22 and 23, we make the following conjecture.

Conjecture 1. Let x, d and r be integers with $d \geq r \geq 1$. If x, d and r satisfy the appropriate inequality below, then any d-regular multigraph has an $(r,r+1)$-factorization with x $(r,r+1)$-factors.

(i) d and r are even and $\frac{3d}{3r+2} \leq x \leq \frac{d}{r}$;

(ii) d is even and r is odd, and $\frac{d}{r+1} \leq x \leq \frac{3d}{3r+1}$;

(iii) d is odd and r is even, and $\frac{3d-1}{3r+2} \leq x \leq \frac{d}{r}$;

(iv) d is odd and r is odd, and $\frac{d}{r+1} \leq x \leq \frac{3d+1}{3r+1}$.

In the constructions in the papers of Era [6] and Egawa [5] which form the basis of the proof of the next theorem, Theorem 24, we may observe that $x = \left\lfloor \frac{d}{r} \right\rfloor$ if r is even (cases (i) and (iii) of Conjecture 1) and $x = \left\lceil \frac{d}{r+1} \right\rceil$ if r is odd (cases (ii) and (iv) of Conjecture 1).

6.2 An analogue for d-regular multigraphs of the Era-Egawa theorem

We have not been able to obtain a complete analogue of Theorem 11 for d-regular multigraphs. Let $\phi_m(r)$ be the smallest integer such that, if $d \geq \phi_m(r)$, then any d-regular multigraph has an $(r, r+1)$-factorization. The subscript m in $\phi_m(r)$ stands for 'multigraph'.

Theorem 24. *$\phi_m(0) = 0$, $\phi_m(2) = 2$ and, if r is even, $r \geq 4$, then*

$$\frac{3}{2}\,r^2 - 2r \leq \phi_m(r) \leq 2r^2 - 3r\,.$$

If r is odd, then

$$\phi_m(r) = r^2 + 1\,.$$

Theorem 24 shows that $\phi_m(4) = 20$; it also shows that $\phi_m(r) = \phi(r)$ when r is odd, but Theorem 11 and Theorem 24 show that $\phi(r) \neq \phi_m(r)$ when r is even, $r \geq 4$.

Proof of Theorem 24. Let us remark that Era's paper [6] refers for its definitions to Harary [10] for whom 'graph' means 'simple graph', and so Era's paper is almost explicitly about simple graphs. Egawa's paper [5] is explicitly about simple graphs. Nonetheless everything in Era's paper works perfectly for multigraphs (but not pseudographs), and some of Egawa's paper works for multigraphs.

If r is odd, we learn by reading Era's Proposition 2 and taking his 'graphs' to be 'multigraphs' that $r^2 + 1 \leq \phi_m(r) \leq r^2 + 2$, and the proof is completed by reading part (ii) of Egawa's theorem with 'graphs' meaning 'multigraphs'.

Now suppose that r is even. The result is easy if $r = 0$ or 2, so let $r \geq 4$. Part (i) of Egawa's theorem does not apply here, as it only works for simple graphs. The upper bound $\phi_m(r) \leq 2r^2 - 3r$ comes from Era's Proposition 1 read with 'graphs' meaning 'multigraphs'. It remains to prove the lower bound.

We first show that if $d = \frac{3}{2}r^2 - ar - 2$, where $a = 3$ or 4, and if G is a regular multigraph of order 3 with three multiple edges, each of multiplicity $\frac{3}{4}r^2 - \frac{ar}{2} - 1$, then G is not the edge-disjoint union of $(r, r+1)$-factors. The only possible $(r, r+1)$-factors in this case are the regular multigraph of degree r in which each edge has multiplicity $\frac{r}{2}$, and an $(r, r+1)$-factor in which one edge has multiplicity $\frac{r}{2} + 1$, and the other two edges have multiplicity $\frac{r}{2}$. Since G is regular, the non-regular factors have to come in sets of three arranged so that their union gives a regular multigraph of order 3 and degree $3r + 2$, each edge having multiplicty $\frac{3}{2}r + 1$. But we note that if $0 \leq x \leq \frac{1}{2}(r-4)$ then $r \nmid \left(\frac{3}{2}r^2 - ar - 2)\right) - x(3r+2)$. When $x = \frac{1}{2}(r-4)$ then $\left(\frac{3}{2}r^2 - ar - 2\right) - x(3r+2) = (5-a)r + 2 < 3r + 2$ since $a \in \{3, 4\}$. Therefore G has no $(r, r+1)$-factorization.

Next we show that if $d = \frac{3}{2}r^2 - 2r - 1$ then there is a d-regular multigraph H which has no $(r, r+1)$-factorization. Let H be the multigraph with order 6, vertex set $\{a, b, c, a', b', c'\}$, and edge-set consisting of a bridge cc', edges ab and $a'b'$ having multiplicity $\frac{3}{4}r^2 - r$, and edges ac, bc, $a'c'$, $b'c'$ having multiplicity $\frac{3}{4}r^2 - r - 1$. In any $(r, r+1)$-factor F containing the bridge cc', the submultigraph induced by $\{a, b, c, d\}$ is one of the multigraphs shown in Figure 1.

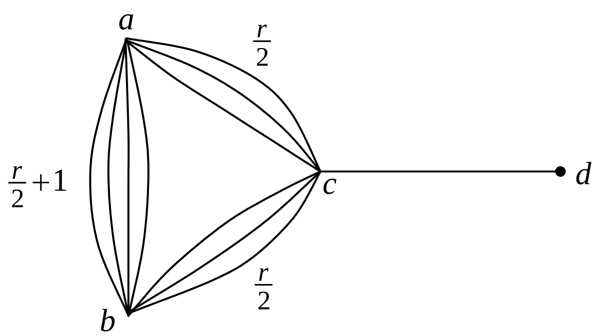
Suppose that an $(r, r+1)$-factorization of H contained a factor F with A as an induced submultigraph. Then the union of the remaining factors would be a multigraph whose restriction to $\{a, b, c\}$ would be a regular multigraph of degree $\frac{3}{2}r^2 - 3r - 2$. But, as we just showed, such a multigraph has no $(r, r+1)$-factorization. Thus we may suppose that no submultigraph of F is isomorphic to A.

Suppose the $(r, r+1)$-factor F had B as an induced submultigraph. Then the union of the remaining factors would be a multigraph whose restriction to $\{a, b, c\}$ would be a multigraph J with edge ab having multiplicity $\frac{3}{4}r^2 - \frac{3}{2}r$ and edges ac and bc having multiplicity $\frac{3}{4}r^2 - \frac{3}{2}r - 1$. The removal of an $(r, r+1)$-factor from J consisting of an edge ab with multiplicity $\frac{r}{2} + 1$ and edges ab and ac with multiplicity $\frac{r}{2}$ would leave a regular multigraph of degree $\frac{3}{2}r^2 - 4r - 2$. But we have shown that such a multigraph does not have an $(r, r+1)$-factorization. Thus we may also suppose that no subgraph of F is isomorphic to B.

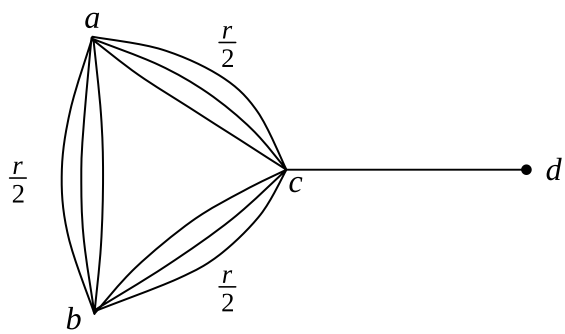
Finally, suppose the $(r, r+1)$-factor F had C as an induced subgraph. Then the union of the remaining factors would have edge ab with multiplicity $\frac{3}{4}r^2 - \frac{3}{2}r - 1$ and edges bc and ca with multiplicity $\frac{3}{4}r^2 - \frac{3}{2}r$. Thus the components of this union would each be isomorphic to the multigraph J in the paragraph above. But we showed that J has no $(r, r+1)$-factorization.

It follows that H has no $(r, r+1)$-factorization, so $\phi_m(r) \geq \frac{3}{2}r^2 - 2r$, as asserted. $\square$

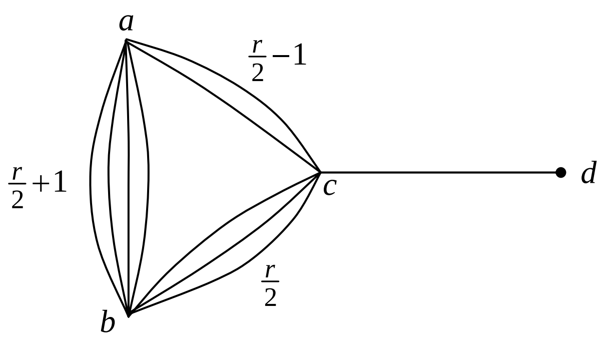
A:



B:



C:



It seems likely that the lower bound $\frac{3}{2}r^2 - 2r \leq \phi_m(r)$ in Theorem 24 is the correct value of $\phi_m(r)$. The next theorem shows that the value $\frac{3}{2}r^2 - 2r$ is probably at least not far from being the correct value of $\phi_m(r)$.

Theorem 25. *If r and d are both even, $r \geq 4,\ d \geq \frac{3}{2}r^2 - 2r - 2$, then any d-regular multigraph has an $(r, r+1)$-factorization.*

Proof of Theorem 25. If we read 'graph' to mean 'multigraph', then the statement and proof of Proposition 1 part (1) of Era's paper [6] gives this result. □

To complete the evaluation of $\phi_m(r)$, we need to show that the inequality $\frac{3}{2}r^2 - 2r \leq \phi_m(r)$ is true if r is even and d is odd.

7 $(d, d+1)$-multigraphs (without loops)

7.1 Bounds on the number of $(r, r+1)$-factors

The analogue of Lemma 21 in this case is:

Lemma 26. *Let G be a $(d, d+1)$-multigraph of odd order $2n+1$, let $1 \leq r \leq d$, and suppose that G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors. Then*

(a) $\quad x \leq \frac{(d+1)(2n+1)}{r(2n+1)+1} \quad$ *if r is odd,*

(b) $\quad x \geq \frac{d(2n+1)}{(r+1)(2n+1)-1} \quad$ *if r is even.*

Proof. (a) This is similar to the proof of Lemma 21(a), noting that the total number of edges is at most $\frac{1}{2}(d+1)(2n+1)$.

(b) This is similar to the proof of Lemma 21(b), noting that the total number of edges is at least $\frac{d(2n+1)}{2}$. □

Using Lemma 26 and the argument for Theorem 15(iii) we obtain:

Theorem 27. *Let G be a $(d, d+1)$-multigraph, let $1 \leq r < d$, and suppose that G has an $(r, r+1)$-factorization into x $(r, r+1)$-factors. Then*

(i) If $|V(G)|$ is even and r is even, then

$$\frac{d+1}{r+1} < x \leq \frac{d}{r};$$

(ii) If $|V(G)|$ is even and r is odd, then

$$\frac{d+1}{r+1} \le x < \frac{d}{r}\,;$$

(iii) If $|V(G)| = 2n+1$ and r is even, then

$$\frac{d(2n+1)}{(r+1)(2n+1)-1} \le x \le \frac{d}{r}\,;$$

(iv) If $|V(G)| = 2n+1$ and r is odd, then

$$\frac{d+1}{r+1} \le x \le \frac{(d+1)(2n+1)}{(r+1)(2n+1)+1}\,.$$

There are values of x satisfying the inequalities of Theorem 27 and $(d, d+1)$-multigraphs G for which G does not have an $(r, r+1)$-factorization into x $(r, r+1)$-factors.

Theorem 28. *Let x satisfy the inequalities of Theorem 27, but*

(i) $x < \frac{3d+2}{3r+2}$ *if d and r are even;*

(ii) $x > \frac{3d}{3r+1}$ *if d is even and r is odd;*

(iii) $x < \frac{3d+3}{3r+2}$ *if d is odd and r is even;*

(iv) $x > \frac{3d+1}{3r+1}$ *if d and r are odd.*

Then there are $(d, d+1)$-multigraphs which do not have an $(r, r+1)$-factorization into x $(r, r+1)$-factors.

Proof. We can apply the bounds of Theorem 23 since a d-regular multigraph is also a $(d, d+1)$-multigraph. Similarly we can also apply the bounds of Theorem 23 slightly indirectly, with $d+1$ replacing d, since a $(d+1)$-regular multigraph is also a $(d, d+1)$-multigraph. Thus, if d and r are even, then if

$$x < \max\left\{\frac{3d}{3r+2}\,,\ \frac{3(d+1)-1}{3r+2}\right\} = \frac{3d+2}{3r+2}\,,$$

there are $(d, d+1)$-multigraphs which do not have an $(r, r+1)$-factorization into x $(r, r+1)$-factors. The similar bound if d is even and r is odd is

$$x > \min\left\{\frac{3d}{3r+1}\,,\ \frac{3(d+1)+1}{3r+1}\right\} = \frac{3d}{3r+1}\,.$$

If d is odd and r is even, the bound is

$$x < \max\left\{\frac{3d-1}{3r+2}\,,\ \frac{3(d+1)}{3r+2}\right\} = \frac{3d+3}{3r+2}\ .$$

If d is odd and r is odd, the bound is

$$x > \min\left\{\frac{3d+1}{3r+1}\,,\ \frac{3(d+1)}{3r+1}\right\} = \frac{3d+1}{3r+1}\ .$$

$\square$

7.2 An analogue for $(d, d+1)$-multigraphs of the Era-Egawa theorem

Let $\psi_m(r)$ be the smallest integer such that, if $d \geq \psi_m(r)$, then any $(d, d+1)$-multigraph has an $(r, r+1)$-factorization. Theorem 3 (the theorem of Akiyama and Kano, and of Cai) shows that

$$\psi_m(r) \leq \begin{cases} 3r^2 + r & \text{if } r \text{ is even,} \\ 3(r+1)^2 & \text{if } r \text{ is odd.} \end{cases}$$

We prove the following analogue of Theorems 17 and 24, giving bounds for $\psi_m(r)$ which are better, but still not best possible.

Theorem 29. *Let $r \in \mathbb{Z}$, $r \geq 1$. Then*
$\frac{3}{2}r^2 - 2r \leq \psi_m(r) \leq 2r^2 + r - 1$
if r is even, and
$r(r+1) + 1 \leq \psi_m(r) \leq 2r^2 + 3r - 1$
if r is odd.

Proof. The proof uses various results which we have yet to prove.

The upper bound follows from Corollary 31 (or Lemma 30) if r is even, and from Lemmas 32 and 34 if r is odd.

When r is even, the lower bound follows from the same lower bound in Theorem 24. Similarly, when r is odd, the lower bound follows from the same lower bound in Theorem 17. $\square$

Lemma 30. *Let r be even and let $d = qr + s - 1$ for some $s \geq 1$ where q, r and s are integers. If $q \geq 2s$ then any $(d, d+1)$-multigraph G has an $(r, r+1)$-factorization with q $(r, r+1)$-factors.*

Corollary 31. *If r is even then $\psi_m(r) \leq 2r^2 + r - 1$.*

Proof. $2r^2 + r - 1 = \max_{1 \le s \le r}(2s)r + s - 1$. $\square$

Proof of Lemma 30.

We remark that we could restrict s so that $s \le r$, but the argument works without this restriction; this fact is used in the proof of Lemma 39.

By Theorem 1, G has a $(d-s, d-s+1)$-factor, i.e. a $(qr-1, qr)$-factor J. Let J be maximal and let $H = G - E(J)$. Then the vertices of H have degrees (in H) in the set $\{s-1,\ s,\ s+1\}$.

Let W be the set of vertices of minimum degree, $qr-1$, in J. Then, in H, W is an independent set, because if two vertices of W are joined in H, then the edge can be removed from H and put into J, contradicting the maximality of J. By Theorem 18(a)(ii), J has an $(r-1, r)$-factorization into q $(r-1, r)$-factors, $A_1, ..., A_q$. For $1 \le i \le q$, let W_i be the set of vertices of A_i of degree (in A_i) $r-1$. Then the W_i are pairwise disjoint sets, and $W = W_1 \cup ... \cup W_q$.

Now let B be a bipartite graph with bipartition $(W, V(H) - W)$ whose edges are the edges of H which are incident with the vertices of W. Then the vertices in W have degrees s or $s+1$, and the vertices of $V(H) - W$ have degrees at most s. By Theorem 12, B has an equitable edge-colouring with s colours. From any one of these colour sets, we can choose a matching M from W into $V(H) - W$. Let M_i be the subset of M consisting of those edges of M which are incident with W_i. The factors $A_i \cup M_i$ are all $(r, r+1)$-factors of G.

We must now add all the edges of $H - M$ to these $(r, r+1)$-factors so that all edges are used, but all the factors remain $(r, r+1)$-factors. We do this by properly colouring the edges of H greedily with q colours $c_1, ..., c_q$, starting with the edges of M_i which we colour c_i $(1 \le i \le q)$. If we wish to colour an uncoloured edge e joining two vertices u and v, then at most $s-1$ colours occur on the edges of one of u and v, say u, and at most s colours occur on the edges on v (this is because W is an independent set). Since $q \ge 2s$, there is a colour available to colour e with.

When H is properly edge-coloured with $c_1, ..., c_q$, we let A_i^+ be A_i together with all the edges of H coloured c_i $(1 \le i \le q)$. Then $A_1^+, ..., A_q^+$ is the required $(r, r+1)$-factorization of G. $\square$

Lemma 32. *Let d and r both be odd positive integers with $d \ge 2r^2 - 3$. Then any $(d, d+1)$-multigraph has an $(r, r+1)$-factorization.*

Proof. Let $d + 1 = (r+1)q + s + 2$ where $0 \le s < r$, let $N = q + s - r + 2$ and let $M = \frac{r+1}{2} - \frac{s}{2} - 1$. Then s is even, so M is a non-negative integer and $s < r$. Since $d \ge 2r^2 - 3$, we have that $(r+1)q + s + 1 \ge 2r^2 - 3$, so

$(r+1)q \geq 2r^2 - s - 4 \geq 2r^2 - r - 3 = (2r-3)(r+1)$, so $q \geq 2r-3$. Therefore

$$N - 2M = q - 2r + 2s + 3 \geq 2s \geq 0\,,$$

so $N \geq 2M$.

Note also that $(r+1)N + 2rM = (r+1)q + s + 2 = d + 1$.

We may also note that, as $s < r$, so $s+1 < r+1$, and so $q = \left\lfloor \frac{d}{r+1} \right\rfloor$.

Let G be a $(d, d+1)$-multigraph and $\tilde{G}$ a $(d+1)$-regular multigraph with G as an induced submultigraph, i.e. deleting the vertices of $V(\tilde{G}) - V(G)$ in $\tilde{G}$ gives G. [We shall use the tilde to denote submultigraphs of $\tilde{G}$ which contain vertices of $V(\tilde{G}) - V(G)$. For a submultigraph $\tilde{Q}$ of $\tilde{G}$, its restriction to G will be denoted by Q, i.e. by omitting the tilde. Note that $\tilde{Q} = Q$ if and only if $V(\tilde{Q}) = V(Q)$.] By Theorem 4 (Petersen's theorem), since $d + 1 = (r+1)N + 2rM$, we can decompose $\tilde{G}$ into N $(r+1)$-factors $\tilde{D}_1, ..., \tilde{D}_N$ and M $2r$-factors $\tilde{H}_1, ..., \tilde{H}_M$. If $M = 0$ then $D_1, ..., D_N$ is the required $(r, r+1)$-factorization, so we shall suppose from now that $M > 0$.

For each i, $1 \leq i \leq M$, consider the components $\tilde{C}$ of $\tilde{H}_i$. Since $2r$ is even, each such component is Eulerian. Starting at any edge on the Eulerian circuit, we can choose a direction, and then, going round the circuit in this direction, we can colour the edges alternately α and β. If $|E(\tilde{C})|$ is even then both the α-submultigraph $A(\tilde{C})$ and the β-submultigraph $B(\tilde{C})$ are r-regular. The number $|E(\tilde{C})|$ is odd if $|V(\tilde{C}|$ is odd since r is odd and $|E(\tilde{C})| = \frac{1}{2}\sum_{v \in V(\tilde{C})} d_{\tilde{C}}(v) = \frac{1}{2} \times |V(\tilde{C})| \times 2r = |V(\tilde{C})| \times r$. In this case $A(\tilde{C})$ is an $(r, r+1)$-multigraph which has exactly one vertex, the initial vertex of the colouring, say $s(\tilde{C})$, of degree $r+1$, and $B(\tilde{C})$ is an $(r-1,\, r)$-multigraph which has exactly one vertex, again $s(\tilde{C})$, of degree $r-1$. We can choose which vertex of the component to have as $s(\tilde{C})$; unless $C = \tilde{C}$ (so that $\tilde{C}$ contributes no edges to $\tilde{G} - G$) we choose the vertices $s(\tilde{C})$ to lie in $V(\tilde{G}) - V(G)$.

From now on, if we refer to the vertex $s(C)$ or the set $V(C)$ (rather than $s(\tilde{C})$ or $V(\tilde{C})$), we shall be assuming that $C = \tilde{C}$ and $|V(C)|$ is odd. However $A(C)$ and $B(C)$ will be the restrictions of $A(\tilde{C})$ and $B(\tilde{C})$ to G.

Recall that $N \geq 2M$. For $1 \leq i \leq M$ and for each component $\tilde{C}$ of $\tilde{H}_i$, we associate $A(\tilde{C})$ or $B(\tilde{C})$ with the $(r+1)$-factor $\tilde{D}_{2i-1}$ or the $(r+1)$-factor $\tilde{D}_{2i}$ as follows.

(a) If $C = \tilde{C}$ and $|V(C)|$ is odd and, for all or all bar one vertices $v \in V(C)$, $d_{D_{2i-1}}(v) = r$, then we associate $A(\tilde{C})$ with $\tilde{D}_{2i-1}$ and associate $B(\tilde{C})$ with $\tilde{D}_{2i}$.

(b) If $C \neq \tilde{C}$ and $|V(A(C))|$ $(= |V(B(C))|\,)$ is odd and it should happen that

$$d_{D_j}(v) = r \ \Leftrightarrow\ d_{Q(C)}(v) = r \qquad\qquad (\forall v \in V(C))$$

for some $j \in \{2i-1,\, 2i\}$ and some $Q(C) \in \{A(C), B(C)\}$, then we associate D_j with $R(C)$ and D_k with $Q(C)$, where $\{k\} = \{2i-1,\, 2i\} \setminus \{j\}$ and $R(C) = \{A(C),\, B(C)\} \setminus \{Q(C)\}$.

(c) Otherwise we associate $A(\tilde{C})$ with $\tilde{D}_{2i}$ and $B(\tilde{C})$ with $\tilde{D}_{2i-1}$.

Note that, in case (b), as $C \neq \tilde{C}$, if $|V(\tilde{C})|$ is odd then $s(\tilde{C})$ will be in $V(\tilde{G}) - V(G)$. Thus, whether $|V(\tilde{C})|$ is even or odd, all vertices of $A(C)$ and $B(C)$ will have degrees $r-1$ or r. Since $|V(A(C))|$ $(= |V(B(C))|\,)$ is odd, and since r is odd, not all vertices of $A(C)$ or $B(C)$ can have degree r, so at least one of each will have degree $r-1$. Moreover, if a vertex has degree $r-1$ in $A(C)$ (or $B(C)$), then it has degree r in $B(C)$ (or $A(C)$ respectively), and has degree $r+1$ in D_{2i-1} and D_{2i}. Similarly if a vertex has degree r in one of D_{2i-1} and D_{2i}, then it has degree $r+1$ in the other, and has degree r in $A(C)$ and $B(C)$. It follows from all this that, in case (b),

$$d_{D_j}(v) = r \Rightarrow d_{D_j}(v) + d_{R(C)}(v) = 2r+1 \qquad (\forall v \in V(A(C)))$$

and

$$d_{D_k}(v) = r \Rightarrow d_{D_k}(v) + d_{Q(C)}(v) = 2r+1 \qquad (\forall v \in V(B(C))).$$

If $C = \tilde{C}$ and $|V(C)|$ is odd, then we choose $s(C)$ according to the following rules.

(1) If $d_{D_{2i-1}}(v) = r+1$ for exactly one vertex $v \in V(C)$, then $s(C)$ is chosen to be some other vertex $w \in V(C)$. [Then $d_{D_{2i}}(s(C)) = r+1$.]

(2) If $d_{D_{2i-1}}(v) = r+1$ for more than one vertex $v \in V(C)$ and if $d_{D_{2i}}(w) = r$ for some vertex $w \in V(C)$, then we choose $s(C) = w$. [Then $d_{D_{2i-1}}(s(C)) = r+1$.]

(3) If $d_{D_{2i-1}}(v) = r+1$ for more than one vertex $v \in V(C)$ and if $d_{D_{2i}}(w) = r+1$ for all $w \in V(C)$, then we choose $s(C)$ so that $d_{D_{2i-1}}(s(C)) = r+1$.

(4) Otherwise we choose $s(C) \in V(C)$ arbitrarily.

For $1 \leq i \leq M$, we let $\tilde{P}_{2i-1}$ be the union of $\tilde{D}_{2i-1}$ and all its associated multigraphs ($A(\tilde{C})$ or $B(\tilde{C})$, chosen as above) and we let $\tilde{P}_{2i}$ be the union of $\tilde{D}_{2i}$ and its associated multigraphs. Consider a multigraph P_j $(1 \leq j \leq 2M)$. Of course P_j is a submultigraph of G. Because of the way P_j is defined, each vertex has degree $2r+2$, $2r+1$ or $2r$. We can see this as follows. Clearly no vertex has degree greater than $2r+2$. It is also clear that no vertex has degree less than $2r-1$. But also, a vertex v cannot have degree $2r-1$ in P_j. For suppose otherwise. Then $d_{D_j}(v) = r$ and $C = \tilde{C}$, $|V(C)|$ is odd and $d_{B(C)}(v) = r-1$ (so $v = s(C)$). Moreover, P_j contains $B(C)$ as a subgraph. Either $j = 2i-1$ or $j = 2i$ for some i, $1 \leq i \leq M$. By construction, if $j = 2i-1$ then, since $B(C)$ is associated with D_{2i-1}, there are at least two vertices $v \in V(C)$ such that $d_{D_{2i-1}}(v) = r+1$, and so $s(C)$ is chosen according to (2). Therefore $d_{D_{2i-1}}(s(C)) = r+1$, and so $d_{P_j}(s(C)) = 2r$, i.e. $d_{P_j}(v) = 2r$, a contradiction. Now suppose that $j = 2i$. Then $B(C)$ is associated with D_{2i}. This happens if, for all or all bar one vertices $v \in V(C)$, $d_{D_{2i-1}}(v) = r$. For such vertices, $d_{D_{2i}}(v) = r+1$. If all vertices $v \in V(C)$ satisfy $d_{D_{2i-1}}(v) = r$, then $d_{D_{2i}}(v) = r+1$ $(\forall v \in V(C))$, so $d_{D_{2i}}(s(C)) = r+1$. If exactly one vertex $v \in V(C)$ satisfies $d_{D_{2i-1}}(v) = r$, then we chose $s(C)$ by (1), so that $d_{D_{2i}}(s(C)) = r+1$ again. Thus in either case $d_{P_j}(s(C)) = 2r$, a contradiction. Finally we remark that a vertex v can have degree $2r+2$ in P_j only if $d_{D_j}(v) = r+1$ and $C = \tilde{C}$, $|V(C)|$ is odd and $d_{A(C)}(v) = r+1$ (so $v = s(C)$). In this case $j = 2i$ for some i, $1 \leq i \leq M$.

Now pair off the vertices of P_j of degree $2r+1$ and join each pair by an extra edge to form a multigraph P_j^*. Then P_j^* has degrees $2r$ or $2r+2$, so each component K^* of P_j^* is Eulerian. Similarly to before, starting with any edge we can colour the edges of an Eulerian circuit of K^* alternately γ and δ. We can also again choose the initial vertex. We obtain two submultigraphs, the γ-coloured one, $\Gamma(P_j^*)$, and the δ-coloured one, $\Delta(P_j^*)$. If $|E(K^*)|$ is even, then, at each vertex v of K^*, either both have γ- and δ-degree $r+1$, or both have γ- and δ-degree r. Moreover, in the multigraphs $\Gamma(P_j)$ and $\Delta(P_j)$ obtained by removing the extra edges, if a vertex v had degree r in $\Gamma(P_j^*)$ and $\Delta(P_j^*)$, then it still has degree r. Thus $\Gamma(P_j)$ and $\Delta(P_j)$ are both $(r, r+1)$-factors.

Let K be the component K^* with the extra edges removed. Now suppose that $|E(K^*)|$ is odd and that K has a vertex v_0 of degree $2r+1$. Then we can choose the extra edge on v_0 to be the initial edge of the Eulerian circuit of K^*, so that the degree in $\Gamma(P^*)$ of v_0 is $r+2$ and the degree in $\Delta(P^*)$ of v_0 is r. Then the degree in $\Gamma(P)$ of v_0 is $r+1$ and the degree in $\Delta(P)$ is r. The degrees of all the other vertices in $\Gamma(P)$ and $\Delta(P)$ are r or $r+1$.

Now suppose that $|E(K^*)|$ is odd and that all the vertices of K have degree $2r$ or $2r+2$. Then $K^* = K$ and no extra edges were added in. In fact, this case cannot occur. For suppose it did occur. If K has a vertex of degree $2r+2$ then, by the construction $j = 2i$ for some i, $d_{D_{2i}}(v) = r+1$ and $d_{A(C)}(v) = r+1$ for some component $\tilde{C}$ of $\tilde{H}_i$; moreover $C = \tilde{C}$, $|V(C)|$ is odd and $v = s(C)$. If, for some vertex $v \in V(C) - s(C)$, $d_{D_{2i}}(v) = r+1$

then $d_{P_j}(v) = 2r+1$ since $d_{A(C)}(v) = r$, a contradiction. So we may suppose that $d_{D_{2i}}(v) = r$ for all $v \in V(C) - s(C)$. But then $d_{D_{2i-1}}(v) = r+1$ for all $v \in V(C) - s(C)$, and so $s(C)$ was chosen according to (2), so that $d_{D_{2i}}(s(C)) = r$. Then $d_{P_j}(s(C)) = 2r+1$, again a contradiction. We need finally to consider the possibility that all the vertices of K have degree $2r$. Since we are supposing that r is odd and $|E(K)|$ is odd, it follows that $|V(K)|$ is odd, and so K incorporates an odd order multigraph, $A(C)$ or $B(C)$ for some component $\tilde{C}$ of $\tilde{H}_i$. First suppose that $C \neq \tilde{C}$. In this case (b) and the remark about (b) directly after (c) apply, and show that this possibility cannot arise. Now suppose that $C = \tilde{C}$. For only one $j \in \{2i-1, 2i\}$ can it be true that $d_{D_j}(v) = r$ for all $v \in V(C)$, and, by the construction and the choice of $s(C)$, $A(C)$ is associated with such a D_j, so that $d_{P_j}(v) = 2r+1$ for at least one $v \in V(C)$; thus $d_{P_j}(v) = 2r+1$ for at least one $v \in V(K)$, a contradiction.

Thus it follows that the multigraphs P_i $(1 \leq i \leq M)$ can each be factorized into two $(r, r+1)$-factors. Together with the $(r, r+1)$-factors $D_{2M+1}, ..., D_N$, they give an $(r, r+1)$-factorization of G, as required. □

We need the next lemma in the proof of Lemma 34. First we give some notation. If the edges of a pseudograph G are coloured α and β, let G_α be the spanning subgraph of G whose edges are those coloured α in G; similarly for G_β. We call an edge-colouring of G with α and β *equalized* if $||E(G_\alpha)| - |E(G_\beta)|| \leq 1$. For $v \in V(G)$, let $E(G_\alpha(v))$ be the set of edges incident with v that are coloured α; similarly for $E(G_\beta(v))$. We also recall some definitions. A *circuit* is a connected pseudograph whose vertices all have even degree, and a *trail* is a connected pseudograph with exactly two vertices having odd degree. Note that a circuit is "Eulerian" in the sense that we can write down a sequence $v_0, e_0, v_1, e_1, v_2, \ldots, v_{i-1}, e_{i-1}, v_i, e_i, v_{i+1}, \ldots, v_x, e_x, v_{x+1}$ of vertices and edges in such a way that each edge occurs exactly once, vertices may be repeated, e_i is incident with v_i and v_{i+1}, and $v_0 = v_{x+1}$. A trail has the same property except that $v_0 \neq v_{x+1}$.

Lemma 33. *Let a pseudograph G have an equitable edge-colouring with two colours, α and β. Then G has an edge-colouring with α and β which is both equitable and equalized.*

Proof. Let G be given an equitable edge-colouring with α and β, and suppose that $|E(G_\alpha)| \geq |E(G_\beta)| + 2$. Let $v_1, ..., v_k$ be the vertices v such that $|E(G_\alpha(v))| = |E(G_\beta(v))| + 1$ and let $w_1, ..., w_\ell$ be the vertices w such that $|E(G_\alpha(w))| + 1 = |E(G_\beta(w))|$. For $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$, join v_{2i-1} and v_{2i} by an edge coloured β, and, for $1 \leq i \leq \lfloor \frac{\ell}{2} \rfloor$, join w_{2i-1} and w_{2i} by an edge coloured α. Since $v_1, \ldots, v_k$, $w_1, \ldots, w_\ell$ are the only vertices of odd degree, $k + \ell$ is even, and so k is odd if and only if ℓ is odd. If k and ℓ are both odd, then introduce a further vertex u and join v_k to u by an edge coloured β and join w_ℓ to u by an edge coloured α.

Let G^+ be the graph obtained. Each vertex of G^+ has an equal number of α-coloured edges as β-coloured edges incident with it. We shall show first that G^+ has an edge-disjoint decomposition into circuits C which have the property that the (α, β)-edge-colouring of G^+ restricted to C is also equitable. Thus at each vertex of C, the number of α-coloured edges of C equals the number of β-coloured edges. The decomposition is obtained by separating out equitably coloured circuits one by one.

To see that this can be done, suppose that we have removed circuits $C_1, \ldots, C_h$, and that $G^* = G^+ \setminus E(C_1 \cup \ldots \cup C_h)$ and $E(G^*) \neq \phi$. Note that G^* has the property that at each vertex there are as many edges coloured α as there are edges coloured β. Let $v_0 \in V(G^*)$, let $v_0v_1 \in E(G^*)$ and suppose that v_0v_1 is coloured α. We may "grow" a trail $v_0, v_1, v_2, \ldots, v_i$ edge by edge with the edges coloured alternately α and β. Since each vertex has as many α-coloured edges on it as it has β-coloured edges, we may continue to grow the trail until we reach v_0 again with a final edge coloured β (we may have earlier "passed through" v_0, reaching v_0 first with an α-coloured edge). We then have a circuit C_{h+1}. [This is not the only way, but it seems to be the simplest way to obtain a circuit.] Continuing this inductive process, we eventually decompose G^+ into edge-disjoint circuits.

The additional edges that we added in to form G^+ from G may now be removed. Thinking of the sequence property of circuits, we see that removing some edges of a circuit will divide the circuit into edge-disjoint sections, and each section is in fact a trail. Each circuit in our circuit decomposition of G^+ becomes in G either a circuit (if no edges are removed) or an edge-disjoint union of trails. Thus G itself becomes the union of even circuits and trails, each circuit and trail being alternately coloured α and β. Note that each vertex of G will have odd degree in at most one trail (i.e. be the end-vertex of at most one trail). Also note that if we interchange the colours on any trail, then the edge-colouring of G will still be equitable. Since $||E(G_\alpha)| - |E(G_\beta)|| \geq 2$, there is at least one trail T_1 with an odd number of edges, with each end edge coloured α. We interchange the colours on the edges of T_1. Then G now has one more β-coloured edge than previously, and one fewer α-coloured edge. If this revised edge-colouring of G is still not equalized, there will be a second trail T_2 with both ends coloured α. We interchange the colours on the edges of T_2. Continuing like this, we eventually obtain an equalized and equitable edge-colouring of G. □

Lemma 34. *Let d and r be positive integers with d even, r odd and $d \geq$*

$2r^2+3r-1$. *Then any* $(d,d+1)$*-multigraph has an* $(r,r+1)$*-factorization.*

Proof. Let $d=(r+1)q+s$, where $0\le s\le r$, let $N=q+s-r+1$ and let $M=\frac{r+1}{2}-\frac{s}{2}$. Then s is even, so $s\le r-1$ and M is a non-negative integer. Since $d\ge 2r^2+3r-1$ we have that $(r+1)q+s\ge 2r^2+3r-1$, so that $(r+1)q\ge 2r^2+3r-s-1\ge 2r^2+3r-(r-1)-1=2r^2+2r=2r(r+1)$, so $q\ge 2r$. Therefore $N-2M=q+s-r+1-r-1+s=q-2r+2s\ge 2s\ge 0$, so $N\ge 2M$. Note also that $(r+1)(N-1)+2rM=(r+1)q+s=d$.

We may note that $q=\left\lfloor\frac{d}{r+1}\right\rfloor$.

Let G be a $(d,d+1)$-multigraph. Let H be a maximal $(d-1,d)$-factor of G, and let $F=G-E(H)$. Then F is a graph whose components are isolated vertices, or paths with one or two edges.

Let $\tilde{H}$ be a d-regular multigraph with H as an induced submultigraph. [We shall again use the tilde to denote submultigraphs of $\tilde{H}$ which contain vertices of $V(\tilde{H})-V(H)$. For a submultigraph $\tilde{Q}$ of $\tilde{H}$, its restriction to H will be denoted by Q.] By Theorem 4 (Petersen's theorem), since $d=(r+1)(N-1)+2rM$, we can decompose $\tilde{H}$ into $(N-1)$ $(r+1)$-factors $\tilde{D}_2,...,\tilde{D}_N$ and M $2r$-factors $\tilde{H}_1,...,\tilde{H}_M$.

Following the argument of Lemma 32, we can decompose each of $H_i\cup D_{2i-1}\cup D_{2i}$ $(2\le i\le M)$ into four $(r,r+1)$-factors. We shall show that we can decompose $F\cup H_1\cup D_2$ into three $(r,r+1)$-factors. These, together with the $(r,r+1)$-factors $D_{2M+1},...,D_N$ constitute the $(r,r+1)$-factorization of G that we require.

Let the components of the multigraph H_1 be $C_1,C_2,...,C_k$. For $1\le i\le k$, as we saw in Lemma 32, unless $|E(C_i)|$ is odd and $d_{C_i}(v)=2r\quad(\forall v\in V(C_i))$, C_i can be given an equitable edge-colouring with two colours, say α and β. By Lemma 33, we can make this edge-colouring be equalized as well as equitable. In the exceptional case, there is one vertex, $s(C_i)$, such that, for $v\ne s(C_i)$, $|\alpha(v)|=|\beta(v)|$, and, for $v=s(C_i)$, $||\alpha(v)|-|\beta(v)||=2$; we shall suppose that $|\alpha(s(C_i))|=|\beta(s(C_i))|+2$. [Here $\alpha(v)$ and $\beta(v)$ mean the set of edges of C_i incident with v coloured α or coloured β respectively.] Furthermore, we can choose any vertex $v\in V(C_i)$ to be $s(C_i)$. We let $A(C_i)$ and $B(C_i)$ be the spanning subgraphs of C_i whose edges are the edges of C_i coloured α and the edges coloured β respectively. Since C_i is a $(2r-1,2r)$-multigraph, it follows that, except in the exceptional case, $A(C_i)$ and $B(C_i)$ are both $(r-1,r)$-multigraphs; moreover, since the edge-colouring of C_i was equalized, each of $A(C_i)$ and $B(C_i)$ have at least one vertex of degree r. In the exceptional case, we may suppose that $B(C_i)$ is an $(r-1,r)$-multigraph with exactly one vertex, $s(C_i)$, of degree $r-1$, and $A(C_i)$ is an $(r,r+1)$-multigraph with exactly one vertex, again $s(C_i)$, of degree $r+1$.

Next we adjoin each edge of F to $A(C_i)$ or to $B(C_i)$ for some i, $1\le i\le k$, as follows. Firstly, if C_i contains a vertex v of degree $2r-1$, then an edge e_v of F is incident with v; if v has degree $r-1$ in $A(C_i)$ then we adjoin e_v to $A(C_i)$, and if v has degree $r-1$ in $B(C_i)$ then we adjoin e_v to $B(C_i)$.

I. If some edge $e = uv$ of F has a vertex (say v) in common with an exceptional component C_i (i.e. a $2r$-regular component C_i with an odd number of edges, so that $s(C_i)$ is defined; also $v \in V(C_i)$), then, for one such edge, we choose $s(C_i) = v$, and we adjoin e to $B(C_i)$. If the other end-vertex, u, of e was in a submultigraph $A(C_j)$ for some j, then we interchange the labels $A(C_i)$ and $B(C_i)$. Then $s(C_i)$ has degree $r-1$ in $A(C_i)$ and is the end-vertex of e.

We adjoin all further edges of F to the multigraph $\bigcup_{1\leq i\leq k} B(C_i)$.

II. Again, if at this stage some edge $e = uv$ of F has a vertex (or vertices) in common with an exceptional component C_i (or exceptional components C_i and C_j), then, for one such edge, we choose the vertex $s(C_i)$ (or vertices $s(C_i)$ and $s(C_j)$) to be u or v (or u and v) provided they are not already chosen under I.

Let $\mathscr{A}$ be the multigraph $\bigcup_{1\leq i\leq k} A(C_i)$ with some edges of F adjoined, as just described, and let $\mathscr{B}$ be the multigraph $\bigcup_{1\leq i\leq k} B(C_i)$ with the remaining edges of F adjoined, again as just described. All vertices of $\mathscr{A}$ will have degree r or $r+1$, and all vertices of $\mathscr{B}$ will also have degree r or $r+1$, except for vertices $s(C)$ in components $B(C)$, where C is an exceptional component and no edge of F has a vertex in the set $V(C)$. In forming $\mathscr{A}$, a component $A(C_i)$ may be connected to a component $A(C_j)$ by an edge of F, so the number of components of $\mathscr{A}$ may be less than k. Note that some of the components of $\mathscr{A}$ may be derived from exceptional components C of H_1 with no edges of F having any vertices in $V(C)$; similarly for some of the components of $\mathscr{B}$.

At this point we have one of the three $(r, r+1)$-multigraphs we aim to construct from D_2, F and H_1, namely $\mathscr{A}$.

Let $\mathscr{B}''$ be the set of components of $\mathscr{B}$ of the type $B(C)$, where C is an exceptional component of H_1 with an odd number of edges which has no vertex in common with any edge of F; let $\mathscr{B}' = \mathscr{B} - \mathscr{B}''$.

Let $\mathscr{L}$ be the set of all components of $D_2 \cup \mathscr{B}$. Let $\mathscr{L}''$ be a maximal set, $\{L_1, ..., L_t\}$, of components L_i of $D_2 \cup \mathscr{B}$ such that each L_i contains at least one of the components of $\mathscr{B}''$. Then $\mathscr{L}'' \subseteq \mathscr{L}$. Let $\mathscr{L}' = \mathscr{L} - \mathscr{L}''$.

Let D_2' be the restriction of D_2 to the vertex set $V(G) - V(\bigcup_{1\leq i\leq t} L_i)$. Clearly $V(D_2')$ induces a set of components of $\mathscr{B}$; we let $\mathscr{B}_1$ be this set of components. Then $\mathscr{B}_1 \subseteq \mathscr{B}'$, so $\mathscr{B}_1$ is an $(r, r+1)$-multigraph. We also note that D_2' is an $(r, r+1)$-multigraph.

Now consider the degrees of the vertices of $\mathscr{L}''$. In order to have a vertex v of degree $2r-1$, the contribution from D_2 would have to be r and from the relevant $B(C)$ the contribution would be $r-1$; there would be no contribution from F. Then $v = s(C)$ and $B(C) \in \mathscr{B}''$. As no edge of F is incident with v, $d_{D_2}(v) = r+1$, a contradiction. Thus the degrees of $\mathscr{L}''$ are $2r$, $2r+1$ or $2r+2$. The contribution of a vertex $w \in B(C) - s(C)$ to $d_{\mathscr{L}''}(w)$ from $B(C)$ is r and from D_2 is $r+1$ (for the same reason as for $v = s(C)$). Thus $d_{\mathscr{L}''}(w) = 2r+1$. Hence, by a by now well rehearsed argument, $\mathscr{L}''$ is the union of two $(r, r+1)$-multigraphs. These, together with $\mathscr{B}_1$ and D_2', give the two further $(r, r+1)$-factors we require. □

7.3 The number of $(r, r+1)$-factors

In Theorem 15 we showed how many $(r, r+1)$-factors an $(r, r+1)$-factorization of a $(d, d+1)$-simple graph can have. We are unable at present to give a complete answer to the same question for $(d, d+1)$-multigraphs, but in this section we give a partial answer to this question: we are able to give some of the values of x for which a $(d, d+1)$-multigraph has an $(r, r+1)$-factorization with x $(r, r+1)$-factors.

In Section 7.2 we gave three lemmas all to the effect that, under certain conditions, a $(d, d+1)$-multigraph has an $(r, r+1)$-factorization. The argument in this section mainly involves generalizing these three lemmas. The proofs of the generalizations of the three lemmas all just involve tinkering with the various parameters, and in each case only affect the very beginning of the proof. So the proofs we give here just explain the initial changes. For the complete proofs, the reader needs to combine the discussion of the parameters given here with the proofs of the lemmas given in Section 7.2.

The first generalization is of Lemma 32.

Lemma 35. *Let d, r and k be positive integers with d and r both odd, and with*

$$d \geq 2kr^2 + 3kr - 3r - 4 + k\,.$$

Let

$$q = \left\lfloor \frac{d}{r+1} \right\rfloor\,.$$

Then any $(d, d+1)$-multigraph has an $(r, r+1)$-factorization into $q+k$ $(r, r+1)$-factors.

Proof. Let $d+1 = (r+1)q + s + 2$, where $0 \leq s \leq r$. Since d and r are both odd, it follows that s is even, so $s < r$, and so $s+1 < r+1$. Thus the equations $d = (r+1)q + s + 1$ and $q = \left\lfloor \frac{d}{r+1} \right\rfloor$ are compatible.

Let N and M be integers with $N = q+s-kr+2$ and $M = \frac{1}{2}k(r+1)-\frac{1}{2}s-1$. Then $(r+1)N + 2rM = q(r+1) + s + 2 = d + 1$ and $N + 2M = q + k$.

Since

$$(r+1)q + r \geq (r+1)q + s + 1 = d \geq 2kr^2 + 3kr - 3r - 4 + k\,,$$

it follows that

$$(r+1)q \geq (r+1)(2kr + k - 4)$$

so that $q \geq 2kr + k - 4$. Therefore

$$\begin{aligned} N - 2M &= q + 2s - 2kr - k + 4 \\ &\geq q - 2kr - k + 4 \qquad \text{(since } s \geq 0) \\ &\geq 0\,. \end{aligned}$$

The proof now proceeds verbatim as in the proof of Lemma 32, and produces an $(r, r+1)$-factorization of G into $q + k(= N + 2M)$ $(r, r+1)$-factors. $\square$

Generalizing Lemma 35 slightly, we have:

Theorem 36. *Let d, r, k be positive integers, with d and r odd and with $d \geq 2kr^2+3kr-3r-4+k$. Let $q = \left\lfloor \frac{d}{r+1} \right\rfloor$. Then any $(d, d+1)$-multigraph has an $(r, r+1)$-factorization into x $(r, r+1)$-factors for each x, $q+1 \leq x \leq q+k$.*

Proof. This follows from Lemma 35 since the function $2kr^2+3kr-3r-4+k$ increases monotonically with k for $k, r \geq 1$. $\square$

The next generalization is of Lemma 34.

Lemma 37. *Let d, r and k be positive integers with d even, r odd, and with*

$$d \geq 2kr^2 + 3kr + k - 2\,.$$

Let

$$q = \left\lfloor \frac{d}{r+1} \right\rfloor\,.$$

Then any $(d, d+1)$-multigraph has an $(r, r+1)$-factorization into $q + k$ $(r, r+1)$-factors.

Proof. Let $d = (r+1)q + s$, where $0 \leq s \leq r$. Observe that this accords with the equation $q = \left\lfloor \frac{d}{r+1} \right\rfloor$. Note that s is even, so in fact $s < r$.

Let N and M be integers with $N = q+s-kr+1$ and $M = \frac{1}{2}k(r+1)-\frac{1}{2}s$. Then

$$(r+1)(N-1) + 2rM = q(r+1) + s = d$$

and

$$N + 2M - 1 = q + k\,.$$

Since

$$(r+1)q+r-1 \geq (r+1)q+s = d \geq 2kr^2+3kr+k-2\,,$$

it follows that

$$(r+1)q \geq (r+1)(2kr+k-1)\,,$$

so that

$$q \geq 2kr+k-1\,.$$

Therefore

$$N-2M = q-(2kr+k-1)+2s \geq 2s \geq 0\,.$$

The proof now proceeds word for word as in the proof of Lemma 34, and produces an $(r,r+1)$-factorization of G into $q+k$ $(=N+2m-1)$ $(r,r+1)$-factors. $\square$

Generalizing this slightly, we have:

Lemma 38. *Let d, r and k be positive integers with d even, r odd, and with $d \geq 2kr^2+3kr+k-2$. Let $q = \left\lfloor \frac{d}{r+1} \right\rfloor$. Then any $(d,d+1)$-multigraph has an $(r,r+1)$-factorization into x $(r,r+1)$-factors for each x, $q+1 \leq x \leq q+k$.*

Proof. This follows from Lemma 37 since the function $2kr^2+3kr+k-2$ increases monotonically with k for $k, r \geq 1$. $\square$

As a consequence of Lemma 30, we have the following corollary.

Lemma 39. *Let d, r and k be integers with $k \geq 0$, r even, $r \geq 2$ and $d \geq 2r^2(k+1)+r(k+1)-1$. Let $q = \left\lfloor \frac{d}{r} \right\rfloor$. Then any $(d,d+1)$-multigraph has an $(r,r+1)$-factorization into x $(r,r+1)$-factors for each x, $q-k \leq x \leq q$.*

Proof. Let G be a $(d,d+1)$-multigraph. We may suppose that

$$d = (2r(k+1)+y+k+1-j)r+jr+z-1\,,$$

where $y \geq 0$, $1 \leq z \leq r$ and $0 \leq j \leq k$. Then $q = 2r(k+1)+y+k+1$. Put $q^* = 2r(k+1)+y+k+1-j$ and $s^* = jr+z$. Then $d = q^*r+s^*-1$. Note that $q^* \geq 2s^*$, so G has an $(r,r+1)$-factorization into $q^* = 2r(k+1)+y+k+1-j$ $(r,r+1)$-factors, by Lemma 30. Thus G has an $(r,r+1)$-factorization into x $(r,r+1)$-factors for $q-k \leq x \leq q$. $\square$

It is convenient to re-express Lemma 39 in the following way.

Theorem 40. *Let d, k, r be positive integers with r even and $d \geq 2r^2k+rk-1$. Let $q = \left\lfloor \frac{d}{r} \right\rfloor - k$. Then any $(d,d+1)$-multigraph has an $(r,r+1)$-factorization into x $(r,r+1)$-factors for each x, $q+1 \leq x \leq q+k$.*

Proof. Let G be a $(d, d+1)$-multigraph. From Lemma 39, with r, k, d and q as in that lemma, it follows that if $q' = \lfloor \frac{d}{r} \rfloor - k - 1$, then G has an $(r, r+1)$-factorization with x $(r, r+1)$-factors for each x,

$$q' + 1 \left(= \left\lfloor \frac{d}{r} \right\rfloor - k = q - k \right) \leq x \leq q' + k + 1 \left(= \left\lfloor \frac{d}{r} \right\rfloor = q \right) .$$

Putting $k' = k + 1$, it follows that if k' and r are positive integers with r even and $d \geq 2r^2k' + rk' - 1$, and if $q' = \lfloor \frac{d}{r} \rfloor - k'$, then G has an $(r, r+1)$-factorization with x $(r, r+1)$-factors, for each x, $q'+1 \leq x \leq q'+k$. Theorem 40 now follows by replacing q' and k' by q and k. □

Collecting information from Theorems 35, 38 and 40 together, we obtain the following theorem.

Theorem 41. *Let $d \geq 2kr^2 + 3kr + k + 2$, where $r \geq 1$ and $k \geq 1$. Let*

$$q = \begin{cases} \lfloor \frac{d}{r+1} \rfloor & if \quad r \quad is \ odd, \\ \lfloor \frac{d}{r} \rfloor - k & if \quad r \quad is \ even. \end{cases}$$

Then any $(d, d+1)$-multigraph has an $(r, r+1)$-factorization with $(r, r+1)$-factors for each x, $q + 1 \leq x \leq q + k$.

Proof. This follows directly from Lemmas 35, 38 and Theorem 40 since

$$\begin{aligned} & 2kr^2 + 3kr + k - 2 \\ = \ & \max\{2kr^2 + 3kr - 3r - 4 + k,\ 2kr^2 + 3kr + k - 2,\ 2kr^2 + kr - 1\} . \end{aligned}$$

□

8 Postscript: equitable edge-colourings of multigraphs

The close relationship between equitable edge-colourings and $(r, r+1)$-factorizations of $(d, d+1)$-graphs is well-illustrated by the quick deduction of Theorem 15(i) from Theorem 14. Theorem 14 in fact is a very nice result about equitable colourings. Possibly Theorem 14 could be strengthened:

Conjecture 2. Let k be a positive integer and let G be a simple graph. If the k-core of G is a forest, then G has an equitable colouring with k colours.

The best result about equitable colourings of pseudographs is due to Häggkvist and Johansson [11]. It has proved to be extremely useful (see [8], [12], [16]).

Theorem 42. (Häggkvist and Johansson). *Let ℓ be an even integer and let G be a connected pseudograph whose degrees are $\ell k - 2$, $\ell k - 1$ or ℓk. Then G has an equitable edge-colouring with k colours if and only if G has some vertex of degree $\ell k - 1$ or else the number of vertices of degree $\ell k - 2$ is either 0 or at least 2, and not odd if $k = 2$.*

Finally we note the following easy deduction from Theorem 41.

Theorem 43. *Let G be a $(d, d+1)$-multigraph, where $d \geq 2kr^2 + 3kr + k + 2$, and $r \geq 1$, $k \geq 1$. Then any $(d, d+1)$-multigraph has an equitable edge-colouring with x colours for each x satisfying*

$$\begin{cases} \left\lfloor \frac{d}{r+1} \right\rfloor + 1 \leq x \leq \left\lfloor \frac{d}{r+1} \right\rfloor + k & \textit{if } r \textit{ is odd,} \\ \left\lfloor \frac{d}{r} \right\rfloor - k + 1 \leq x \leq \left\lfloor \frac{d}{r} \right\rfloor & \textit{if } r \textit{ is even.} \end{cases}$$

It is very likely that this result can be extended considerably.

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